Groupoids, C*-Algebras and Index Theory

Nigel Higson

Lecture at FIM, July 10, 2004

1 Introduction

My goal in this talk is to introduce some topics in Alain Connes' noncommutative geometry, organized around the notion of groupoid and involving for the most part elaborations of the index theory of Atiyah and Singer.

2 Groupoids and Noncommutative Geometry

Groupoids figure prominently in Alain Connes' noncommutative geometry, where noncommutative algebras serve as the coordinate rings for a variety of highly singular spaces — for example the space of leaves of a foliation or the space of orbits of a group action on a manifold.

2.1 Definition. A groupoid is a small category (the collections of all morphisms and all objects are sets) in which every morphism is invertible.

It is often convenient to present a groupoid by specifying its set of objects, B, and the set G of all morphisms, together with the following *structure maps*:

(i) The *source* and *range* maps $s, r\colon G \to B$, which map each morphism to its source and range.

(ii) The *composition* map $\circ\colon G^{(2)} \to G$, where $G^{(2)}$ is the set of composable pairs of morphisms in G:

$$G^{(2)} = \{(\gamma_1, \gamma_2) \in G \times G : r(\gamma_2) = s(\gamma_1)\}.$$

(iii) The *unit* map $e\colon B \to G$ which maps each object to the corresponding identity morphism

(iv) The *inverse* map $i\colon G \to G$ which sends each object to its inverse.

2.2 Example. Suppose that group A acts on a set B. Build a groupoid $A \ltimes B$ as follows. The object space is B and the morphism space is the set of triples

$$A \ltimes B = \{\, (b_2, a, b_1) \in B \times A \times B : a \cdot b_1 = b_2 \,\}.$$

The source and range maps are

$$s(b_2, a, b_1) = b_1 \quad \text{and} \quad r(b_2, a, b_1) = b_2$$

while the composition is

$$(b_3, a_2, b_2) \circ (b_2, a_1, b_1) = (b_3, a_2 a_1, b_1).$$

The identity at b is $(b, 1, b)$ and the inverse of (b_2, a, b_1) is (b_1, a^{-1}, b_2). This is the *crossed product groupoid* associated to the action of A on B.

Noncommutative geometry adopts what might be called the *quotient space picture* of groupoid theory. We focus on the object space B of a groupoid, and we think of the morphisms in the groupoid as defining an equivalence relation on B: two objects are equivalent if there is a morphism between them. Two objects might be equivalent for more than one reason, and the groupoid keeps track of this.

It is customary in mathematics to form the quotient space from an equivalence relation, but even in rather simple examples the ordinary quotient space of general topology can be highly singular, and for example not at all a manifold. The groupoid serves as a smooth stand-in for the quotient space in these situations, and using it one can study for example the cohomology of the quotient space, and even its geometry.

2.3 Example. The crossed-product groupoid construction gives a perfect illustration of this. Let us consider, for instance, the action of the group $\mathbb{Z}$ on the unit circle $\mathbb{T}$ in which the generator of $\mathbb{Z}$ acts by rotation of the circle through an angle which is an irrational multiple of π. The quotient space $\mathbb{T}/\mathbb{Z}$ is a disaster as a topological space — there are no non-trivial open sets at all — and the groupoid $\mathbb{Z} \ltimes \mathbb{T}$ serves as a stand-in. This is in fact a fundamental example in noncommutative geometry.

Of particular interest among groupoids, and especially easy to handle, are the smooth groupoids:

2.4 Definition. A *smooth groupoid* is a groupoid for which the set G of all morphisms and the set B of all objects are smooth manifolds; for which the source and range maps $s, r\colon G \to B$ are submersions; and for which the other remaining structure maps (composition, units, inverses) are smooth.

2.5 Remark. It is a consequence of the fact that r and s are submersions that the set $G^{(2)}$ of composable pairs of morphisms, is a smooth submanifold of $G \times G$.

Before continuing, we need to sketch out what might be called the *families picture* of groupoids, which is a little different in perspective from the quotient space picture. In the families picture we view a groupoid as, first and foremost, the family of smooth manifolds

$$G_x = \{\gamma \in G : s(\gamma) = x\}$$

parametrized by $x \in B$. If $\eta\colon x \to y$ is a morphism in G, then there is an associated diffeomorphism

$$R_\eta\colon G_y \to G_x$$

defined by $R_\eta(\gamma) = \gamma \circ \eta$. In this way we view the groupoid G as a smooth family of smooth manifolds, on which acts the collection of all the intertwining diffeomorphisms R_η.

Of interest in many contexts are familes of smoothing kernels $k_x(\gamma_2, \gamma_1)$ defined on the fibers of G, which are equivariant with respect to the action of the diffeomorphisms R_η: thus $k_y(\gamma_2, \gamma_1) = k_x(\gamma_2 \circ \eta, \gamma_1 \circ \eta)$ when $\eta\colon x \to y$. Such families correspond to smooth functions on G. Indeed from a smooth function $f\colon G \to \mathbb{C}$ we obtain an equivariant family of smoothing kernels k_x on the spaces G_x by the formula $k_x(\gamma_2, \gamma_1) = f(\gamma_2\gamma_1^{-1})$.

In other to consider kernel functions as operators (for instance on the spaces $L^2(G_x)$) we need to speicify measures on the fibers G_x.

2.6 Definition. A *Haar system* on a smooth groupoid G is a family of smooth measures μ_x on the fibers G_x of G such that

(i) If f is a smooth, compactly supported function on G then $\int_{G_x} f\, d\mu_x$ is a smooth function on B; and

(ii) If $\gamma\colon x \to y$ is a morphism in G, then the right-translation operator $R_\gamma\colon G_y \to G_x$ is measure-preserving.

Haar systems may be proved to exist in much the same way that Haar measures are proved to exist on Lie groups. Any two Haar systems $\{\mu_x\}$ and $\{\mu'_x\}$ differ by a smooth, positive function f on the object space B: thus $\mu_x = f(x)\mu'_x$, for all x. For our purposes this means that there is an essentially unique choice of Haar system, as there is in the Lie group case. We shall assume from now on that attached to each smooth groupoid there is a fixed Haar system.

