Index Theory

Erik van Erp and Nigel Higson
(Notes by Konrad Aguilar)

September 2016

Abstract

These are notes of lectures on the Atiyah-Singer index theorem given during a masterclass in Bedlewo. The general aim is to survey the K-theoretic and local proofs of the index theorem.

Contents

1 05 September 2016, Nigel Higson

In this lecture we shall discuss Fredholm Operators, Differential Operators, and the (top order) Symbol of an Elliptic Operator.

1.1 Fredholm Operators

Definition 1.1. Fix vector spaces V, W typically infinite-dimensional. Let $T : V \longrightarrow W$ be a linear operator. We say T is *Fredholm* if $\dim(\ker T) < \infty$ and $\dim(\operatorname{coker} T) < \infty$, where coker is the co-kernel. If T is Fredholm, then the *index* of T is the integer

$$\operatorname{Index}(T) = \dim(\ker T) - \dim(\operatorname{coker} T).$$

Example 1.2. The shift operator

$$\begin{bmatrix} 0 & 0 & 0 & \\ 1 & 0 & 0 & \ddots \\ 0 & 1 & 0 & \ddots \\ 0 & 0 & 1 & \ddots \end{bmatrix}$$

on just about any sequence space is Fredholm, and its index is -1.

Lemma 1.3. *If* $F : V \longrightarrow W$ *has finite rank, then* T *is Fredholm* $\iff$ $T + F$ *is Fredholm. That is,* $\operatorname{Index}(T) = \operatorname{Index}(T + F)$. □

In the Banach space context, there is an interesting variation on the above lemma:

Lemma 1.4. *Given a bounded linear operator* $T \colon V \to V$ *that is also Fredholm (in the above purely algebraic sense), if* $S \colon V \to W$ *is a bounded linear operator with sufficiently small norm (depending on* T*) then* $T + S$ *is also Frehdolm and moreover* $\operatorname{Index}(T) = \operatorname{Index}(T + S)$. □

Thus the set of Fredholm bounded linear operators from one Banach space to another is an open subset (in the norm topology) of the space of all bounded linear operators, and the Fredholm index is a locally constant integer-valued function on it.

So, (typically) the index assumes all values.

Definition 1.5. Fix an infinite-dimesional Hilbert space H. We shall denote by Fred the space of all Fredholm operators on H (with the norm topology), and by Fred_0 the subspace of all Fredholm operators of Index 0.

Both Fred and Fred_0 are reasonable topological spaces (for example they have the homotopy type of CW-complexes). The homotopy-theoretic structure of Fred_0 can be illuminated by considering the map

$$\{\text{compact operators}\} \times \{\text{invertible operators}\} \xrightarrow{\text{onto}} \mathrm{Fred}_0$$

defined by

$$(K, A) \longmapsto K + A,$$

or, better, the map

$$\boxed{\{\text{operators of the form identity plus compact}\} \times \{\text{invertible operators}\} \xrightarrow{\text{onto}} \text{Fred}_0}$$

defined by the formula

$$((I+K), A) \longmapsto (I+K)A$$

The group

$$GL_\infty = \{\text{invertible operators of the form } I + \text{compact}\}$$

acts freely on the left-hand side via the formula

$$B \cdot (I+K, A) = ((I+K)B^{-1}, BA)$$

and the boxed map is a principal GL_∞ fibration (roughly speaking the right-hand side is the quotient by the group action, which is has no pathologies). Now the space on the left-hand side of the boxed formula is a contractible topological space, so, from homotopy theory we obtain isomorphisms of homotopy groups

$$\pi_k(\text{Fred}_0) \xrightarrow{\cong} \pi_{k-1}(GL_\infty).$$

for all k.

The group GL_∞ is very interesting from a homotopy theory point of view. Its homotopy groups agree with those of the direct limit

$$\bigcup_n GL_n(\mathbb{C})$$

in which $GL_n(\mathbb{C})$ is embedded into $GL_{n+1}(\mathbb{C})$ via

$$X \longmapsto \begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix}.$$

In fact each individual homotopy group $\pi_{k-1}(GL_\infty)$ identifies with the group $\pi_{k-1}(GL_n(\mathbb{C}))$ for any sufficiently large k (and "sufficiently large" is not so large; just bigger than $n/2$). Bott's famous periodicity theorem says that

$$\pi_{k-1}(GL_\infty) = \begin{cases} \mathbb{Z} & k \text{ even} \\ 0 & k \text{ odd.} \end{cases}$$

Atiyah and Hirzebruch defined

$$K(X) = [X, \text{Fred}]$$

where the square bracket notation means homotopy classes of maps (actually they defined $K(X)$ a bit differently, and it's a theorem that their definition is equivalent to the one above). This is K-theory. We find from Bott's theorem that

$$K(S^k) = \begin{cases} \mathbb{Z} & k \text{ odd} \\ \mathbb{Z} \oplus \mathbb{Z} & k \text{ even.} \end{cases}$$

So while individual Fredholm operators have an interesting integer invariant, *families* of Fredholm operators parametrized by even-dimensional spheres have a second, even more interesting, integer invariant.

Two footnotes about the definition of K-theory.

1) In the definition as stated, X should be compact (compact Hausdorff). If Z is a *locally compact* Hausdorff space then experience shows it is better define $K(Z)$ to be the space of homotopy slasses of maps $Z \to \text{Fred}$ that are invertible operator-valued outside a compact set.

2) In the definition we can allow the Hilbert space H to vary with $x \in X$ or $z \in Z$. That is we can define a K-theory group using continuous families $\{T_x : H_x \longrightarrow H_x\}$ of Fredholm operators (that are invertible outside a compact set). The details are best arranged using the concept of continuous field of Hilbert space, and first arranged this way by Kasparov.

1.2 Differential Operators

Examples on the plane are

- $D = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ (this is really just a directional derivative).
- $D = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ (this is the Cauchy-Riemann operator from complex analysis).

Despite their similar appearance, they behave very differently from an analytic point of view. Let's consider the equation

$$Du = v$$

in which v is given and the task is to study all solutions u.

Definition 1.6. A linear partial differential operator D is *hypoelliptic* if all the solutions u of the above equation are smooth (that is, infinitely differentiable) in any open set where v is smooth.

This condition is very close to the Fredholm condition, in the sense that the techniques to prove it are usually very close to (some of) the techniques used to prove a differential operator is Fredholm. Returning to the two exaxmples, the first is definitely not hypoelliptic. The operator is differentiation in the direction $x = y$, and the derivative in this direction tells us nothing about the derivative in the orthogonal direction. On the other hand, it's a famous fact that Cauchy-Riemann equation $Du = 0$ are smooth (and a slightly less famous fact that more generally the Cauchy-Riemann operator is hypoelliptic).

1.3 A First Look at the Symbol

Is there a way to tell at a glance whether or not a linear partial differential operator D is hypoelliptic (and likely to lead to a Fredholm operator, perhaps after imposing boundary conditions of a suitable sort)? Yes! We need only examine the (principal) symbol of D which is a *function* (a simpler object than an operator) and examine the values of this function one point at a time.

We'll give here a down-to-earth treatment of the symbol here, and then a fancier treatment, suitable for the discussion of operators and symbols on manifolds, in a while.

Start with an operator

$$D = \sum_{|\alpha| \leq p} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$$

of order p or less on an open set U in $\mathbb{R}^n$. Here α is a multiindex $(\alpha_1, \ldots, \alpha_n)$ with nonnegative integer entries,

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

and of course

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Thank heavens for multiindex notation ...

Definition 1.7. The *order* p symbol of D is the function

$$\sigma_p(D) \colon U \times \mathbb{R}^n \longrightarrow \mathbb{C}$$

defined by the formula

$$\sigma_p(D) \colon (x, \xi) \longmapsto \sum_{|\alpha| = p} a_\alpha(x) \xi^\alpha$$

Remark 1.8. It's important to note that we've dropped all the terms in the expression for D except the terms of degree exactly order p. The lower order terms do not contribute to the symbol.

Example 1.9. The symbols of the two operators that we began with are

- $(x, y, \xi, \eta) \longmapsto \xi + \eta$
- $(x, y, \xi, \eta) \longmapsto \xi + i\eta$

The big definition:

Definition 1.10. An operator D of order less than or equal to p is *elliptic of order* p if at every point (x, ξ) with $\xi \neq 0$ the symbol value $\sigma_p(x, \xi)$ is invertible.

Remark 1.11. Saying that a scalar is "invertible," as we did in the above definition, is an awkard way of saying that it is nonzero. But soon we'll be studying operators whose coefficients $a_\alpha(x)$ are not scalar-valued functions, but *matrix*-valued functions. In this context, the definition of the symbol is the same, but now the symbol is a matrix-valued function, and the criterion for ellipticity is exactly as in the definition above—that the symbol take values in invertible matrices whenever ξ is nonzero.

Example 1.12. The Cauchy-Riemann operator $\partial/\partial x + i\partial/\partial y$ is elliptic; the directional derivative operator $\partial/\partial x + \partial/\partial y$ is not.

Here's a important and quite substantial result:

Theorem 1.13. *If an operator* D *is elliptic, then it is hypoelliptic.*

We shall discuss aspects of the proof of the big theorem in Lecture 3. We shall obtain our Fredholm operators from the pool of elliptic partial differential operators, starting with the examples to be discussed tomorrow.

1.4 Differential Operators on Manifolds

Now let M be a smooth manifold w/o boundary. We want to define and study differential operators on M. We can simply say that such operators have local coordinate expressions like the ones considered above. But for variety (and because it is eventually more efficient to do so) we shall describe a fancier approach.

We start with differential operators of order zero, which are dull:

Definition 1.14. A *differential operator of order* 0 is a complex-linear map

$$D\colon C^\infty(M) \longrightarrow C^\infty(M)$$

that commutes with all operators given by pointwise multiplication by smooth functions. That is, an order 0 operator is a $C^\infty(M)$-module map on $\mathbb{C}^\infty(M)$.

Order $k+1$ (or less) operators are defined in terms of order k (or less) operators:

Definition 1.15. A *differential operator of order* $k+1$ is a complex-linear map

$$D\colon C^\infty(M) \longrightarrow C^\infty(M)$$

with the property that for every smooth function f, viewed as a pointwise multiplication operator on $C^\infty(M)$, the commutator

$$[D, f]\colon C^\infty(M) \longrightarrow C^\infty(M)$$

is a differential operator of order k.

Example 1.16. If M is an open subset of $\mathbb{R}^n$, then the differential operators given by the fancy definition are precisely the same as the differential operators given earlier by the multiindex formula.

Example 1.17. Every differential operator of order 1 on any M has the form

$$D = X + h$$

where X is a vector field on M, that is, a derivation of the algebra $C^\infty(M)$, and h is an order zero operator, that is, a $C^\infty(M)$-module map. In fact $h = D(1)$, so the assertion (an exercise for you) is that if D is a differential operator of order 1, then $D - D(1)$ is a derivation.

Lemma 1.18. *The composition of an order p operator with an order q operator is an order $p+q$ operator.* □

Lemma 1.19. *The commutator $[D, E]$ of an order p operator with an order q operator is an order $p+q-1$ operator.* □

It will be important to consider linear partial differential operators in a slightly broader context that corresponds in the local coordinate, multiindex picture to considering operators whose coefficient functions $a_\alpha(x)$ are matrix-valued functions. The essential point is that we want to consider operators

$$D : \mathcal{E} \longrightarrow \mathcal{F}$$

where $\mathcal{E}$ and $\mathcal{F}$ are modules over the ring $C^\infty(M)$, rather than $C^\infty(M)$ itself. The main examples to keep in mind are spaces of differential forms over M.