2.7 Definition. Let G be a smooth groupoid. The *convolution algebra* of G is the spae $C_c^\infty(G)$ of smooth, compactly supported functions on G, equipped with the following associative convolution product:

$$f_1 \star f_2(\gamma) = \int_{G_{s(\gamma)}} f_1(\gamma \circ \eta^{-1}) f_2(\eta)\, d\mu_{s(\gamma)}(\eta).$$

Thus $C_c^\infty(G)$ is made into an associative algebra, consisting of equivariant families of smoothing operators on the fibers of G.

The operation

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

makes $C_c^\infty(G)$ into a $*$-algebra. For many applications in geometry and topology it is useful to complete this $*$-algebra so as to obtain a C^*-algebra:

2.8 Definition. Let G be a smooth groupoid with right Haar system. Define representations

$$\lambda_x \colon C_c^\infty(G) \to \mathcal{B}(L^2(G_x))$$

by the formulas

$$\lambda_x(f)h(\gamma) = f \star h(\gamma) = \int_{G_{s(\gamma)}} f(\gamma \circ \eta^{-1}) h(\eta)\, d\mu_{s(\gamma)}(\eta).$$

The *reduced groupoid* C^**-algebra* of G, denoted $C_\lambda^*(G)$, is the completion of $C_c^\infty(G)$ in the norm

$$\|f\| = \sup_x \|\lambda_x(f)\|_{\mathcal{B}(L^2(G_x))}.$$

Returning to the quotient space picture, the groupoid C^*-algebra is Alain Connes' substitute for the algebra of continuous, complex-valued functions on the quotient space associated to G.

2.9 Example. The C^*-algebra of the crossed product groupoid associated to an irrational rotation action of $\mathbb{Z}$ on the circle is the famous *irrational rotation algebra*, or *noncommutative torus* A_α. It is the C^*-algebra generated by two unitary elements U and V subject to the relation $UV = \exp(i\alpha)VU$, where α is the angle of rotation.

Let us consider in more detail the irrational rotation action of $\mathbb{Z}$ on the unit circle, and the associated crossed product groupoid $G = \mathbb{Z} \ltimes \mathbb{T}$. The C^*-algebra $A_\alpha = C^*_\lambda(G)$ is an algebra consisting of continuous operator-valued functions on the circle which are invariant under the irrational rotation action of $\mathbb{Z}$ on the circle. This might at first seem surprising since of course every continuous, scalar function on the circle which is invariant under an irrational rotation must be constant. However the fact that our functions are operator valued allows for some interesting possibilities, as follows. First, the spaces G_x all identify with $\mathbb{Z}$. The functions which constitute elements of $C^*_\lambda(G)$ are continuous functions from the unit circle into $\mathcal{B}(\ell^2(\mathbb{Z}))$ which are equivariant under the action of $\mathbb{Z}$ which combines irrational rotation on the circle with translation in $\ell^2(\mathbb{Z})$. Two examples are the functions U and V given by the formulas

$$U_z\phi(n) = \phi(n+1) \quad \text{and} \quad V_z\phi(n) = \exp(i\alpha n)z\phi(n),$$

where $z \in \mathbb{T}$, $n \in \mathbb{Z}$, and $\phi \in \ell^2(\mathbb{Z})$. They satisfy the relation $UV = \exp(i\alpha)VU$, and they generate the C^*-algebra $C^*_\lambda(G)$.

2.10 Remark. The families picture is especially appropriate when we consider the *von Neumann algebra* of a groupoid G. This is a certain von Neumann algebra completion of $C^*_\lambda(G)$, and it consists of *all* bounded, measurable and equivariant families of bounded operators on the field of Hilbert spaces $\{L^2(G_x)\}$. These von Neumann algebras figured prominently in Connes' early work on noncommutative geometry.

3 K-Theory and Index Theory

The main cohomological invariant of Connes' non-commutative spaces, or indeed of C^*-algebras in general, is K-theory. The K-theory groups of a C^*-algebra can be defined[1] as the homotopy groups of the stable general linear group $GL_\infty(A)$:

$$K_j(A) = \pi_{j-1}(GL_\infty(A)).$$

[1]Most of the C^*-algebras of interest to us fail to be unital, and for these the definition of $GL_\infty(A)$ has to be appropriately tailored.

The famous Bott periodicity theorem generalizes to C^*-algebra K-theory and asserts that $K_j(A) \cong K_{j+2}(A)$, for all j, using which we can extend the definition of $K_j(A)$ to all integer indices j. If A is the commutative C^*-algebra of continuous, complex-valued functions on a compact space X then $K_*(A)$ is isomorphic to the Atiyah-Hirzebruch topological K-theory $K^*(X)$.

Elements of C^*-algebra K-theory groups are obtained from constructions which produce invertible matrices over C^*-algebras, loops of invertible matrices, and so on, and an important source of these constructions is index theory. Suppose for example that D is an elliptic linear partial differential operator on a closed manifold M (we will review the theory of these in the next section). If D is self-adjoint then we may apply to it Hilbert space spectral theory and form functions, $f(D)$, of D. It follows from regularity theory for elliptic operators that if f is a rapidly decreasing function then $f(D)$ is in fact a smoothing operator on M, and in particular a compact operator on $L^2(M)$.[2] In fact if f is any continuous function which vanishes at infinity then $f(D)$ is a compact operator. Using this fact it is possible to construct an element $\mathrm{Ind}(D) \in K_*(\mathcal{K}(L^2(M))$, where $\mathcal{K}(L^2(M))$ denotes the C^*-algebra of compact operators on $L^2(M)$. Which particular K_j-group the class $\mathrm{Ind}(D)$ belongs to depends on the symmetry of D. If we assume, for example, that D has the "supersymmetric" form

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

which is common in geometry then $[D]$ belongs to the K_0-group. Now, the group $K_0(\mathcal{K})$ identifies with $\mathbb{Z}$, and under this identification $\mathrm{Ind}(D)$ corresponds to none other than the Fredholm index of D_+.