Working in this generality, we can repeat the definitions above, with small modifications. An order zero operator is any $C^\infty(M)$ module map from $\mathcal{E}$ to $\mathcal{F}$. An order $k+1$ operator is, exactly as in the definition above, a complex-linear map D all of whose commutators $[D, f]$ with *functions* (not arbitrary order zero operators) are order k operators. The composition of an order p and an order q operator (when the composition makes sense) will still be an operator of order $p + q$. So for example if $\mathcal{E} = \mathcal{F}$, then we obtain an algebra of differential operators on $\mathcal{E}$.

But as it stands the setup is too general to be of value: it will be impossible to prove theorems in this degree of generality. To remedy this we insist that $\mathcal{E}$ and $\mathcal{F}$ be the global sections of locally free, finitely generated sheaves of modules over the sheaf of smooth functions on M. In other words, for those who prefer vector bundles over sheaves, $\mathcal{E}$ and $\mathcal{F}$ are the modules of sections of smooth vector bundles over M. We shall be able to develop a reasonable theory in this generality, which encompasses many interesting geometric examples.

1.5 The Symbol

To discuss the symbol of an operator of the general type considered above, we shall need to consider "values" of elements in the modules $\mathcal{E}$ and $\mathcal{F}$ at points $x \in M$. This is done as follows:

Definition 1.20. The *fiber* of $\mathcal{F}$ at $x \in M$ is the vector space quotient

$$\mathcal{F}|_x = \mathcal{F}/I_x\mathcal{F},$$

where $I_x \subseteq C^\infty(M)$ is the ideal of all smooth functions on M vanishing at x.

For the modules we are considering, this is a finite-dimensional vector space. The symbol of an order p differential operator

$$D\colon \mathcal{E} \longrightarrow F$$

will be a family of polynomial functions

$$\sigma_p(D)_x\colon T_x^*M \longrightarrow \mathrm{Hom}(\mathcal{E}|_x, \mathcal{F}|_x),$$

one for each point $x \in M$. Here T_x^*M is the cotangent vector space at $x \in M$ (more on this in a moment), and the target of the map above is the space of linear transformations between two finite-dimensional vector spaces. In effect, the target is the space of matrices of some particular shape $m \times n$.

The fancy definition of symbol will be in agreement with the concrete one we gave before. The functions ξ_i that were used before will become coordinate functions on T_x^*M, and the agreement will be

$$\begin{aligned}\sigma_p(D)_x &= \sigma_p(D)(x,\xi)\\ &= \sum_{|\alpha|=p} a_\alpha(x)\xi^\alpha.\end{aligned}$$

Now let us discuss the cotangent vector space T_x^*M. It can be defined in various ways, but a convenient way for us is to set

$$T_x^*M = I_x/I_x^2.$$

That is, the cotangent vector space is the quotient of the vanishing ideal I_x by its square (the ideal of functions vanishing to order two at x). If $y_1, \ldots, y_n$ are local coordinates centered at x, then their images in the above quotient vector space, which are written $dy_1, \ldots, dy_n$, constitute a basis for the cotangent space.

Definition 1.21. Let D be a differential operator of order p, as above. The *order* p *symbol* of D, is the family of polynomial functions

$$\sigma_p(D)_x\colon T_x^*M \longrightarrow \mathrm{Hom}(\mathcal{E}|_x, \mathcal{F}|_x),$$

defined by

$$\sigma_p(D)_x\colon df \longmapsto \frac{1}{p!}\,[..[[D,f],f],\cdots,f]$$

in which there occur p commutators with the function f, viewed as an operator on $\mathcal{E}$ and on $\mathcal{F}$.

Example 1.22. By far the most important case for us will be the case of an operator of order 1, in which case the formula is simpler:

$$\sigma_p(D)_x\colon df \longmapsto [D,f]$$

The operator $[D,f]\colon \mathcal{E} \to \mathcal{F}$ has order zero, which is to say that it is a $C^\infty(M)$-module map, so that it induces a map

$$[D,f]\colon \mathcal{E}|_x \longrightarrow \mathcal{F}|_x,$$

as the definition requires.

Example 1.23. Another simple case to check is the case where $M = \mathbb{R}$ (the case of $\mathbb{R}^n$ is the same, but with more notation involved). Consider say

$$D = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x).$$

Then

$$[D,f] = 2a(x)f'(x)\frac{d}{dx} + b(x)f'(x) + f''(x)$$

and

$$[[D,f],f] = 2a(x)f'(x)^2.$$

So the symbol is

$$\sigma_2(D)_x = a(x)\xi^2,$$

where ξ is the coordinate function on T_x^*M that maps df to $f'(x)$ (so the right hand side of the display is a function on T_x^*M with values in the scalars, or in other words with values in the space of linear operators from a one-dimensional vector space to itself). As a result, we find that we are in agreement with the concrete notion of symbol given earlier.

Example 1.24. A more serious example is the de Rham differential

$$d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M),$$

which is a differential operator of order one. The fiber at $x \in M$ of $\Omega^k(M)$ identifies with the exteriof power $\wedge^k T_x^*M$ (this is a finite-dimensional vector space). The symbol can therefore be studied as a map

$$\sigma(D)_x : T_x^*M \longrightarrow \mathrm{Hom}\big(\wedge^k T_x^*M, \wedge^{k+1} T_x^*M\big)$$

(we're dropping the subscript 1 in our notation for the symbol). It is a polynomial map, and in fact a linear map because d is a first order operator. One computes that

$$\sigma(d)_x(\xi)\colon \omega \longmapsto \xi \wedge \omega.$$

It is an excellent exercise, almost an essential exercise, to carefully make sense of all of this: to make the identifications explicit, and verify the formula.

1.6 An Extended Example

Think of this section as an extended exercise, with plenty of hints. We're going to take a fairly close look at one operator on the 2-sphere (in fact it is the Dirac operator, which will be discussed in a much more general context in the next lecture).

The 2-sphere as a homogeneous space

It will be convenient to describe the 2-sphere not in the usual coordinate way as

$$S^2 = \{ (x, y, z) : x^2 + y^2 + z^2 = 1 \}$$

but as a homogeneous space for the group $SU(2)$.

To do so, recall the following preliminaries. The Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is linearly spanned over $\mathbb{R}$ by the three matrices

$$X = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The group $SU(2)$ acts on $\mathfrak{su}(2)$ in the usual way by conjugation (this is the adjoint action, in the Lie-theory language). The action preserves the real inner product

$$\langle W_1, W_2 \rangle = \frac{1}{2} \operatorname{Trace}(W_1^* W_2),$$

where the factor of $1/2$ makes $\{ X, Y, Z \}$ an orthonormal basis, and so we get a homomorphism

$$SU(2) \longrightarrow SO(3)$$

if we use the $\{ X, Y, Z \}$ basis to identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$. This is the famous spin double covering of $SO(3)$.

The isotropy subgroup for $X \in \mathfrak{su}(2)$ under the conjugation action is the diagonal subgroup $T \subseteq SU(2)$, and the action of $SU(2)$ is transitive on the unit sphere in $\mathfrak{su}(2)$, so we obtain an identification of the homogeneous space $SU(2)/T$ with the 2-sphere. In particular, we obtain an identification of algebras

$$C^\infty(S^2) = \left\{ f \colon SU(2) \to \mathbb{C} : f(u \exp(tX) = f(u) \quad \forall u \in SU(2) \ \ \forall t \in \mathbb{R} \right\}$$

(it is not written explicitly above, but from now on we'll be dealing exclusively with *smooth* functions on the 2-sphere).

Remark 1.25. While we're on the subject of the $\mathfrak{su}(2)$, let us make note of the commutation relations

$$[X, Y] = 2Z, \quad [Y, Z] = 2X \quad \text{and} \quad [Z, X] = 2Y,$$

which we shall use later on. We shall also use the really simple formula

$$\exp(tX) = \begin{bmatrix} \exp(it) & 0 \\ 0 & \exp(-it) \end{bmatrix}.$$

Modules

Our operator will act not on scalar functions, but between two modules over the smooth functions on the two-sphere, as we discussed above. The modules will be as follows:

$$\mathcal{S}^+ = \left\{ f\colon SU(2) \to \mathbb{C} : f(u\exp(tX)) = \exp(it)f(u) \right\}$$

and

$$\mathcal{S}^- = \left\{ f\colon SU(2) \to \mathbb{C} : f(u\exp(tX)) = \exp(-it)f(u) \right\}.$$

They are indeed modules, since the action by pointwise multiplication of the ring of functions on $SU(2)$ for which $f(u) = f(u\exp(tX))$ leaves invariant the spaces $\mathcal{S}^\pm$.

Differential Operator

Our operator will be a combination of the following elementary operators:

Definition 1.26. If W is any matrix in the Lie algebra $\mathfrak{su}(2)$, and if f is any smooth function on $SU(2)$, then let us define

$$(Wf)(u) = \frac{d}{ds}\Big|_{s=0} f\big(u\exp(sW)\big),$$

which is of course another smooth function on $SU(2)$.

A warning: these operators do not individually preserve the spaces $\mathcal{S}^\pm$, or map one of them into the other. They are however very simply related to the Lie bracket on $\mathfrak{su}(2)$:

Lemma 1.27. *If $[W_1, W_2] = W_3$ in $\mathfrak{su}(2)$, and if f is any smooth function on $SU(2)$, then*

$$W_1(W_2f) - W_2(W_1f) = W_3f. \qquad \square$$

Using the operators in the definition, we obtain a more computationall useful description of the spaces $\mathcal{S}^\pm$, as follows:

Lemma 1.28. $\mathcal{S}^\pm = \{ f\colon SU(2) \to \mathbb{C} : Xf = \pm if \}$. $\square$

We are now almost ready to define our operator. We need one more computation:

Lemma 1.29. *If $f \in \mathcal{S}^+$, then $Yf + iZf \in \mathcal{S}^-$.*

Proof. If $f \in \mathcal{S}^+$, then according to the previous lemmas

$$\begin{aligned} X(Y + iZ)f &= (Y + iZ)Xf + ([X, Y] + i[X, Z])f \\ &= i(Y + iZ)f + (2Z - 2iY)f, \end{aligned}$$

or in other words

$$X(Y + iZ)f = -i(Y + iZ)f,$$

as required. $\square$

Lemma 1.30. *The formula*

$$Df = (Y + iZ)f$$

defines a first-order differential operator

$$D \colon \mathcal{S}^+ \longrightarrow \mathcal{S}^-$$

on the 2-sphere.

Proof. According to the definition, we need to show that if f is a smooth function on the 2-sphere, or in other words a right $U(1)$-invariant function on $SU(2)$, then the commutator

$$[D, f] \colon \mathcal{S}^+ \longrightarrow \mathcal{S}^-$$

commutes with pointwise multiplication by other functions on the 2-sphere. We compute that

$$[D, f] = Yf + iZf$$

where the function $Yf + iZf$ on $SU(2)$ is acting by by pointwise multiplication. Certainly this commutes with all other pointwise multiplication operators. □

Symbol

The computation of the first order symbol of D is mostly easy, since it involves only the operators $[D, f]$ that we have already computed. Namely, if x is a point on the 2-sphere, then the symbol

$$\sigma(D)_x \colon T^*_x M \longmapsto \mathrm{Hom}(\mathcal{S}^+|_x, \mathcal{S}^-|_x)$$

is the map

$$\sigma(D)_x \colon df \longmapsto \Big[(Yf + iZf) \colon \mathcal{S}^+|_x \to \mathcal{S}^-|_x \Big]$$

But we need to identify the fiber spaces $\mathcal{S}^\pm|_x$ in order to make the formula a bit more useful.