Suppose now that G is a smooth groupoid with compact object space, and assume that $\{D_x\}$ is a smooth, equivariant family of self-adjoint differential operators on the fibers G_x of G. If we apply to the operators D_x a rapidly decreasing function f then each $f(D_x)$ is a smoothing operator, and so is represented by a kernel function $k_x(\gamma_2, \gamma_1)$. These kernels vary smoothly with x, and are invariant under right translations, in the sense that if $\eta\colon x \to y$ then

$$k_y(\gamma_2, \gamma_1) = k_x(\gamma_2 \circ \eta, \gamma_1 \circ \eta).$$

Accordingly they give rise to a single smooth function f_D on G via the formula $f_D(\gamma) = k_x(\gamma, e)$, where $x = s(\gamma)$, that ew employed earlier.

[2]Here and elsewhere, we are going to neglect in our notation the fact that D may act not on scalar functions but on sections of some vector bundle.

3.1 Theorem. *Let* G *be a smooth groupoid with compact object space* B*. Let* $D = \{D_x\}$ *be a smooth, equivariant family of elliptic, first order differential operators on the fibers* G_x *of* G*. If* f *is a continuous function on* $\mathbb{R}$ *which vanishes at infinity then the above construction defines an element* $f_D \in C^*_\lambda(G)$.

3.2 Remarks. The extra hypothesis, that the operators D_x are of first order, is used to satisfy a support condition — recall that $C^*_\lambda(G)$ is a completion of the *compactly supported* smooth functions on M. For a general continuous function f, the kernel k_x associated to the operator $f(D_x)$ may not be a smooth function, but rather just a distribution. So the statement of the theorem has to be interpreted with some care. What *is* true is that if f has compactly supported Fourier transform then the function f_D on G, assembled from the kernels $k_x(\gamma_2, \gamma_1)$, is smooth and compactly supported. The interpretation of the statement $f_D \in C^*_\lambda(G)$, for a general continuous function f which vanishes at infinity, is made via an approximation argument.

3.3 Example. Associate to a smooth, closed manifold M the *pair groupoid* $M \times M$ with object space M, source and range maps

$$s(m_2, m_1) = m_1 \quad \text{and} \quad r(m_2, m_1) = m_2,$$

and composition

$$(m_3, m_2) \circ (m_2, m_1) = (m_3, m_1).$$

The identity morphism at $m \in M$ is (m, m), and the inverse of (m_2, m_1) is (m_1, m_2). Looking at the pair groupoid $G = M \times M$ from the families point of view, we find that all the fibers G_m identify with M, in such as way that the intertwining maps $R_\eta \colon G_y \to G_x$ are all the identity map on M. So a single operator D on M gives rise to a (effectively constant) equivariant family of operators on the fibers of the pair groupoid. Applying the theorem we obtain the statement $f(D) \in \mathcal{K}(L^2(M))$ that we noted earlier.

A richer collection of examples is provided by the theory of foliations. In order to avoid some complications we shall consider here only foliated manifolds (M, F) with *trivial holonomy*. It is not important, in this survey, to know the meaning of this hypothesis. But for example if the leaves of the foliation are simply connected then the foliation automatically has trivial holonomy.

3.4 Definition. Let (M, F) be a foliated manifold, with trivial holonomy. The *foliation groupoid* $G(M, F)$ is, in the families picture, the union

$$G(M, F) = \cup_{m \in M} L_m,$$

where L_m is the leaf of the foliation which contains $m \in M$. Thus $G(M, F)$ may alternately be thought of as the space

$$G(M, F) = \{ (m_2, m_1) \in M \times M \mid m_1 \text{ and } m_2 \text{ belong to the same leaf} \}.$$

The source and range maps are the projection onto the m_1 and m_2-coordinates, respectively. Composition is given by the formula $(m_3, m_2) \circ (m_2, m_1) = (m_3, m_1)$; the identity morphism at m is (m, m), and the inverse of (m_2, m_1) is (m_1, m_2).

The reason for the hypothesis of trivial holonomy is that using it we can give $G(M, F)$ a very natural smooth groupoid structure, which however we shall not describe here. (In general the definition of $G(M, F)$ requires some modification — one must replace the leaves L_m by their "holonomy covers.") We may therefore form the groupoid C^*-algebra, which is called the *foliation* C^**-algebra* and is denoted $C^*_\lambda(M, F)$.

Now if (M, F) is a compact, foliated manifold, and if D is a leafwise elliptic operator on M (meaning that D restricts to each of the leaves of M, and is elliptic there), then by means of K-theory constructions we hinted at earlier we obtain a class $\mathrm{Ind}(D) \in K_*(C^*_\lambda(M, F))$. This is the K-theoretic *index* of the leafwise elliptic operator D, and the subject of Connes' index theorem for foliations. It has many of the familiar properties from classical index theory. For example (for those familiar with some index theory) the index of the leafwise signature operator is a (leafwise) homotopy invariant, while the leafwise index of the Dirac operator is zero in the presence of positive (leafwise) scalar curvature.

If the leaves of the foliated manifold (M, F) are actually the fibers of a submersion $M \to Z$ then the foliation C^*-algebra is very closely related to $C_0(Z)$, the algebra of continuous functions on the base, and indeed $K_*(C^*_\lambda(M, F))$ is isomorphic to $K_*(C_0(Z))$, which is in turn isomorphic to the Atiyah-Hirzebruch K-theory $K^*(Z)$. In this case the element $\mathrm{Ind}(D)$ is given by the Atiyah-Singer index of families construction. In general it is a rather more elaborate object.

4 The Tangent Groupoid

The constructions in the last section raise two issues:

- Develop index theory in a variety of contexts; for example, for foliations..
- Develop tools to compute the K-theory groups of groupoid C^*-algebras.

In this section we shall consider the first. We shall look at classical index theory from a groupoid point of view. The approach generalizes very easily to foliations and other situations.

Let M be a smooth manifold. We are going to define the *tangent groupoid* of M, which is a smooth groupoid $\mathbb{T}M$ whose object space is the product $M \times \mathbb{R}$. In the families picture, the tangent groupoid of M consists of repeated copies of M, together with the tangent spaces $T_m M$. These are joined together to form the fibers of a single smooth map $s\colon \mathbb{T}M \to M \times \mathbb{R}$. In order to describe how this is done we need to review two rather simpler constructions, which we shall combine to form $\mathbb{T}M$.

The *tangent bundle* of M, TM, can be thought of as a family of groups — the vector spaces $T_m M$ — parametrized by M, and in this way it can be thought of as a groupoid with object space M. Thus the source and range maps are[3]

$$s(X, m) = m \quad \text{and} \quad r(X, m) = m,$$

while composition is given by the formula

$$(X, m, 0) \circ (Y, m, 0) = (X + Y, m, 0).$$

The identity map at $m \in M$ is the morphism $(0, m)$, and the inverse of (X, m) is $(-X, m)$. The tangent bundle is a smooth groupoid.

The *pair groupoid* on M was introduced in the last section. It is $M \times M$, with object space M, source and range maps

$$s(m_2, m_1) = m_1 \quad \text{and} \quad r(m_2, m_1) = m_2,$$

and composition

$$(m_3, m_2) \circ (m_2, m_1) = (m_3, m_1).$$

The identity morphism at $m \in M$ is (m, m), and the inverse of (m_2, m_1) is (m_1, m_2).

4.1 Definition. Let M be a smooth, open manifold. The *tangent groupoid* of M is the groupoid $\mathbb{T}M$ constructed as the disjoint union of groupoids $G = \cup_{t\in\mathbb{R}} G_t$, where $G_0 = TM$ and $G_t = M \times M$, when $t \neq 0$.

Thus the object space for $\mathbb{T}M$ is the disjoint union of the object spaces for the groupoids G_t, and since the object space for each G_t is M, we can identify the

[3] We denote elements of TM as pairs, (X, m), where $m \in M$ and $X \in T_m M$.

object space of $\mathbb{T}M$ with $M \times \mathbb{R}$. The source, range and other structure maps for $\mathbb{T}M$ are defined "fiberwise," using the structure maps in each G_t.

We are going to topologize $\mathbb{T}M$. To do so, it will be convenient to write to write elements of $G_0 = TM$ as triples $(X, m, 0)$, where $m \in M$ and $X \in T_m M$. When $t \neq 0$ we shall write elements of $G_t = M \times M$ as triples (m_2, m_1, t), where $m_1, m_2 \in M$. Thus we shall regard $\mathbb{T}M$ as the space

$$\mathbb{T}M = TM \times \{0\} \cup M \times M \times \mathbb{R}^\times.$$

We shall think of a triple (m_2, m_1, t) as an "approximate tangent vector" which is close to a real tangent vector $X \in TM$ if the difference quotient $|f(m_2) - f(m_1)|/t$ is close to $X(f)$ on smooth functions $f \in C^\infty(M)$.

4.2 Definition. Let M be a smooth manifold. The space $\mathbb{T}M$ is equipped with the weakest topology (the one with the fewest open sets) such that for each $f \in C^\infty(M)$ the map

$$\begin{cases} (X, m, 0) \mapsto X(f) \\ (m_2, m_1, t) \mapsto \dfrac{f(m_2) - f(m_1)}{t} \end{cases}$$

from $\mathbb{T}M$ into $\mathbb{R}$, is continuous, and in addition the maps $s, r\colon \mathbb{T}M \to M \times \mathbb{R}$ are continuous.

The topology on $\mathbb{T}M$ is Hausdorff. Moreover it is locally Euclidean:

4.3 Lemma. *Let M be a smooth manifold. If W is an open subset of M then the set*

$$\mathbb{T}W = TW \times \{0\} \cup W \times W \times \mathbb{R}^\times$$

is an open subset of $\mathbb{T}M$. Moreover if $\phi\colon W \to \mathbb{R}^n$ is a diffeomorphism onto an open subset then the map $\Phi\colon \mathbb{T}W \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined by the fomulas

$$\begin{cases} \Phi(X, m, 0) = (D\phi(X), \phi(m), 0) \\ \Phi(m_2, m_1, t) = \big(\dfrac{\phi(m_2) - \phi(m_1)}{t}, \phi(m_1), t\big) \end{cases}$$

is a homeomorphism onto an open subset. □

4.4 Remark. We denote by $D\phi\colon T_m U \to \mathbb{R}^n$ the derivative of ϕ at $m \in U$ (we are identifying the tangent space $\mathbb{R}^n$ at the point $\phi(m)$ with $\mathbb{R}^n$ itself).

4.5 Definition. Let us call the map $\Phi\colon \mathbb{T}W \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ the *standard local coordinate chart* on $\mathbb{T}M$ associated to the local coordinate chart ϕ on M.

The standard local coordinate charts determine a C^∞ atlas of charts for the manifold $\mathbb{T}M$:

4.6 Lemma. *Let W and V be open subsets of a smooth manifold M, and let $\Phi\colon \mathbb{T}W \to \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ and $\Psi\colon \mathbb{T}V \to \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ be the standard local coordiante charts associated to local coordinate charts $\phi\colon W \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^n$. The composition $\Psi\circ\Phi^{-1}$ is a smooth map from one open subset of $\mathbb{R}^n\times\mathbb{R}^n\times R$ to another.*

Proof. The inverse Φ^{-1} is given by the formula

$$\Phi^{-1}(v_2, v_1, t) = \begin{cases} (D\phi^{-1}_{v_1}(v_2), \phi^{-1}(v_1), 0) & \text{if } t = 0 \\ (\phi^{-1}(tv_2 + v_1), \phi^{-1}(v_1), t) & \text{if } t \neq 0 \end{cases}$$

Using the notation $\theta = \psi\circ\phi^{-1}$, the composition $\Theta = \Psi\circ\Phi^{-1}$ is therefore given by the formula

$$\Theta(w_2, w_1, t) = \begin{cases} (D\theta_{w_1}(w_2), \theta(w_1), 0) & \text{if } t = 0 \\ (\dfrac{\theta(tw_2 + w_1) - \theta(w_1)}{t}, \theta(w_1), t) & \text{if } t \neq 0. \end{cases}$$