Recall that

$$\mathcal{S}^\pm|_x = \mathcal{S}^\pm / I_x \mathcal{S}^\pm$$

where I_x is the ideal of smooth functions on the 2-sphere that vanish at x. The point x corresponds to some left coset uT in $SU(2)$, and I_x corresponds to the functions on $SU(2)$ that are invariant under the right T-action and that vanish on that coset. We find that

$$I_x \mathcal{S}^\pm = \{ f \in \mathcal{S}^\pm : f|_{uT} = 0 \}$$

So

$$\begin{aligned} \mathcal{S}^\pm|_x &= \{ f \colon uT \to \mathbb{C} : Xf = \pm i f \} \\ &= \{ f \colon uT \to \mathbb{C} : f(u \exp(tX)) = \exp(\pm it) f \} \end{aligned}$$

which is a one-dimensional vector space, isomorphic to $\mathbb{C}$ via evaluation at $u \in uT$. Of course, the choice of u within the coset is not canonical, so our identification of the fibers with $\mathbb{C}$ is a bit arbitrary, but let us make it anyway. Under the identification, the symbol becomes

$$\sigma(D)_x \colon df \longmapsto (Yf)(u) + i(Zf)(u),$$

where the object on the right is a complex number, to be viewed as a linear operator from $\mathbb{C}$ to $\mathbb{C}$ by multiplication.

Now if $(Yf)(u) = 0$ and $(Zf)(u) = 0$, then Yf and Zf are zero on the whole coset uT, because f is right T-invariant, and Xf is zero on uT too, so f vanishes to second order on $uT \subseteq SU(2)$, and therefore $df = 0$. We find that if $\sigma(D)_x df$ fails to be invertible, it is because $df = 0$. So D is elliptic.

A Family of Operators

If we define

$$\mathcal{S}_n^{\pm} = \{ f \colon SU(2) \to \mathbb{C} : Xf = (n \pm i)f \},$$

then we can define an operator

$$D_n \colon \mathcal{S}_n^+ \longrightarrow \mathcal{S}_n^-$$

by exactly the same formula $Y + iZ$ as before (we just need to check that $Y + iZ$ does indeed map $\mathcal{S}_n^+$ into $\mathcal{S}_n^-$, which is done in the same was as the $n = 0$ case).

Point by point in S^2, the symbol of D_n is the same as the symbol of D, if we identify $\mathcal{S}_n^{\pm}|_x$ with $\mathbb{C}$ by picking the same element of $SU(2)$ that maps to x as we did for D. Therefore the operators D_n are all elliptic. But overall the symbol is different: for instance the K-theory classes associated to the symbols $\sigma(D_n)$ that we shall describe in the next section are distinct from one another.

The Fredholm index of the operator D_n certainly *does* depend on n. In fact we shall see that

$$\text{Index}(D_n) = -n.$$

1.7 A Final Remark

Suppose that M is now a closed manifold (for a reason that we'll point out below) and that D is an elliptic linear partial differential operator on D.

We don't know it yet, but the assumptions imply that D is a Fredholm operator, say in the algebraic sense as a complex linear map

$$D \colon \mathcal{E} \longrightarrow \mathcal{F}.$$

So D has a Fredholm index, which is an integer.

There is another discrete (that is, resembling an integer) invariant that one can attach to D. It is made from the symbol of D, and it uses the K-theory ideas we described in the first part of the lecture.

To begin, the various cotangent vector spaces T_x^*M assembled to form a single smooth manifold T^*M of twice the dimension on M. This is the (total space of the) cotangent bundle, of course.

Now the symbol attaches to each point of each cotangent vector space T_x^*M a linear operator from $\mathcal{E}|_x$ to the vector space $\mathcal{F}|_x$. These are finite-dimensional vector spaces, so it is automatic that the linear operators that constitute the symbol are Fredholm. Moreover for all but a compact subset of T^*M (namely the "zero section" consisting of the zero vectors in all the spaces T^*M, which is compact because M is compact) the operators that constitute the symmbol are invertible. This is so because of the definition of ellipticity.

So in summary, the symbol can be viewed as a family of Frehdolm operators parametrized by the locally compact space T^*M that is invertible outside of a compact set. And as a result there is an associated K-theory class

$$[\sigma(D)] \in K(T^*M).$$

An observation made very early on, essentially by Gelfand, is that *the Fredholm index of* D *depends only on the symbol class* $[\sigma(D)]$. So the index problem begins to assume a definite form: *understand* K*-theory and the properties of the symbol class well enough to determine a formula for the index of* D *in terms of the symbol class.*

2 06 September 2016, Erik van Erp

2.1 Dirac Operators

Before we consider Dirac Operators on manifolds, we present the "model" Dirac Operator on $\mathbb{R}^n$. The Dirac operator on $\mathbb{R}^n$ is constructed inductively.

- $n = 1$: the Dirac operator on $\mathbb{R}$ is $D = -i\frac{d}{dx}$.

  The symbol of D is $\sigma = -i\xi$, which is invertible if $\xi \neq 0$, so D is an elliptic operator. If u, v are two distributions on $\mathbb{R}$ such that $Du = v$, then $u(x) = c + \int_0^x v(t)dt$. Thus, u is "more regular" than v. In particular, we see that D is hypo-elliptic: If v is C^∞ on some open set $(a, b) \subset \mathbb{R}$, then u is also C^∞ on (a, b).

- $n = 2$: the Dirac operator on $\mathbb{R}^2$ is the Cauchy-Riemann operator $D = \frac{\partial}{\partial x} + i\frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{z}}$. The symbol of D is $\sigma = \xi + i\eta$. Since $\sigma = 0$ iff $(\xi, \eta) = (0, 0)$, we see that D is elliptic.

- $n = 3$: ... we will proceed inductively.

The Dirac operator on $\mathbb{R}^n$ is the map $D : C^\infty\left(\mathbb{R}^n, \mathbb{C}^{2^r}\right) \longmapsto C^\infty\left(\mathbb{R}^n, \mathbb{C}^{2^r}\right)$ defined by

$$D = \sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j} \tag{2.1}$$

Here $n = 2r$ or $n = 2r + 1$, and E_j are $2^r \times 2^r$ complex-valued matrices that satsify:

- $E_j^2 = -I_n$ for $j \in \{1, \ldots, n\}$, where I_n is the $n \times n$ identity matrix.
- $E_j E_k = -E_k E_j$ for $j, k \in \{1, \ldots, n\}$ and $j \neq k$.

The matrices E_j are defined inductively in the following way.

- $n = 1$ $(r = 0) : E_1 = (-i)$.
- $n = 2$ $(r = 1) : E_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
- $n = 3$ $(r = 2)$: E_1, E_2 are the same as in the case $n = 2$, and $E_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$.
- $n = 4$ $(r = 2)$: For $j = 1, 2, 3$ we replace the 2×2 matrix E_j from the case $n = 3$ by the $2^r \times 2^r = 4 \times 4$ matrix $\begin{bmatrix} 0 & E_j \\ E_j & 0 \end{bmatrix}$, and add $E_4 = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}$, where I_2 is the 2×2 identity matrix.
- $n = 5$ $(r = 2)$: E_1, E_2, E_3, E_4 are as in the case $n = 4$. Add $E_5 = \begin{bmatrix} -iI_2 & 0 \\ 0 & iI_2 \end{bmatrix}$.

Etcetera.

Remark 2.1. $D^2 = \sum -I_{2^r} \otimes \frac{\partial^2}{\partial x_j^2} = \begin{bmatrix} \Delta & & & \\ & \Delta & & \\ & & \ddots & \\ & & & \Delta \end{bmatrix}$, where Δ is the Laplacian.

<u>Aim</u>: We want to define a Dirac operator D on a manifold and not just on $\mathbb{R}^n$. Consider the "model" Laplacian $\Delta = \sum_{j=1}^n -\frac{\partial}{\partial x_j^2}$ on $\mathbb{R}^n$. What, fundamentally, is the reason that a Laplacian Δ exists on <u>Riemannian</u> manifolds? The reason is that the structure group of a Riemannian manifold is $O(n)$, the group of orthogonal matrices of size $n \times n$. This means that we can identity the tangent space $T_pM \approx \mathbb{R}^n$ with the standard Euclidean space, <u>up</u> <u>to</u> an action of the group $O(n)$ on $\mathbb{R}^n$.

$\Longrightarrow$ therefore Δ is well-defined as an operator on each fiber T_pM

$\Longrightarrow$ which implies, in turn, that the *highest order part* of Δ is well-defined on M,

$$\Delta = \sum -g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \text{ lower order terms.}$$

The difficulty is that the "model" Dirac operator D on $\mathbb{R}^n$ is *not* $SO(n)$-invariant, where $SO(n)$ is the group of special orthogonal matrices of size $n \times n$. The first modification is that we need the more sophisticated notion of equivariance to replace invariance.

Equivariant vector bundle. Let M be a manifold with a group action $G \times M \to M$. A vector bundle $E \to M$ is an G *equivariant vector bundle* if the group G acts on E in a way that lifts the action of G on M. This means that if $g \in G$, then there is a vector bundle homomorphism $\tilde{g} : E \to E$ that lifts the action $g : M \to M$,

$$
\begin{tikzcd}
E \arrow[d] & G \arrow[l, hook'] & E \arrow[r, "\tilde{g}"] \arrow[d] & E \arrow[d] \\
M & G \arrow[l, hook'] & M \arrow[r, "g"'] & M
\end{tikzcd}
$$

We will assume that M, E, G and all the actions are C^∞. If E is a G equivariant vector bundle, then $g \in G$ acts on sections $s \in C^\infty(E)$ by

$$(g, s)(x) = \tilde{g}(s(g^{-1}x))$$

An operator $D : C^\infty(E) \to C^\infty(E)$ is called G equivariant if it commutes with the action of $g \in G$ on sections,

$$
\begin{tikzcd}
C^\infty(E) \arrow[r, "g"] \arrow[d, "D"'] & C^\infty(E) \arrow[d, "D"'] \\
C^\infty(E) \arrow[r, "g"'] & C^\infty(E)
\end{tikzcd}
$$

Unfortunately, D is not $SO(n)$-equivariant. However D is equivariant for an action of the spin group. Recall that for $n \geq 3$ the fundamental group of $SO(n)$ is the two element group $\mathbb{Z}/2\mathbb{Z}$. The spin group $\mathrm{Spin}(n)$ is the connected double cover of $SO(n)$ (for $n \geq 2$), which is simply connected if $n \geq 3$,

$$
\begin{tikzcd}
\mathrm{Spin}(n) \arrow[d, "2:1"] \\
SO(n)
\end{tikzcd}
$$

The group $\mathrm{Spin}(n)$ acts on $\mathbb{R}^n$ (via its map to $SO(n)$), and we will see that it also acts on $\mathbb{C}^{2^r}$ (by an action that does not factor through $SO(n)$).

Theorem 2.2. *The Dirac Operator* $D = \sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j}$ *on* $\mathbb{R}^n$ *is* $\mathrm{Spin}(n)$*-equivariant.*

Proof. The proof uses facts about Clifford algebras that will be discussed in the next section. $\square$

2.2 Clifford Algebras

The Clifford algebra C_n is the universal $\mathbb{R}$-algebra with n generators $e_1, \ldots, e_n$ and relations

$$e_j^2 = -1 \qquad e_j e_k = -e_k e_j \qquad j \neq k$$

Note that these relations are satisfied by the matrices E_j used in the definition of the Dirac operator on $\mathbb{R}^n$ in Equation (2.1).

The relations easily imply that C_n is spanned (as an $\mathbb{R}$ vector space) by the 2^n products $e_{i_1} e_{i_2} \cdots e_{i_p}$ with $i_1 < i_2 < \cdots < i_p$ (including 1 for $p = 0$). Exercise(tricky): $\dim(C_n) = 2^n$, i.e., the products $e_{i_1} e_{i_2} \cdots e_{i_p}$ are linearly independent.