By a version of the Taylor expansion, there is a smooth, matrix-valued function $\widetilde{\theta}(w_2, w_1)$ on $\mathbb{R}^n\times\mathbb{R}^n$ such that

$$\frac{\theta(tw_2 + w_1) - \theta(w_1)}{t} = \widetilde{\theta}(w_2, w_1)w_2,$$

while $\widetilde{\theta}(0, w_1)$ is the derivative of θ at w. So we see that

$$\Theta(w_2, w_1, t) = \begin{cases} \big(D\theta_{w_1}(w_2), \theta(w_1), 0\big) & \text{if } t = 0 \\ \big(\widetilde{\theta}(tw_2, w_1)w_2, \theta(w_1), t\big) & \text{if } t \neq 0. \end{cases}$$

This is clearly a smooth function. □

We have therefore obtained a smooth manifold $\mathbb{T}M$. It is clear that the source map $s\colon \mathbb{T}M \to M\times\mathbb{R}$ is a submersion since in the local coordinates of Lemma **??** it is a coordinate projection. To verify that $\mathbb{T}M$ is in fact a smooth groupoid, it is convenient to consider first the case where $M = \mathbb{R}^n$, in the following way:

4.7 Example. The map $\Phi\colon \mathbb{TR}^n \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined by

$$\begin{cases} \Phi\colon (v_2, v_1, 0) \mapsto (v_2, v_1, 0) \\ \Phi\colon (v_2, v_1, t) \mapsto \big(\frac{v_2 - v_1}{t}, v_1, t\big) \quad (t \neq 0) \end{cases}$$

is a diffeomorphism. Indeed it is the (globally defined) standard coordinate chart on $\mathbb{TR}^n$ associated to the "identity" chart $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$. Now consider the space

$$G = \{\, (w_2, a, w_1) \,:\, w_1, w_2 \in \mathbb{R}^n \times \mathbb{R}, a \in \mathbb{R}^n, w_2 = a \triangle w_1 \,\},$$

where the operation $\triangle$ is defined by

$$a \triangle (v, t) = (v + ta, t).$$

Thus the $\triangle$ operation defines an action of the group $A = \mathbb{R}^n$ on $\mathbb{R}^n \times \mathbb{R}$, and our space G is the corresponding crossed product groupoid $A \ltimes (\mathbb{R}^n \times \mathbb{R})$. The smooth manifold G identifies with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ by dropping w_2 from (w_2, a, w_1). Using this, we can consider the diffeomorphism Φ to be a diffeomorphism from $\mathbb{TR}^n$ to G by the formulas

$$\Phi(v_2, v_1, 0) = \big((v_1, 0), v_2, (v_1, 0)\big)$$
$$\Phi(v_2, v_1, t) = \Big((v_2, t), \frac{v_2 - v_1}{t}, (v_1, t)\Big) \quad (t \neq 0).$$

But it is evident that Φ is actually an isomorphism of groupoids (in other words, Φ is compatible with all the groupoid structure maps). It therefore follows that a $\mathbb{TR}^n$ is a smooth groupoid, as required, since G is certainly a smooth groupoid.

To summarize:

4.8 Proposition. *Denote by* $G = A \ltimes \mathbb{R}^{n,1}$ *the transformation groupoid associated to the action of the group* $A = \mathbb{R}^n$ *on the space* $\mathbb{R}^{n,1} = \mathbb{R}^n \times \mathbb{R}$ *given by the formula*

$$a \triangle (v, t) = (v + ta, t) \qquad (\, a \in \mathbb{R}^n \text{ and } (v, t) \in \mathbb{R}^{n,1}).$$

The map $\Phi\colon \mathbb{TR}^n \to G$ *which is given by the formulas*

$$\Phi(v_2, v_1, 0) = \big((v_1, 0), v_2, (v_1, 0)\big)$$
$$\Phi(v_2, v_1, t) = \Big((v_2, t), \frac{v_2 - v_1}{t}, (v_1, t)\Big) \quad (t \neq 0).$$

is an isomorphism of smooth groupoids. □

4.9 Remark. The groupoid $\mathbb{T}\mathbb{R}^n$ only depends on the smooth structure of $\mathbb{R}^n$, whereas, superficially at least, the groupoid $G = A \ltimes \mathbb{R}^{n,1}$ depends very much on the vector space structure of $\mathbb{R}^n$. The proposition shows that this dependence is an illusion.

4.10 Proposition. *If M is any smooth manifold, then $\mathbb{T}M$ is a smooth groupoid.*

Proof. Since smoothness is a local property, we can check this in a coordinate neighbourhood W. Since the construction of $\mathbb{T}W$ is coordinate-independent we can assume that $W = \mathbb{R}^n$, and thereby reduce to the example just considered. □

To understand what the tangent groupoid is good for, we need to recall something about the regularity theory of order k, elliptic linear partial differential operators.

4.11 Definition. If D is a linear operator on M then for each point m of M we can form the *model operator* D_m, which is the translation-invariant, homogeneous operator on T_mM which best approximates D at the point m. Thus if, in local coordinates, $D = \sum_{|\alpha|\leq k} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$, where the a_α are coefficient functions, then $D_m = \sum_{|\alpha|=k} a_\alpha(m) \frac{\partial^\alpha}{\partial x^\alpha}$.

The theory of translation-invariant, homogeneous operators is easily developed using Fourier theory:

4.12 Proposition. *Let D be a translation-invariant, self-adjoint homogeneous partial differential operator on a vector space V. The following are equivalent:*

(i) D *is hypoelliptic (that is, if* $Du = v$ *in the sense of distributions, and if* v *is smooth, then* u *is smooth).*

(ii) *If* f *is a rapidly decreasing, then* $f(D)$ *is a smoothing operator.*

(iii) *The Fourier transform of* D*, which is multiplication operator on say* $L^2(V^*)$*, is represented by a function* σ *on* V^* *which is invertible everywhere expect the origin.* □

The main theorem in elliptic regularity theory says that if D is a general operator on a manifold, and if each model operator D_m is hypoelliptic, then D itself is hypoelliptic. In fact:

4.13 Theorem. *Let D be a self-adjoint, linear partial differential operator on a closed manifold M. Assume that each model operator D_m is hypoelliptic. Then:*

(i) D *is hypoelliptic.*

(ii) *If* f *is a rapidly decreasing, then* $f(D)$ *is a smoothing operator.*

4.14 Definition. If each D_m is hypoelliptic then D is said to be elliptic. The *symbol* of D is collection of model operators of D, or equivalently the function σ on T^*M obtained from the Fourier transforms of all the model operators.