Consider the subspace

$$\mathfrak{g} = \mathrm{span}\{e_i e_j : i < j,\} \subset C_n$$

The subspace $\mathfrak{g}$ is closed under commutators in C_n, and is therefore a Lie algebra. This can be easily checked by direct calculation. For example,

$$[e_i e_j, e_j e_k] = e_i e_j e_j e_k - e_j e_k e_i e_j = -2 e_i e_k.$$

In fact, we have an isomorphism of Lie algebras $\mathfrak{g} \cong \mathfrak{so}(n)$, where $\mathfrak{so}(n)$ is the Lie algebra of skew $n \times n$ real-valued matrices. This isomorphism is given by

$$\frac{1}{2} e_i e_j \longmapsto \begin{bmatrix} & & 1 \\ \cdots & -1 & \\ & \vdots & \end{bmatrix}$$

where the matrix on the right has -1 in the i^{th} row and j^{th} column, and 1 in j^{th} row and i^{th} colum, 0 elsewhere. Furthermore, we can define the following Lie algebra representation

$$d\rho : \mathfrak{g} \longrightarrow \mathrm{End}\,(\mathbb{R}^n) \qquad d\rho(\alpha)v = \alpha v - v\alpha \quad \alpha \in \mathfrak{g}, v \in \mathbb{R}^n$$

where we view $\mathbb{R}^n = \mathrm{span}\{e_1, \ldots, e_n\}$ as a subset of C_n. The fact that $\alpha v - v\alpha \in \mathbb{R}^n$ is established by direct calculation. For example, $[e_i e_j, e_j] = e_i e_j e_i - e_j e_i e_j = -2e_i$.

Note that $SO(n) = \exp \mathfrak{so}(n) \subset M_n\mathbb{R}$, the space of $n \times n$ real-valued matrices. Likewise, we can exponentiate the Lie algebra $\mathfrak{g}$ in the Clifford algebra C_n and we obtain a Lie group,

$$G = \exp \mathfrak{g} = \left\{ g = \sum_{k=0}^{\infty} \frac{1}{k!} \alpha^k : \alpha \in \mathfrak{g} \right\} \subset C_n$$

The group G is represented on $\mathbb{R}^n$ by conjugation

$$\rho : G \longmapsto \mathrm{End}\,(\mathbb{R}^n) \qquad \rho(g)v = g v g^{-1} \in C_n. \tag{2.2}$$

Then $d\rho$ is the Lie algebra representation induced by ρ, and so we have in fact

$$\rho : G \longmapsto SO(n)$$

To establish that G is isomorphic to $\mathrm{Spin}(n)$ we only need to prove that ρ is not one-to-one. The following caclulation does this.

<u>**Calculation**</u>: Consider $te_1e_2 \in \mathfrak{g}, t \in \mathbb{R}$. Note that $(e_1e_2)^2 = e_1e_2e_1e_2 = -1$. Therefore,

$$\exp(te_1e_2) = \cos(t) + \sin(t)e_1e_2$$

For $t = \pi$, we have $\exp(\pi e_1e_2) = -1 \in C_n$. Therefore $-1 \in G$. From Expression (2.2), we see that $\rho(-1) = I_n \in SO(n)$, which establishes that $G \cong \mathrm{Spin}(n)$.

<u>**Back to the Dirac operator**</u>: Consider the map

$$c : \mathbb{R}^n \longmapsto \mathrm{End}\left(\mathbb{C}^{2^r}\right) \qquad c(v) = \sum_{j=1}^{n} v_j E_j \tag{2.3}$$

where $n = 2r$ or $n = 2r + 1$, and E_j are the $2^r \times 2^r$ matrices as in Equation (2.1). Since $c(e_j) = E_j$, and the matrices E_j satisfy the defining relations of the Clifford algebra C_n, by universality of C_n we see that c extends to a representation of the Clifford algebra

$$c : C_n \longrightarrow \mathrm{End}\left(\mathbb{C}^{2^r}\right)$$

This representation, in turn, restricts to a (unitary) representation of the spin group,

$$c : \mathrm{Spin}(n) \longrightarrow \mathrm{End}\left(\mathbb{C}^{2^r}\right)$$

This is the *spinor representation*.

<u>**Now**</u>:

- $G = \mathrm{Spin}(n)$ acts on $\mathbb{R}^n$ (by conjugation $g \mapsto gvg^{-1} \in C_n$).
- G acts on $\mathbb{C}^{2^r}$ via $c(g)$ and on $\mathrm{End}\left(\mathbb{C}^{2^r}\right)$ via $T \mapsto c(g)Tc(g)^{-1}$.

It is now immediately clear that the $\mathbb{R}$ linear map

$$c : \mathbb{R}^n \longmapsto \mathrm{End}\left(\mathbb{C}^{2^r}\right) \qquad c(v) = \sum_{j=1}^{n} v_j E_j$$

is $\mathrm{Spin}(n)$-equivariant, by inspecting the commuting diagram

$$\begin{tikzcd} v \arrow[r, mapsto] \arrow[d, mapsto, "g"'] & c(v) = \sum_{j=1}^{n} v_j E_j \arrow[d, mapsto, "g"] \\ gvg^{-1} \arrow[r, mapsto] & c(gvg^{-1}) = c(g)c(v)c(g)^{-1} \end{tikzcd}$$

But the map c is just the symbol of the Dirac operator on $\mathbb{R}^n$,

$$D = \sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j} : C^\infty\left(\mathbb{R}^n, \mathbb{C}^{2^r}\right) \longmapsto C^\infty\left(\mathbb{R}^n, \mathbb{C}^{2^r}\right)$$

The symbol of D is $\sigma = \sum \xi_j E_j = c(\xi)$, with $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. In other words, the map c is the symbol of D. Since the symbol of D is $\mathrm{Spin}(n)$-equivariant, and D is a constant coefficient operator on $\mathbb{R}^n$, it follows that D is $\mathrm{Spin}(n)$-equivariant. This establishes Theorem (2.2).

□

Next, let M be a Riemannian Manifold oriented $\rightsquigarrow SO(n)$ structure group. The frame bundle

$$F = \{(p, v_1, \ldots, v_n) : p \in M, (v_1, \ldots, v_n) = \text{ oriented orthonormal basis for } T_pM\}.$$

with $F/SO(n) = M$ fiber bundle over M and $TM = F \underset{SO(n)}{\times} \mathbb{R}^n$ in which $(f, v) \sim (f.g, g^{-1}v)$.

Spin structure on M:

$$\begin{tikzcd} & P \arrow[dl] \arrow[dd, two heads, "2:1"] & = \text{ double cover of } F \text{ with connected fibers} \\ M & & \\ & F \arrow[ul] & \end{tikzcd}$$

3 07 September 2016, Nigel Higson

Our goal today is to move from elliptic partial differential operators to Fredholm operators. We'll reach the goal in Theorem 3.17, but along the way there will be some important milestones. Good news for the functional analysts: today there will be Hilbert spaces . . .

So we'll start with a linear partial differential operator (PDO)

$$D : \mathcal{E} \to \mathcal{F}$$

on some manifold M, as in Lecture 1. For the most part you can think of $\mathcal{E}, \mathcal{F}$ as the spaces of smooth functions on M, but as we mentioned earlier the additional generality discussed in Lecture 1 is important.

We're going to relocate D to the Hilbert space context, and the first step is to manufacture Hilbert spaces out of $\mathcal{E}$ and $\mathcal{F}$. We shall assume given

- a pointwise inner product

$$\overline{\mathcal{E}} \underset{C^\infty(M)}{\otimes} \mathcal{E} \xrightarrow{<\,,\,>} C^\infty(M),$$

where $\overline{\mathcal{E}}$ is denotes the complex conjugate space of $\mathcal{E}$, and

- a smooth measure on M.

We get from these things a Hilbert space, which we denote $\mathcal{E}_{L^2}$. And similarly we get $\mathcal{F}_{L^2}$. We want to consider our operator D as an unbounded operator

$$D : \mathcal{E}_{L^2} \longrightarrow \mathcal{F}_{L^2}$$

initially with doman $\mathcal{E}_{\mathrm{comp}}$ (the compactly supported elements of $\mathcal{E}$). You can think of $\mathcal{E}_{\mathrm{comp}}$ as the compactly supported functions on M.

Along the *initial domain* $\mathcal{E}_{\mathrm{comp}}$ we'll also be concerned with two other domains, both of which are more convenient than $\mathcal{E}_{\mathrm{comp}}$ for functional analysis. First, we denote by

$$D^\diamond : \mathcal{F} \longrightarrow \mathcal{E}$$

the *formal adjoint* of D, which is the PDO defined by

$$\langle Dw_1, w_2\rangle_{\mathcal{F}_{L^2}} = \langle w_1, D^\diamond w_2\rangle_{\mathcal{E}_{L^2}}$$

for all $w_1 \in \mathcal{E}_{\mathrm{comp}}, w_2 \in \mathcal{F}_{\mathrm{comp}}$. It is a fact (which comes down to the integration by parts formula) that:

Lemma 3.1. *There is a unique formal adjoint* $D^\diamond$ *as above.* □

The first of our two domains us the *minimal domain*:

$$\Big\{u \in \mathcal{E}_{L^2} : u = \lim_{n\to\infty} u_n, (u_n)_{n\in\mathbb{N}} \subset \mathcal{E}_{\mathrm{comp}} \text{ and } v = \lim_{n\to\infty} Du_n \text{ exists}\Big\}.$$

This is larger than the initial domain, and D extends to it by setting $v = Du$. (For the functional analysts, this is the so-called minimal closed domain, hence the name.) The second is the *maximal domain*:

$$\big\{u \in \mathcal{E}_{L^2} : \exists v \in \mathcal{F}_{L^2}, \langle u, D^\diamond w\rangle_{\mathcal{E}_{L^2}} = \langle v, w\rangle_{\mathcal{F}_{L^2}}, \text{ for all } w \in \mathcal{F}_{\mathrm{comp}}\big\}.$$

This is is larger than the minimial domain, and again D extends by setting $Du = v$.

Our first helpful (and not trivial) result is this:

Theorem 3.2. *If* D *has order* 1 *and is compactly supported, then the Maximal and Minimal domains agree.*

The theorem is not true for higher order (this is a challenging exercise).

3.1 Self-adjoint and essentially self-adjoint operators

Let H be a Hilbert space. Suppose $D : H \longrightarrow H$ is unbounded such that

$$\langle Du, v\rangle = \langle u, Dv\rangle$$

for all $u, v \in \mathrm{dom}(D)$, the domain of D. In the abstract Hilbert space context one says that D is *symmetric*. In the PDO context, where $\mathrm{dom}(D)$ will be the initial domain $\mathcal{E}_{\mathrm{comp}}$, the symmetry condition in the display is just the *formal self-adjointness* condition $D = D^{\diamond}$.

Symmetric operators were studied by von Neumann, who worked out that the good symmetric operators from the point of view of spectral theory are those which have the additional property that the operators

$$(D \pm iI) : \mathrm{dom}(D) \longrightarrow H,$$

have dense range, in which case we have bounded operators $(D \pm iI)^{-1}$ since the operators in the display are always injective, and indeed bounded below. Starting with these *resolvent operators*

$$(D \pm iI)^{-1} : H \longrightarrow H$$

we get a *von Neumann symbol*

$$C_0(\mathbb{R}) \longrightarrow B(H),$$

a C^*-algebra homomorphism that maps $(x \pm i)^1$ to the resolvent operators (this property characterizes the symbol). The symbol extends (uniquely)to a C^*-algebra homomorphism

$$C_b(\mathbb{R}) \longrightarrow B(H).$$

So as long as the symbol exists, we have operators

$$e^{itD}, e^{-tD^2} \text{ (when t is nonnegative)}, D(I + D^2)^{-1/2}, \text{ etc.}$$

The good operators have an official name: *essentially self-adjoint*.