Now the fundamental problem in index theory, solved by Atiyah and Singer, is to compute the index of an elliptic operator in terms of the symbol. The tangent groupoid fits very nicely into index theory because of the following result:

4.15 Proposition. *Let* M *be a smooth manifold and let* D *be a linear partial differential operator on* M *of order* k*. The family of operators on the fibers of the tangent groupoid which is given by the formulas*

$$D_{(m,t)} = t^k D \quad \textit{when } t \neq 0; \quad \textit{and} \quad D_{m,0} = D_m$$

is smooth and equivariant.

So in some sense the tangent groupoid allows us to smoothly interpolate between the symbol of an operator (the family of model operators D_m) and the operator itself (which appears here at $t = 1$).

To make use of this observation, let us go back to the definition of the groupoid $G = \mathbb{T}M$, which we constructed as a disjoint union of groupoids G_t. Each G_t is a smooth groupoid in its own right, and so each has a groupoid C^*-algebra $C^*_\lambda(G_t)$. When $t = 0$ we obtain, by Fourier transform, the C^*-algebra $C_0(T^*M)$ of continuous functions, vanishing at infinity, on the cotangent bundle T^*M. When $t \neq 0$ we obtain the compact operators $\mathcal{K}(L^2(M))$.

The fact that the groupoids G_t fit together smoothly to form a single smooth groupoid implies[4] that their groupoid C^*-algebras fit together to form what is called a *continuous field* of C^*-algebras. But if $\{A_t\}$ is any continuous field of C^*-algebras, then the K-theory groups $K_j(A_t)$ have the property that any $x_{t_0} \in K_j(A_{t_0})$ can be canonically prolonged to a family $x_t \in K_j(A_t)$, for all t near t_0. (In fancy language, the $K_j(A_t)$ form the stalks of a pre-sheaf over $\mathbb{R}$.)

In our present case, since all the G_t are identical for $t \neq 0$, this prolongation process determines a homomorphism

$$K_*(C^*_\lambda(G_0)) \to K_*(C^*_\lambda(G_1)).$$

[4] To be accurate, this is a common, but not completely general, fact about smooth families of groupoids.

4.16 Definition. The *analytic index map* is the homomorphism

$$\mathrm{Ind}\colon K^0(T^*M) \to K_0(\mathcal{K}(L^2(M))) \cong \mathbb{Z}$$

obtained from the above construction by identifying $C^*_\lambda(G_0)$ with $C_0(T^*M)$ and $C^*_\lambda(G_1)$ with $\mathcal{K}(L^2(M))$.

The symbol of D — the family of model operators D_m — defines an element in $K_*(C^*_\lambda(G_0))$ by the process hinted at in the last section, and D itself defines an element of $K^*(C^*_\lambda(G_1))$. The former is the same as the *symbol class* $[\sigma] \in K^0(T^*M)$ which appears in the work of Atiyah and Singer; the latter identifies with the Fredholm index of D, as discussed in the previous section. It follows from the interpolation property of the tangent groupoid that the analytic index map takes the symbol class to the Fredholm index.

The construction of the analytic index map, which puts the index problem squarely in the context of K-theory, is a major step in the K-theory proof of the Atiyah-Singer index theorem. There are other ways to define it — for example Atiyah and Singer originally approached the construction through the theory of pseudodifferential operators. The groupoid approach has the advantage that it extends very naturally to more complex situations, for example to foliations.

The reader is referred to Connes' book for an interesting approach to the remaining parts of the proof of the Atiyah-Singer index theorem via groupoid theory.

5 The Baum-Connes Conjecture

Let M be a smooth, connected manifold and denote by $\tilde{M}$ its universal covering space. Now form the tangent groupoid $\mathbb{T}\tilde{M}$. The group $\pi = \pi_1(M)$ acts properly and freely by diffeomorphisms on $\tilde{M}$, and hence π acts on $\mathbb{T}\tilde{M}$, also properly and freely.

5.1 Definition. Denote by $\mathbb{T}_\pi M$ the quotient space obtained by dividing $\mathbb{T}\tilde{M}$ by the action of π.

The space $\mathbb{T}_\pi M$ is a smooth manifold and indeed a groupoid with object space $M \times \mathbb{R}$. It is not the tangent groupoid for M. Like $\mathbb{T}M$, the groupoid $\mathbb{T}_\pi M$ can be thought of as a family of groupoids over $\mathbb{R}$, and like $\mathbb{T}M$ the groupoid over $0 \in \mathbb{R}$ is TM, the tangent bundle of M. However the groupoid over $t \neq 0$ is not the pair groupoid $M \times M$ but the quotient $\tilde{M} \times_\pi \tilde{M}$ of the cartesian product $\tilde{M} \times \tilde{M}$ by the diagonal action of π.

We can think of $\tilde{M} \times_\pi \tilde{M}$ as the space of triples (m_2, α, m_1), where α is a homotopy class of paths in M connecting m_1 to m_2. From this point of view, the groupoid operations are easy to describe: the source and range maps send (m_2, α, m_1) to m_1 and m_2, respectively, while composition is given by the formula

$$(m_3, \alpha_2, m_2) \circ (m_2, \alpha_1, m_1) = (m_3, \alpha_2\alpha_1, m_1),$$

where $\alpha_2\alpha_1$ denotes concatination of paths.

From the families point of view, elements of the groupoid algebra correspond to equivariant (appropriately supported) smoothing operators on the universal cover $\tilde{M}$. The C^*-algebra of the groupoid is faithfully represented on $L^2(\tilde{M})$ as the norm closure of the algebra of π-equivariant smoothing operators on $\tilde{M}$.