If D is a PDO, then "essentially self-adjoint" means formally self-adjoint, plus Maximal domain = Minimal domain.

Remark 3.3. The use of the term "symbol" here is not standard, but it is justified (we think) by the following comparision. Assume for simplicity that D is a formally self-adjoint operator of order q acting on a closed manifold M, and assume further, also for simplicity, that $\mathcal{E} = C^\infty(M)$. The (PDO) symbol is a smooth function

$$\sigma : T^*M \longrightarrow i^q \mathbb{R}$$

it induces a C^*-algebra homomorphism

$$\sigma^* : C_0(i^q \mathbb{R}) \longrightarrow C_b(T^*M).$$

Incidentally ellipticity implies that the symbol is a proper function in the topological sense, and therefore it corresponds to a C*-homomorphism

$$\sigma^* \colon C_0(i^q\,\mathbb{R}) \longrightarrow C_0(T^*M).$$

There is therefore an interesting formal analogy between the von Neumann symbol and the PDO symbol ...

Remark 3.4. There is a helpful 2×2 matrix trick that can be used to reduce the study of operators for which $D \neq D^\diamond$ to the formally self-adjoint case. Consider

$$D_+ : \mathcal{E}^+ \longrightarrow \mathcal{E}^-$$

where perhaps $D_+ \neq D_+^\diamond$. Define $D_- := D_+^\diamond$, and form

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix} : \mathcal{E}^+ \oplus \mathcal{E}^- \longrightarrow \mathcal{E}^+ \oplus \mathcal{E}^-.$$

This operator is formally self-adjoint.

3.2 Elliptic operators

From now on we shall be studying *elliptic* PDO's

$$D : \mathcal{E} \longrightarrow \mathcal{F}$$

of some order q, mostly but not exclusively on closed manifolds. The first of two fundamental theorems about these operators is as follows:

Theorem 3.5. *If M is compact and D elliptic, then the Maximal and Minimal domains agree. So if in addition D is formally self-adjoint, then it is essentially self-adjoint.*

Remark 3.6. In the non-compact case, the theorem is true with the following modification: if $f \in C^\infty_{\mathrm{comp}}(M)$, then

$$f \cdot \{\text{maximal domain}\} = f \cdot \{\text{minimal domain}\}.$$

The next theorem is even more fundamental:

Theorem 3.7. *If M is compact, then all the minimal domains of all elliptic operators D of the same order q that act on $\mathcal{E}$ are equal to one another. Moreovoer the common minimal domain of elliptic operators of order q is the intersection of all minimal domains of all PDO of order q or less.*

Remark 3.8. If M is not compact, then there is a modification like the one in the previous remark: all domains agree after multiplication pointwise by any smooth, compactly supported function f on M. Moreover the common (in this sense) domain agrees with the interection described above after pointwise multiplication by f.

The theorem summarizes just above everything we need to know about the functional analysis of elliptic operators. The proof is not easy, but we can discuss part of it. The idea is to reduce from the case of general operators to the case of constant coeefficient operators (in some coordinate system), and then apply the following argument:

Proof for constant coefficient operators on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Thanks to Fourier, we have a unitary isomorphism

$$L^2(\mathbb{T}^n) \xrightarrow{\cong} \ell^2(\mathbb{Z}^n).$$

Then, under the constant coefficient assumption the operator $D = \sum a_\alpha \partial^\alpha$ corresponds poinwise multiplication by $\sum_{|\alpha|=q} a_\alpha (in)^\alpha$. These multiplication operators are easy to analyze directly. □

3.3 Rellich and Sobolev

The following results of Rellich and Sobolev pertaining to the common domain will lead us towards the conclusion Elliptic $\implies$ Fredholm.

Definition 3.9. Assume that M is compact. Fix $q > 0$. Let $\mathcal{E}_{L^2_q}$ be the common domain of Theorem (3.7) for elliptic PDO of order q. This is called the q*th* L^2-*Sobolev space* associated to $\mathcal{E}$.

Lemma 3.10 (Rellich lemma)**.** *If* M *is compact and* $q > 0$*, then the inclusion*

$$\mathcal{E}_{L^2_q} \longrightarrow \mathcal{E}_{L^2}$$

is/factors through a compact operator. Moreover, it is/factors through a Schatten class operator of type $\dim(M)/q + \varepsilon$ *for all* $\varepsilon > 0$*.*

Remark 3.11. If M were not compact, the lemma would say instead that the composition

$$\mathcal{E}_{L^2_q} \longrightarrow \mathcal{E}_{L^2_q} \xrightarrow{f} \mathcal{E}_{L^2}$$

is compact and indeed Schatten class, as above, for any $f \in C^\infty_{\text{comp}}(M)$.

Theorem 3.12 (Sobolev embedding theorem)**.** *If* M *is compact, then*

$$\bigcap_q \mathcal{E}_{L^2_q} = \mathcal{E}.$$

Remark 3.13. If M were not compact, we would have

$$f \cdot \bigcap_q \mathcal{E}_{L^2_q} = f \cdot \mathcal{E}$$

for $f \in C^\infty_{\text{comp}}(M)$.

3.4 Eigenfunctions, Hypoellipticity, Hodge, and the Fredholm property

We're ready to draw some important conclusions after all the hard work of the previous section (that we didn't actually do).

3.4.1 Eigenfunctions and eigenvalues

If M is compact and D is elliptic and formally self-adjoint, then $(D + iI)^{-1}$ is bounded, normal, and compact (by the Rellich lemma (Lemma (3.10))).

So, by Hilbert's theory of compact operators,

Theorem 3.14. *If M is compact and D is elliptic and formally self-adjoint, then there is a basis for $\mathcal{E}_{L^2}$ consisting of eigenfunctions for D. Moreover the eigenvalues converge to infinity (in absolute value).* □

Moreover, due to Sobolev, the eigenfunctions are smooth, since from the eigenvalue equation it is clear that they lie in the (maximal) domains of all powers of D (and all powers are elliptic).

Corollary 3.15. *If M is compact and D is elliptic, then the kernel of D in its maximal domain is equal to its kernel on $\mathcal{E}$, and the kernel is finite-dimensional. Moreovoer the operator is bounded below (in the Hilbert space norm) on the orthogonal complement of the kernel.*

Proof. Use the 2×2 matrix trick to reduce to the formally self-adjoint case, then apply the previous theorem. □

3.4.2 Hypoellipticity

Suppose that M is compact and D is elliptic. Suppose $Du = \nu$, where ν is smooth everywhere. Then, $u \in \mathrm{dom}(D^k)$ for all k, so u is smooth.

The above is an easy version of hypoellipticity. The fuller version, that if ν is smooth on U, then u is smooth on U, is easy to get, too, using smooth cutoff functions to reduce to the case we have just considered. It's a fun exercise.

3.4.3 Hodge Theorem

Theorem 3.16. *Suppose that M is compact and let*

$$D \colon \mathcal{E} \longrightarrow \mathcal{F}$$

be an elliptic PDO. If $\nu \in \mathcal{F}$ and if

$$\nu \perp \ker\bigl(D^{\diamond} \colon \mathcal{F} \to \mathcal{E}\bigr),$$

then $\nu = Du$ for some $u \in \mathcal{E}$.

Proof. Elementary Hilbert space theory tells us that ν, being orthogonal to the kernel of $D^\diamond$, is in the closure of the range of D in $\mathcal{F}_{L^2}$. But by the corollary of the previous section the range of D as it acts on its minimal domain is closed in $\mathcal{F}_{L^2}$. So can write $\nu = Du$ with u in the miminal domain. But hypoellipticity now implies that u is smooth. $\square$

3.4.4 Fredholm Property

From Hilbert and Hodge we see that if

$$D\colon \mathcal{E} \longrightarrow \mathcal{F}$$

is an elliptic operator on a compact manifold, then both $\ker(D)$ (on smooth functions) and $\operatorname{coker}(D)$ (on smooth functions) are finite dimensional. so D is Fredholm. That is,

Theorem 3.17. *If M is compact and D is elliptic, then D is Fredholm.* $\square$

4 08 September 2016, Erik van Erp

Theorem 4.1. *[Atiyah-Singer] If M is a closed (compact without boundary) spin manifold, and D the Dirac operator of M, then*

$$\operatorname{Index} D = \int_M \widehat{A}(TM). \tag{4.1}$$

In this lecture I will describe in detail the meaning of the left hand side and the right hand side of Equation (6.1). We begin with...

4.1 Left Hand Side of Equation (6.1) of Atiyah-Singer

We need to explain what the Dirac operator is for a spin manifold. In Section 06 September we constructed the "model" Dirac operator on $\mathbb{R}^n$. For $n = 2r, 2r+1$:

$$D : C^\infty\left(\mathbb{R}^n, \mathbb{C}^{2^r}\right) \longrightarrow \left(\mathbb{R}^n, \mathbb{C}^{2^r}\right),$$

is defined by

$$D = \sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j}$$

where E_j are $2^r \times 2^r$ matrices with entries $0, \pm 1, \pm i$. The main property of D is that its square D^2 is a Laplacian, and that D is $\mathrm{Spin}(n)$ equivariant.

Spin-manifold: Let M be an oriented Riemannian manifold. Choose a good open cover of M and local trivializations: $TM|_{U_i} \approx U_i \times \mathbb{R}^n$. Then TM is represented by transition functions $\psi_{ij} : U_i \cap U_j \longrightarrow SO(n)$. Locally, we can lift each smooth map ψ_{ij} to the spin group:

$$\begin{tikzcd} U_i \cap U_j \arrow[r, "\psi_{ij}"] \arrow[dr, dotted, "\widetilde{\psi}_{ij}"'] & SO(n) \\ & \mathrm{Spin}(n) \arrow[u, two heads] \end{tikzcd}$$

For each triple intersection $U_i \cap U_j \cap U_k$, we have the cocyle condition $\psi_{ij}\psi_{jk}\psi_{ki} = I \in SO(n)$. This implies that

$$\widetilde{\psi} ijk = \widetilde{\psi}_{ij}\widetilde{\psi}_{jk}\widetilde{\psi}_{ki} = \pm 1 \in \mathrm{Spin}(n)$$

In other words, $\widetilde{\psi} ijk$ defines a 2-cocyle with values in $\mathbb{Z}/2\mathbb{Z}$. The cohomology class of this 2-cocycle does not depend on the choice of trivializations. It is called the second Stieffel-Witney class,

$$w_2(M) = \left[\widetilde{\psi}\right] \in H^2(M, \mathbb{Z}/2\mathbb{Z})$$

If $w_2(M) = 0$, then $\widetilde{\psi}_{ijk}$ is the boundary of a 1-cochain $U_i \cap U_j \to \mathbb{Z}/2\mathbb{Z}$. This 1-cochain tells us how to correct the original choices $\widetilde{\psi}_{ij}$ to obtain maps for which

$$\widetilde{\psi}_{ij}\widetilde{\psi}_{jk}\widetilde{\psi}_{ki} = 1$$

Such a choice of lift $\widetilde{\psi}_{ij}$ is called a *spin structure*.

Spinor bundle: If we have a spin structure $\{\widetilde{\psi}_{ij}\}$, as above, then the spinor representation $c : \mathrm{Spin}(n) \to GL\left(2^r, \mathbb{C}\right)$ of Expression (2.3) gives rise to transition functions

$$c \circ \widetilde{\psi}_{ij} : U_i \cap U_j \to GL\left(2^r, \mathbb{C}\right)$$

The *spinor bundle* $\mathbb{S}$ associated to the spin structure ψ is the complex vector bundle with fiber $\mathbb{C}^{2^r}$ defined by the transition functions $c \circ \widetilde{\psi}_{ij}$.