5.2 Proposition. $K(C^*_\lambda(\tilde{M} \times_\pi \tilde{M})) \cong K(C^*_\lambda(\pi))$.

This is not difficult. In fact the groupoid C^*-algebra turns out to be *Morita equivalent* (in the sense of C^*-algebra theory) to the group C^*-algebra of π. This is a much stronger statement, which certainly implies that the K-theory groups are isomorphic.

Now, by following the procedure described in the preceding section, and by invoking the above proposition, we obtain from the groupoid $\mathbb{T}_\pi M$ a homomorphism of K-theory groups

$$\mu\colon K(T^*M) \to K(C^*_\lambda(\pi)).$$

This map has been extensively studied in C^*-algebra theory, and is known as the *Baum-Connes assembly map*. It can be thought of as associating to a symbol of an elliptic differential operator on M a K-theoretic "index" of the π-equivariant operator on $\tilde{M}$ obtained from this symbol.

It is quite instructive to consider the case where M is a torus T^n. Here π is of course the free abelian group $\mathbb{Z}^n$, and by Fourier theory, $C^*_\lambda(\mathbb{Z}^n)$ is isomorphic to $C(T^n)$ (actually, the torus which appears here is "dual" to the one we began with, but we can identify the two). On the other hand if M is a torus, then the cotangent bundle T^*M is trivial and so by Bott periodicity, the K-theory of T^*M identifies with the K-theory of M. We obtain the following diagram, in which the bottom map is the one induced from the Baum-Connes assembly map by the two vertical isomorphisms.

$$\begin{array}{ccc}
\mathsf{K}(\mathsf{T}^*\mathsf{M}) & \xrightarrow{\mu} & \mathsf{K}_*(\mathsf{C}^*_\lambda(\pi)) \\
{\scriptstyle \text{Bott}}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \text{Fourier}} \\
\mathsf{K}^*(\mathsf{T}^n) & \xrightarrow[\mu']{} & \mathsf{K}^*(\mathsf{T}^n)
\end{array}$$

Now, the remarkable fact about his bottom version of the Baum-Connes assembly map is that it is equal to its own inverse: the composition of μ' with itself is the identity map on $\mathsf{K}(\mathsf{T}^n)$. This is a K-theoretic version of what is known in other contexts as *Fourier-Mukai duality*.

This example suggests to a willing mind the following quite sweeping conjecture.

5.3 Conjecture (Baum and Connes)**.** *If* M *is any aspherical manifold, and* $\pi = \pi_1(\mathsf{M})$*, then the Baum-Connes assembly map*

$$\mu\colon \mathsf{K}^*(\mathsf{T}^*\mathsf{M}) \to \mathsf{K}_*(\mathsf{C}^*_\lambda(\pi))$$

is an isomorphism of abelian groups.

5.4 Remark. In fact the above is part of more general conjecture, which provides a formula for $\mathsf{K}_*(\mathsf{C}^*_\lambda(\mathsf{G}))$ when G is any locally compact (second countable) topological group.[5]

The Baum-Connes conjecture is a close relative of the topological rigidity conjecture discussed in Weinberger's talk (the Borel conjecture). It has some quite striking implications — for example it implies Novikov's conjecture on the homotopy invariance of higher signatures — and for this and other reasons it has been the topic of a great deal of research. One of the main reasons for attacking the Novikov conjecture through the Baum-Connes conjecture is that the major tool of K-theory — the Bott periodicity theorem — or at least a set of techniques related to the Bott Periodicity theorem, can be brought to bear on the problem.

The conjecture is known to be true in a quite a few cases. Roughly speaking they fall into two classes. The *Haagerup property* is a property of locally compact groups which is a strong negation of Kazhdan's property T (groups which have both the Haagerup property and Kazhdan's property T are automatically compact).

[5] This, in turn was formerly part of a still more general conjecture which did the same for arbitrary (second countable) locally compact groupoids. In this ultimate generality however the conjecture is now known to be false.

Examples include all amenable groups, all Coxeter groups, complex hyperbolic groups, and a few exotica, like Thompson's group F. Higson and Kasparov proved the Baum-Connes conjecture for all groups with the Haagerup property. In another direction, Vincent Lafforgue proved the Baum-Connes conjecture in a number of instances which mostly relate to the realm of negative curvature. For instance, building on Lafforgue's work, Yu and Mineyev were able to prove the Baum-Connes conjecture for all word-hyperbolic groups.

The major outstanding problem is to prove the conjecture for lattices in semisimple groups. Not much is known here beyond the rank one case, although Lafforgue's work did settle the case of uniform lattices in $SL(3)$.

Actually, the Novikov conjecture does not require the full strength of the Baum-Connes conjecture, only the (rational) injectivity of the Baum-Connes assembly map. This injectivity has been proved in a much wider class of examples, including for example all linear groups. In fact the C^*-algebra K-theory approach to the Novikov conjecture has proved to be perhaps the most effective one available.

6 Index Theory on Contact Manifolds

We conclude with a variation on the tangent groupoid for a class of hypoelliptic, but not elliptic, operators. This has been worked out recently by Erik van Erp.

Let M be a smooth manifold and let E be a codimension-one subbundle of the tangent bundle. Let N be the quotient of the tangent bundle by E. It is of course a line bundle on M.

The vector spaces $E_m \oplus N_m$ are equipped with a natural Lie algebra structure, in which N_m is central and the Lie bracket of two elements $X, Y \in E_m$ is the element of N_m defined by the following procedure: extend X and Y to vector fields on M; take their Lie bracket $[X, Y]$; and take the image of $[X, Y]_m$ in the quotient space N_m. This prescription does not depend on the extensions of X and Y that we chose.

6.1 Definition. The subbundle E is a *contact structure* on M if each Lie algebra $E_m \oplus N_m$ is a *Heisenberg* Lie algebra. This means that there are basis elements $X_1, Y_1, \dots, X_n, Y_n$ of H_m and Z of N_m such that $[X_i, X_j] = 0$, $[Y_i, Y_j] = 0$, and $[X_i, Y_j] = \delta_{ij} Z$.