Recall that the $\mathbb{R}$-linear map

$$c : \mathbb{R}^n \longrightarrow \mathrm{End}\left(\mathbb{C}^{2^r}\right) \qquad c(v) = \sum_{j=1}^{n} E_j v_j$$

is $\mathrm{Spin}(n)$-equivariant. Therefore we have a well-defined global map

$$c : T_pM \longrightarrow \mathrm{End}(\mathbb{S}_p) \qquad p \in M$$

Furthermore,

$$\left\{ \begin{array}{l} E_j^2 = -I, \forall j \\ E_iE_j = -E_jE_i, \forall i, j, i \neq j \end{array} \right\} \iff c(v)^2 = -\|v\|^2 I, \forall v \in \mathbb{R}^n$$

shows that we also have $c(v)^2 = -\|v\|^2 I$ for $v \in T_pM$. In summary, by choosing a spin structure $\widetilde{\psi}$ on M, we obtain a spinor bundle $\mathbb{S}$ equipped with some extra structure:

$$\text{Spin}^c \text{ structure} \begin{pmatrix} c : T_pM \longrightarrow \text{End}(\mathbb{S}) \\ \\ c(v)^2 = -\|v\|^2 I \\ \\ \dim(\mathbb{S}_p) = 2^r \end{pmatrix}. \tag{4.2}$$

Remark 4.2. We have seen that a spin manifold has a Spin^c structure. But there are other ways in which Spin^c structures arise. For example, a complex manifold is Spin^c:

$T^{1,0}M =$ holomorphic tangent bundle with $\dim_{\mathbb{R}} M = n = 2r$, $\dim_{\mathbb{C}} M = r$
$\mathbb{S} = \bigwedge^0 T^{1,0}M \rightsquigarrow \dim(\mathbb{S}_p) = 2^r$.
$z \in T^{1,0}M = T_pM \xrightarrow{c} \text{End}(\mathbb{S})$
$c(z)\alpha = z \wedge \alpha - z \llcorner \alpha$ with $\alpha \in \mathbb{S}_p$
Exercise: $c(z) = -|z\|^2 I$.

This defines a natural Spin^c structure associated to a complex structure. But there are many other geometric structures that give rise to a Spin^c structure. The following diagram gives an overview.

$$\begin{tikzcd}
(\text{spin}) \arrow[r, Rightarrow] & (\text{Spin}^c) & \text{contact} \arrow[dl, Rightarrow] & \text{symplectic} \arrow[dll, Rightarrow] \\
& \text{stably almost complex} \arrow[u, Rightarrow] & & \\
& \text{almost complex} \arrow[u, Rightarrow] & & \\
& (\text{complex}) \arrow[u, Rightarrow] & &
\end{tikzcd}$$

Now, that we know what a spin manifold is, we move on to the Dirac operator on a spin manifold. The "model" Dirac operator on $\mathbb{R}^n$,

$$D_{\mathbb{R}^n} : C^\infty\left(\mathbb{R}^n, \mathbb{C}^{2^r}\right) \longrightarrow C^\infty\left(\mathbb{R}^n, \mathbb{C}^{2^r}\right)$$

is $\text{Spin}(n)$ equivariant, and therefore for every point $p \in M$ we have a well-defined Dirac operator on the tangent fiber,

$$D_{\mathbb{R}^n} \rightsquigarrow D_p : C^\infty(T_pM, \mathbb{S}_p) \longrightarrow C^\infty(T_pM, \mathbb{S}_p)$$

Now each D_p is an operator on T_pM, not on M. But in a small neighborhood of p, T_pM is a good approximation of M. What the family of operators $\{D_p, p \in$

$\mathfrak{m}\}$ defines is, in fact, the *highest order part* of a differential operator on D on M,

$$\{D_p, p \in M\} \rightsquigarrow D : C^\infty(\mathbb{S}) \longrightarrow C^\infty(\mathbb{S})$$

The choice of the order zero term of D is arbitrary, but it should be chosen such that D is formally self-adjoint. Regardless of the choice of the order zero term, the principal symbol of D is precisely the map $c : T_pM \to \mathrm{End}(\mathbb{S}_p)$. We see from $c(v)^2 = -\|v\| \cdot I$ that $c(v)$ is invertible as long as $v \neq 0$, which means, by definition, that D is elliptic. Therefore as long as we choose the order zero term such that D is formally self-adjoint, it will also be essentially self-adjoint (i.e., the minimal and maximal domains of D are equal, and the closure of D is self-adjoint).

The index that appears at the left hand-side of Equation (6.1) is *not*

$$\begin{aligned} \mathrm{Index} D &= \dim\ker D - \dim \mathrm{coker} D \\ &= \dim\ker D - \dim\ker D^* \end{aligned}$$

Since D is essentially self-adjont, this index is zero. To understand the left hand side of Equation (6.1) we need to consider one final bit of structure.

Grading: We now need to restrict ourselves to even dimensional manifolds. If $n = 2r$, then the inductively defined matrices $E_1, \dots, E_n$ are all off-diagonal,

$$E_1, \dots, E_n = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$$

Thus, the spinor vector space $\mathbb{C}^{2^r} = W$ splits into two vector spaces $W = W^+ \oplus W^-$ such that each E_j maps $W^+ \to W^-$ and $W^- \to W^+$. Therefore each of the subspaces W^+, W^- is invariant under all products E_iE_j. Recall that $\mathrm{Spin}(n) = \exp \mathfrak{g}$, where $\mathfrak{g} = \mathrm{span}\{e_ie_j : i \neq j\}$ in the Clifford algebra C_n. Therefore every element $g \in \mathrm{Spin}(n)$ is represented on W by a matrix of the form $c(g) = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$. In other words, the subspaces W^+ and W^- are invariant under the action of $\mathrm{Spin}(n)$.

Fact: W^+, W^- are *irreducible* representations of $\mathrm{Spin}(n)$ that do not factor through representations of $SO(n)$ (they are so-called $\frac{1}{2}$-spin representations), and W^+ and W^- are the two smallest such representations.

It follows that the split $W = W^+ \oplus W^-$ also exists for the spinor bundle $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ (if $n = 2r$ even). Moreover, by the way it is defined, the Dirac operator maps positive spinors to negative spinors,

$$D_+ : C^\infty(\mathbb{S}^+) \longrightarrow C^\infty(\mathbb{S}^-)$$

Then the "total" Dirac operator D defined before can be represented as the 2×2 matrix of operators

$$D = \begin{bmatrix} 0 & D_+^* \\ D_+ & 0 \end{bmatrix},$$

corresponding to the splitting $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$.

The index on the left hand side of Equation (6.1) is really the index of D_+,

$$\mathrm{Index} D_+ = \dim \ker D_+ - \dim \mathrm{coker} D_+ = \dim \ker D_+ - \dim \ker D_+^*.$$

4.2 Right Hand Side of Equation (6.1) of Atiyah-Singer

$$\mathrm{Index} D = \int_M \widehat{A}(TM).$$

The left hand side IndexD is a *global invariant*. For example, the kernel of D—i.e., the space of solutions of the differential equation $Ds = 0$ for section s in the spinor bundle $\mathbb{S}^+$—is *locally* an infinite dimensional space. $Ds = 0$ is equivalent to $D^*Ds = 0$, and D^*D is a Laplace-type operator. In an open subset of $\mathbb{R}^n$ the space of harmonic spinors (solutions of $D^*Ds = 0$) is infinite dimensional, and the same is true in small open subsets of a manifold.

However, to solve $Ds = 0$ globally, you need to "propagate" a local solution across the manifold. If the manifold is closed, only in very exceptional cases will the local solution propagate in such a way as to "match up" to a globally defined smooth solution. Indeed, as we know, the solution space of $Ds = 0$ is finite dimensional if M is closed.

Thus, the index of a geometrically defined differential operator, like D, depends on the global topology of the manifold.

By contrast, the right hand side of the formula is an integral of a function $\widehat{A}(TM)$, which, as we will see, is a polynomial expression in the Riemannian curvature tensor of M. Curcature is of course a purely *local* phenomenon.

$\widehat{A}$ - class/genus. On a Riemannian manifold we have the Riemannian curvature tensor

$$R \in C^\infty\left(\wedge^2 T^*M \otimes \mathrm{End}(TM)\right)$$

which can be thought of as an $\mathrm{End}(TM)$ valued 2-form. If we choose an orthonormal basis for T_pM at a point $p \in M$, then R is represented concretely as an $n \times n$ skew symmetric matrix of 2-forms (on M). We want to extract differential forms on M from the curvature R.

Definition 4.3. A polynomial $p : \mathfrak{so}(n) \longrightarrow \mathbb{R}$ in the coefficients of $n \times n$ skew-symmetric real matrices is $O(n)$-invariant if

$$p(gAg^{-1}) = p(A) \qquad \forall A \in \mathfrak{so}(n), g \in O(n)$$

How do we get such a polynomial? Skew symmetric matrices can be brough in a normal form:

$$\forall A \in \mathfrak{so}(n), \exists g \in O(n) \text{ such that } gAg^{-1} = \begin{bmatrix} 0 & -x_1 & & & \\ x_1 & 0 & & & \\ & & 0 & -x_2 & \\ & & x_2 & 0 & \\ & & & & \ddots \end{bmatrix}$$

Therefore an invariant polynomial p is entirely determined by its value on matrices in normal form, which is a polynomial in the variables $x_1, \ldots, x_r$. Furthermore, the $O(n)$-action can permute the values $x_1, ..., x_r$, and even exchange each pair $x_j, -x_j$ (i.e., change the sign of each x_j). It follows that p, when evaluated on matrices in normal form, must be a symmetric polynomial in $x_1^2, \ldots, x_r^2$. It is not hard to prove the following result.

Proposition 4.4. *There is a ring isomorphism*

- $\{O(n) -$ *invariant polynomials* $p : \mathfrak{so}(n) \longrightarrow \mathbb{R}\}$
- $\left\{\text{\textit{symmetric polynomials in} } x_1^2, \ldots, x_r^2\right\}$.

We wish to apply an invariant polynomial p to the Riemannian curvature tensor R. While R is (in a local representation) a skew symmetric matrix, its coeffiecients are 2-forms instead of real numbers. But note that

$$\bigwedge\nolimits^{\text{even}} T_p M \text{ is a commutative algebra}$$

Thus, it makes sense to evaluate an invariant polynomial $p : \mathfrak{so}(n) \to \mathbb{R}$ on the coefficients of the curvature matrix R.

Definition 4.5. Given an $O(n)$-invariant polynomial $p : \mathfrak{so}(n) :\longrightarrow \mathbb{R}$. Then

$$p\,(TM) := p\left(\frac{R}{2\pi}\right) \in \Omega^{\bullet}(M).$$

Note that each x_j is replaced by a 2-form, and hence x_j^2 by a 4-form. Thus, $p(R)$ is a form in degrees $0, 4, 8, 12, \ldots$.

<u>Fact</u>: $p(TM)$ is a closed form

<u>Fact</u>: The de Rham cohomology class $[p(TM)] \in H^{\bullet}_{dR}(M, \mathbb{R})$ does not depend on the metric.

It follows that every symmetric polynomial in formal variables $x_1^2, \ldots, x_r^2$ gives rise to a smooth invariant of a closed manifolds. If we orient M, then for every p we can define the real number

$$p(M) := \int_M p(TM) \in \mathbb{R}$$

It is a deep result of Novikov that these numbers are, in fact, *topological* invariants of M.

Remark 4.6. The Pfaffian $\mathrm{Pf} : \mathfrak{so}(n) \to \mathbb{R}$ is the polynomial that, when restricted to matrices in normal form, is defined by $\mathrm{Pf}(x) = x_1 x_2 \cdots x_n$. However, this expression is not $O(n)$-invariant but only $SO(n)-$invariant. Conjugation by a matrix in $O(n)$ can change the sign of any of the variables x_j. But conjugation by a matrix in $SO(n)$ can only change the signs of an *even* number of the variables $x_1, \ldots, x_r$. Thus the product $x_1 x_2 \cdots, x_r$ is invariant for $SO(n)$ but not $O(n)$.

It follows that the differential form $\mathrm{Pf}(R)$ is well-defined on every *oriented* Riemannian manifold. We thus obtain an invariant of closed oriented manifolds,

$$\mathrm{Pf}(M) = \int_M \mathrm{Pf}(\frac{R}{2\pi})$$

The Chern-Gauss-Bonnet theorem identifies this number as the Euler number

$$\int_M \mathrm{Pf}\left(\frac{R}{2\pi}\right) = \text{Euler number} \in \mathbb{Z}$$

This classical theorem is a special case of the Atiyah-Singer theorem. Note that the invariant is always an integer in this case. This justifies, to some extent, the factor 2π in the definition $p(TM) = p(R/2\pi)$.

For surfaces ($n = 2, r = 1$) the Riemann curvature is of the form $R = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}$, where κ is the Gaussian curvature. In this case $\mathrm{Pf}(x) = x_1$ and so $\mathrm{Pf}(R/2\pi) = \kappa/2\pi$. The Chern-Gauss-Bonnet theorem reduces to the much simpler Gauss-Bonnet theorem,

$$\int_M \frac{\kappa}{2\pi} = \text{Euler number}$$

The $\widehat{A}$-genus $\widehat{A}(M) \in \mathbb{R}$ is an invariant of closed manifolds defined in the same way by specifying an invariant polynomial $\widehat{A}(x)$. We will define the $\widehat{A}$ polynomial next time in Section 12 September.

5 09 September 2016, Nigel Higson

Throughout this lecture we shall be dealing with a linear elliptic PDO

$$D_+ : \mathcal{E}^+ \longrightarrow \mathcal{E}^-$$

on a closed manifold M. We'll assume that D is of order one, although today that is no more than a small convenience, and we shall use the 2×2 matrix trick to convert D into a formally self-adjoitn operator

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

acting on $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$.

The Atiyah-Singer index formula expresses the index of D_+ as an integral

$$\text{Index}(D_+) = \int_M \boxed{\text{a certain form on } M}\,, \tag{5.1}$$

at least for Dirac operators, where form comes from the world of geometry (connections and curvature). Our goal today is to calculate that irrespective of geometric considerations

$$\text{Index}(D_+) = \int_M \boxed{\text{a certain function on } M}\; dm. \tag{5.2}$$

An advantage of equation (5.2) over (5.1) is that the equality will result from a rather direct and general calculation, and in principal enough students locked in a room for long enough could actually compute the integral. A huge disadvantage of (5.2) is that the calculation, if done using only the technoloyg we'll develop today, would be extraordinarily complicated: in principal it would involve calculating all the partial derivatives of all the coefficients of D_+ (and a bit more) to order the dimension of M. That's a lot of partial derivatives. In contrast the integral provided by the Atiyah-Singer formula involves only second order derivatives.

What eventually emerges is that for "the" Dirac operator (a term we shall have to explain), not only are the two integrals the same (this is what the index theorem says) but the *integrands* are the same, point by point. So all those higher derivatives do not in fact contribute to the integrand, let alone to the integral. This is the phenomenon of *miraculous cancellations*, as McKean and Singer called it.

5.1 Traces of operators

We begin with a quick review of the the trace in the Hilbert space context. Fix a bounded Hilbert space operator T. If we have a diagram of bounded operators

$$\begin{tikzcd} \mathcal{E}_{L^2} \arrow[rr, "T"] \arrow[dr] & & \mathcal{E}_{L^2} \\ & \mathcal{E}_{L^2_k} \arrow[ur] & \end{tikzcd}$$

with $k \gg 0$ (actually $k > \dim(M)/2$), then T is a Hilbert-Schmidt operator and can be representated as a kernel

$$k_T \in \mathcal{E} \otimes \overline{\mathcal{E}}$$

in the Hilbert space tensor product. If k is still larger ($k > \dim(M)$, in fact), then T is a trace-class operator, and k_T can be represented as a sum

$$k_T = \sum e_j \otimes \bar{f}_j$$

with

$$\sum \|e_j\| \cdot \|f_j\| < \infty$$

The trace of T is then

$$\mathrm{Trace}(T) = \sum \langle f_j, e_j \rangle.$$

Keeping in mind the definition of the inner product (see Lecture 3), we find that the Trace has the form of an integral over M (of the sum of the pointwise inner products of e_j with f_j). We are making (a very small bit of) progress towards equation (5.2).

To make further progress, we need to represent the index as a trace. There are many options here; we're going to examine a strategy originally outlined by Atiyah and Bott (see for example Atiyah's 1966 ICM address). Actually, we'll begin with a modification suggested by Hörmander) that is perhaps easier to understand on a first encounter.

5.2 Heat Equation Approach to the Index Theorem

Use the functional calculus (part of von Neumann's symbol package) to form the family of operators

$$e^{-t\Delta} \colon \mathcal{E}_{L^2} \longrightarrow \mathcal{E}_{L^2},$$

where $\Delta = D^2$ and $t > 0$. The ranges of these operators lie in $\mathcal{E}_{L^2_k}$ for all k, so the operators belong to the trace class. Rather than study the traces of the operators $e^{-t\Delta}$ themselves we shall study $\mathrm{Trace}(\gamma e^{-t\Delta})$, where

$$\gamma = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} : \mathcal{E} \longrightarrow \mathcal{E}.$$

Lemma 5.1. $\mathrm{Trace}(\gamma e^{-t\Delta})$ *is equal to* $\mathrm{Index}(D_+)$ *for all* $t > 0$.

Proof. Let's first show that the index is independent of t. Calculus gives

$$\begin{aligned} \frac{d}{dt}\mathrm{Trace}(\gamma e^{-t\Delta}) &= -\mathrm{Trace}(\gamma \Delta e^{-t\Delta}) \\ &= -\mathrm{Trace}(\gamma D^2 e^{-t\Delta}) \end{aligned}$$

Next, we use the trace property to compute that

$$\begin{aligned} \mathrm{Trace}(\gamma D^2 e^{-t\Delta}) &= \mathrm{Trace}(\gamma D e^{-t\Delta} D) \\ &= \mathrm{Trace}(D\gamma D e^{-t\Delta}). \end{aligned}$$

But D and γ anticomute, so

$$\mathrm{Trace}(D\gamma D e^{-t\Delta}) == \mathrm{Trace}(\gamma D^2 e^{-t\Delta}).$$

As a result,

$$\mathrm{Trace}(\gamma D^2 e^{-t\Delta}) = -\mathrm{Trace}(\gamma D^2 e^{-t\Delta}),$$

so the derivative is zero. Now recall that there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for $\mathcal{E}_{L^2}$ consisting of (smooth) eigenfunctions for D. We have that $e^{-t\Delta}e_n = e^{-t\lambda^2}e_n$, and we see therefore that $e^{-t\Delta}$ converges, even in the trace norm, to the orthogonal projection onto the kernel of D. As a result,

$$\lim_{t\to\infty} \mathrm{Trace}(\gamma e^{-t\Delta}) = \mathrm{Index}(D_+),$$

as required. □

The operators $e^{-t\Delta}$ are closely related to the heat equation

$$\frac{du}{dt} = -\Delta u,$$

and Hörmander's proposal was to take advantage to the theory of this PDE in the service of index theory. The point is that heat equation techniques show that for a second order Δ (subject to some hyptotheses that are certainly satisfied when D is the Dirac operator and $\Delta = D^2$) there is an asymptotic expansion

$$\mathrm{Trace}(e^{-t\Delta}) \sim \sum_{n\geq -\dim(M)} a_n t^{n/2}$$

as $t \to 0$. Moreover each of the coefficients is the integral over M of an explicit (but complicated) expression in the coefficients of Δ and their derivatives. More on this later (in the lecture after the next one).

5.3 Zeta Function Approach to the Index Theorem

Let us turn now to the original proposal of Atiyah and Bott, who suggested that one should study, instead of the operators $e^{-t\Delta}$, the zeta function

$$\mathrm{Trace}\left(\gamma(I+\Delta)^{-s}\right).$$

As with the heat operators $e^{-t\Delta}$, it is easy to see that the trace is a constant function of s, and that the constant is $\mathrm{Index}(D_+)$. Moreover it can be shown (and we shall show it in the lecture after next) that the value at $s = 0$ can be computed by local methods, like the coefficients in the asymptotic expansion. So the the zeta function above offers an alternative to the heat equation approach to the index theorem (and actually, as we have noted, it was the original local approach).

Before we can go any further, we need to address the following important issue: the operators $(I+\Delta)^{-s}$ are only in the trace class when $\mathrm{Real}(s) > \dim(M)/2$, so it does not (yet) make sense to consider the value of the zeta function at $s = 0$. The issue is resolved by the following remarkable result:

Theorem 5.2. *[Minakshisundaram-Pleijel] Let Δ be any (2nd order), positive, invertible, elliptic PDO with scalar symbol,*[1] *then* $\mathrm{Trace}(\Delta^{-s})$ *extends to a meromorphic function on* $\mathbb{C}$ *with only simple poles.*

In the remainder of this lecture we shall sketch a proof of this theorem.

5.3.1 Traces on the Algebra of Differential Operators

The key idea (in the proof that we shall present) is well illustrated by the proof of the following little result:

Proposition 5.3. *There are no nonzero traces on the algebra of differential operators on* M

Lemma 5.4. *There are functions* $A_1, \ldots, B_N$ *and vector fields* $B_1, \ldots, B_N$ *on* M *such that*

1. $\sum [B_i, A_i] = nI$*, where* $\dim(M) = n$*, and*
2. *if* T *is any differential operator of order* q*, then*

$$(n+q)D = \sum [B_i D, A_i] + R,$$

where the remainder R *has order less than* q*.*

Proof. If the manifold is $\mathbb{R}^n$, then let $A_i = x_i$ and $B_i = \frac{\partial}{\partial x_i}$. If the manifold is not $\mathbb{R}^n$, use local coordinate charts and partitions of unity to reduce to the case of $\mathbb{R}^n$. □

Note that in (2), the expression $[B_i D, A_i] = D[B_i, A_i] + B_i[D, A_i]$, in which the first term provides the constant n in $(n+q)$ by (1), and q comes from the second term.

Proof of the Proposition. If τ is any trace on the algebra of differential operators, then applying it to both sides of the identity in the lemma we find that

$$(n+q)\tau(D) = \tau(R)$$

since the trace vanishes on commutators. So if the trace is zero on every operator of order less than q then it vanishes on every operator of order q. But the trace does indeed vanish of every operator of order -1 (there are none). □

[1] Recall that the symbol is a function valued in matrices; we require that the matrices be multiples of the identity. This hypothesis will be used in the following way: if T is any differential operator acting on $\mathcal{E}$, of order q, then the commutator $[T, \Delta]$ has order at most $q+1$.

5.3.2 Algebras of Holomorphic Families of Operators

We're going to prove the meromorphic continuation theorem by applying the idea of the previous section to the algebra[2] $\mathcal{C}$ of families of differential operators (parametrized by $z \in \mathbb{C}$) that have asymptotic expansions of the form

$$T_1\Delta^{az+b_1} + T_2\Delta^{az+b_2} + \cdots ,$$

where a and the b_k are nonpositive integers, and if we define

$$\operatorname{order}\big(T\Delta^{az+b}\big) = \operatorname{order}(T) + 2(az+b),$$

then the order of the terms is strictly decreasing. We won't spell out the nature of the asymptotic expansion, but it will be clear from the proofs below (which supply examples) that the definition ought to be.