If M is a contact manifold then it may be shown that M is locally equivalent to the Heisenberg Lie group H — the simply connected Lie group whose Lie algebra is the Heisenberg Lie algebra — in much the same way that a general manifold is locally equivalent to $\mathbb{R}^n$. Thus there are local diffeomorphisms from M to H which carry the subbundle E onto the subbundle of the tangent bundle of H spanned by the vector fields X_i and Y_j, for $i, j = 1, \dots, n$. This is a version of Darboux theorem from the theory of symplectic manifolds.

Now a Heisenberg Lie group H admits a one-parameter family of endomorphisms, $\alpha_t \colon H \to H$, defined by the following Lie algebra formulas:

$$\alpha_t(X_i) = tX_i, \quad \alpha_t(Y_j) = tY_j, \quad \text{and} \quad \alpha_t(Z) = t^2 Z.$$

When $t \neq 0$ these are of course automorphisms. Using the one-parameter family we can define an action of the group H on the space $H \times \mathbb{R}$ by the formula

$$h \triangle (k, t) = (\alpha_t(h)k, t).$$

This of course mimics a construction that we made in Section **??** in the context of the abelian Lie group $\mathbb{R}^n$. It suggests the following global construction on a contact manifold.

6.2 Definition. Let (M, H) be a contact manifold. Its H-*tangent groupoid* is the groupoid $\mathbb{T}_H M$ constructed as the disjoint union of groupoids $G = \cup_{t \in \mathbb{R}} G_t$, where $G_0 = HM$, the bundle of Heisenberg Lie groups over M associated to the bundle of Lie algebras $E \oplus N$, and $G_t = M \times M$, when $t \neq 0$.

Suppose that M is diffeomorphic to the Heisenberg group H, via a diffeomorphism ϕ which maps the contact bundle of M to the contact bundle of H. If we use the notation

$$\frac{h_1 - h_2}{t} := \alpha(t^{-1})(h_1 h_2^{-1})$$

in the Heisenberg group (being careful, of course, to note that this does always not behave in exactly the same way as its commutative counterpart), then we can borrow a formula from Section **??** and define an isomorphism of groupoids $\Phi \colon \mathbb{T}_H M \to H \ltimes (H \times \mathbb{R})$ by the formulas

$$\begin{cases} \Phi(h, m, 0) = \big((m, 0), D\phi h, (m, 0)\big) \\ \Phi(m_2, m_1, t) = \Big((m_2, t), \dfrac{\phi(m_2) - \phi(m_1)}{t}, (m_1, t)\Big) \quad (t \neq 0). \end{cases}$$

Here $D\phi$ denotes the map on Lie groups induced from the derivative of ϕ, which is a homomorphism at the level of Lie algebras.

The following result has been proved by Erik van Erp:

6.3 Theorem. *There is a smooth groupoid structure on the* H*-tangent groupoid* $\mathbb{T}_H M$ *which is functorial with respect to inclusions of open sets into contact manifolds, and for which the standard charts* Φ *defined above are diffeomorphisms.*

This suggests an index theory of "H-elliptic" operators on contact manifolds, which is based on model operators which are not translation invariant operators on tangent vector spaces but instead are translation invariant operators on the "tangent" Heisenberg Lie groups associated to a contact structure. This theory indeed exists, and it can be approached in a very conceptual way through the tangent groupoid.

The necessary analysis was investigated first by Hormander and then developed by Stein and his coworkers. If D is a linear partial differential operator on a contact manifold M then at each point m of M there is a *model operator* D_m, which is a (right) translation invariant, homogeneous[6] linear partial differential operator on the tangent Heisenberg Lie group $H_m M$.

6.4 Theorem. *Let* D *be a self-adjoint, linear partial differential operator on a contact manifold* (M, H). *Assume that each model operator* D_m *is hypoelliptic. Then:*

(i) D *is hypoelliptic.*

(ii) *If* f *is a rapidly decreasing, then* $f(D)$ *is a smoothing operator.*

In particular, if M *is closed then* D *is Fredholm.*

6.5 Definition. The H-*symbol* of D is collection of model operators of D. Let us say that D is H-*elliptic* if each of the model operators D_m is hypoelliptic.

It should be stressed that that the analysis of H-elliptic operators on contact manifolds is rather more complicated than the standard elliptic theory, due to the more complicated nature of the Fourier transform for Heisenberg groups. In addition, we should note that although H-elliptic operators are hypoelliptic, they are definitely not elliptic.

Since the tangent Heisenberg groups are noncommutative, the H-symbol is a more complicated object than the classical symbol. When D is H-elliptic, the symbol defines an element not in $K^0(T^*M)$, as in the Atiyah-Singer theory, but rather an element in $K_0(C^*_\lambda(HM))$, the K-theory of the groupoid C^*-algebra associated to the bundle of Heisenberg tangent Lie groups.

[6]The term "homogeneous" refers to the fact that D_m transforms in a homogeneous fashion under the scaling automorphisms α_t introduced earlier: $\alpha_t(D_m) = t^k D_m$.

6.6 Theorem (van Erp)**.** *The analytic index map*

$$\mathrm{Ind}\colon \mathsf{K}_0(\mathsf{C}^*_\lambda(\mathsf{HM})) \to \mathsf{K}_0(\mathcal{K}(\mathsf{L}^2(\mathsf{M}))) \cong \mathbb{Z}$$

associated to the H*-tangent groupoid maps the symbol class of an* H*-elliptic operator* D *to the Fredholm index of* D.

On the basis of this result, which neatly packages all of the analysis of H-elliptic operators using K-theory for noncommutative C^*-algebras, van Erp has formulated and proved in his thesis the Atiyah-Singer index theorem for H-elliptic operators on contact manifolds. Indeed, from here, there is a simple reduction to the classical index theorem of Atiyah and Singer, since a noncommutative Thom isomorphism theorem of Connes implies that $\mathsf{K}_*(\mathsf{C}^*_\lambda(\mathsf{HM})) \cong \mathsf{K}^*(\mathsf{T}^*\mathsf{M})$.