Let's take it for granted, for a moment, that the above is indeed an algebra (obviously the issue is how to compute the product of two families, as above). We want to define a trace, argue as in the previous section that the trace must be zero, and then deduce the meromorphic continuation theorem from the vanishing of the trace.

The range of the trace will be a vector space $\mathcal{Q}_N$, rather than the complex scalars (actually, as the notation suggests, there will be a trace for each $N \in \mathbb{N}$). It is defined as follows

(i) Given $N \in \mathbb{N}$, let $\mathcal{A}_N$ be the vector space of functions that are holomorphic in the region $\operatorname{Real}(z) > N$. These spaces form a directed system

$$\mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \cdots$$

and we define $\mathcal{A}$ to be the direct limit. It consists of functions that are defined and analytic in some right half-space in $\mathbb{C}$.

(ii) Next, let $\mathcal{M}_{-N}$ be the space of functions that are meromorphic in the region $\operatorname{Real}(z) > -N$, with only simple poles, and analytic in some right half-space.

(iii) Finally, define $\mathcal{Q}_N$ to be the quotient vector space

$$\mathbb{Q}_N = \mathcal{A}/\mathcal{M}_{-N}.$$

There is an obvious functional

$$\tau_N \colon \mathcal{C} \longrightarrow \mathcal{Q}_N$$

[2] Actually it is technically better to work with the bimodule over the algebra of differential operators that consists of those families with $a = -1$ in the definition below, but we'll allow some slight inaccuracies and work with algebras rather than bimodules. This is because we want to emphasize the relation between the proof of the M-P theorem and the proof of the proposition in the previous section.

that maps $T\Delta^{az+b}$ to its trace function $z \mapsto \mathrm{Trace}(T\Delta^{az+b})$. If the order of $T\Delta^{az+b}$ is sufficiently small then the trace function is defined on $\mathrm{Real}(z) > -N$ and is analytic there (and in particular it is meromorphic there). So the τ_N-trace of $T\Delta^{az+b}$ is zero if the order is sufficiently small, and as a result, the trace τ_N is well defined on all families admitting an asymptotic expansion of the type we are considering (only finitely many terms have a nonzero trace).

It will be obvious that the functional τ_N is indeed a trace once we have consider the issue of whether or not $\mathcal{C}$ is an algebra, which is what we shall turn to next.

5.3.3 Commutators and the Binomial Expansion

We shall use the Cauchy formula

$$\Delta^z = \frac{1}{2\pi i}\int \lambda^z(\lambda-\Delta)^{-1}\,d\lambda$$

to define the complex powers of Δ. The contour of integration is a (downwards oriented) vertical line in the plane separating 0 from the spectrum of Δ.

The fact that our algebra $\mathcal{C}$ is indeed an algebra follows from the following lemma, due to Connes and Moscovici.

Lemma 5.5. *If* C *is any differential operator, then*

$$\begin{aligned}[C,\Delta^z] = \binom{z}{1}\Delta^{z-1}C^{(1)} + \binom{z}{2}\Delta^{z-2}C^{(2)} + \cdots + \binom{z}{k}\Delta^{z-k}C^{(k)} \\ + \int \lambda^z(\lambda-\Delta)^{-1}[\Delta,C]^{(k)}(\lambda-\Delta)^{-k}d\lambda,\end{aligned}$$

where $C^{(1)} = [C,\Delta]$ *and* $C^{(k+)} = [\Delta, C^{(k)}]$

Remark 5.6. It will actually suffice to consider values of z which have large negative real part. For these, all the operators under discussion will be bounded, and indeed trace class.

Proof. Using the commutator identity

$$[C,(\lambda-\Delta)^{-1}] = (\lambda-\Delta)^{-1}[C,\Delta](\lambda-\Delta)^{-1}$$

we find that

$$\begin{aligned}[C,\Delta^z] &= \frac{1}{2\pi i}\int \lambda^z[C,(\lambda-\Delta)^{-1}l]\,d\lambda \\ &= \frac{1}{2\pi i}\int \lambda^z(\lambda-\Delta)^{-1}[C,\Delta](\lambda-\Delta)^{-1}\,d\lambda \\ &= \frac{1}{2\pi i}\int \lambda^z(\lambda-\Delta)^{-2}[C,\Delta]\,d\lambda + \frac{1}{2\pi i}\int \lambda^z(\lambda-\Delta)^{-1}[[C,\Delta],(\lambda-\Delta)^{-1}]\,d\lambda\end{aligned}$$

The first term can be calculated by using the Cauchy integral formula:

$$\frac{1}{2\pi i}\int \lambda^z(\lambda-\Delta)^{-2}[C,\Delta]\,d\lambda = \binom{z}{1}\Delta^{z-1}[C,\Delta]$$

As for the second term can manipulate it using the commutator formula

$$[[C,\Delta],(\lambda-\Delta)^{-1}] = (\lambda-\Delta)^{-1}[[C,\Delta],\Delta](\lambda-\Delta)^{-1}$$

Plugging this into the second integral gives

$$\begin{aligned}\frac{1}{2\pi i}&\int \lambda^z(\lambda-\Delta)^{-1}[[C,\Delta],(\lambda-\Delta)^{-1}]\,d\lambda\\ &= \frac{1}{2\pi i}\int \lambda^z(\lambda-\Delta)^{-2}[[C,\Delta],\Delta](\lambda-\Delta)^{-1}\,d\lambda\\ &= \frac{1}{2\pi i}\int \lambda^z(\lambda-\Delta)^{-3}[[C,\Delta],\Delta]\,d\lambda\\ &\quad + \frac{1}{2\pi i}\int \lambda^z(\lambda-\Delta)^{-2}[[C,\Delta],(\lambda-\Delta)^{-1}]\,d\lambda.\end{aligned}$$

Once again, we can calculate the first term at the bottom using the Cauchy integral formula. As for the second term, we can continue to manipulate it in the same way. After k steps like this we arrive at the formula in the statement of the lemma. □

Proof of the Minakshisundaram-Pleijel theorem. The argument used to prove the vanishing of traces on the algebra of differential operators applies, because the lemma on which it depends applies to the algebra $\mathcal{C}$. The key formula is

$$\mathrm{Trace}(T\Delta^{az+b}) = (n+q+2(az+b))^{-1}\,\mathrm{Trace}(R_z)$$

where $q = \mathrm{order}(T)$ and the family R_z has lower order than $(n+q+2(az+b))$ (note, by the way, that the pole of $(n+q+2(az+b))^{-1}$ is the reason meromorphic functions appear). We find that if the trace τ_N vanishes on all families

$$T_1\Delta^{az+b_1} + T_2\Delta^{az+b_2} + \cdots,$$

with leading order $q+az+b_1$ then it vanishes on all families of leading order $q+az+b_1+1$. On the other hand τ_N vanishes on all families of sufficiently low leading order just by virtue of its definition. So it vanishes on all families. This being true for all N, we see that each function $\mathrm{Trace}(T\Delta^{-z})$ extends to a meromorphic function as required. □

6 12 September 2016, Erik van Erp

6.1 continuation of right hand side of Equation (6.1) of Atiyah Singer

Recall the version of the Atiyah-Singer formula that we are considering.

Theorem 6.1. *If* M *is a closed (compact without boundary) spin manifold, and* D *the Dirac operator of* M*, then*

$$\operatorname{Index} D = \int_M \widehat{A}(TM). \tag{6.1}$$

As we saw in Subsection (4.1), by $\operatorname{Index} D$ we really mean $\operatorname{Index}(D_+)$, where $D_+ : C^\infty(\mathbb{S}^+) \longrightarrow C^\infty(\mathbb{S}^-)$.

We continue with the description of the right hand side of the equation. As before, we let $\dim(M) = n = 2r$, and

$$\begin{aligned} \mathbb{R} &= \text{ Riemannian curvature} \\ &= \text{ (locally) } n \times n \text{ skew matrix of 2-forms,} \end{aligned}$$

As we explained, every $n \times n$ skew matrix with coefficients in $\mathbb{R}$ is similar (via conjugation by an orthogonal matrix) to a matrix of the form

$$\text{skew } \sim \begin{bmatrix} 0 & -x_1 & & & \\ x_1 & 0 & & & \\ & & 0 & -x_2 & \\ & & x_2 & 0 & \\ & & & & \ddots \end{bmatrix}$$

Every symmetric polynomial in the variables $x_1^2, x_2^2, x_3^2, \ldots$ induces an $O(n)$-invariant polynomial $\mathfrak{so}(n) \to \mathbb{R}$, which can be evaluated on the Riemannian curvature matrix of 2-forms.

The $\widehat{A}$-class corresponds to the invariant polynomial

$$\widehat{A} : \mathfrak{so}(n) \longrightarrow \mathbb{R}$$

$$\begin{aligned} \widehat{A}(x_1, x_2, \ldots) &= \prod_{j=1}^{r} \frac{x_j}{e^{x_j} - e^{-x_j}} \rightsquigarrow \text{ even function} \\ &= \prod 1 - \frac{1}{24} x_j^2 + \frac{7}{5760} x_j^4 - \cdots , \end{aligned} \tag{6.2}$$

Note that the series has rational coefficients. As defined here, $\widehat{A}(x)$ is of course a formal power series, and not a polynomial. But on a finite dimensional manifold M the higher powers of x_j correspond to forms of too high a degree, and so those terms will vanish when $\widehat{A}$ is evaluated on the curvature matrix R.

The $\widehat{A}$-genus of M is by definition

$$\widehat{A}(M) = \int_M \widehat{A}\left(\frac{R}{2\pi}\right) \in \mathbb{Q},$$

This is a rational number (because Expression (6.2) has rational coefficients). Note that the number $\widehat{A}(M)$ is defined for any oriented Riemannian manifold,

and like all numbers defined in this way, it is independent of the choice of metric, and in fact a topological invariant of M.
Genus Properties:

(6.3)

1. $\widehat{A}(M \sqcup N) = \widehat{A}(M) + \widehat{A}(N)$. This is clearly true by the way the number is defined. This property hold for all numbers $p(M)$ associated to invariant polynomials $p : \mathfrak{so}(n) \to \mathbb{R}$. Here $\sqcup$ denotes disjoint union.

2. $\widehat{A}(M \times N) = \widehat{A}(M)\widehat{A}(N)$. This property holds essentially because the $\widehat{A}$-plynomial is defined as a *product* of functions in x_j. To see why this is so consider
$$T(M \times N) = p_1^* TM \oplus p_2^* TN$$
$$\begin{tikzcd} & M \times N \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ M & & N \end{tikzcd} \qquad R_{M \times N} = \begin{bmatrix} R_M & 0 \\ 0 & R_N \end{bmatrix}$$
Note that the Pfaffian $Pf(x) = x_1 x_2 \cdots = \Pi x_j$ is also a product of functions in x_j. Indeed, the Euler number satisifies properties (1) and (2), but not the following property (3).

3. $\widehat{A}(\partial M) = 0 \implies \widehat{A}$ is bordism invariant.

   Proof. Because $\widehat{A}(x)$ is a product of the form $\Pi f(x_j)$, we have $\widehat{A}(E \oplus F) = \widehat{A}(E)\widehat{A}(F)$. Moreover, for a trivial vector bundle $\widehat{A}(\text{ trivial bundle }) = 1$. Combining these two facts we see that $\widehat{A}$ is stable, i.e., adding a trivial bundle does not affect the $\widehat{A}$-class of a vector bundle. So, if N is the boundary $N = \partial M$ of a compact manifold M, then the tangent bundle TM restricted to N is TN plus a trivial line bundle (the normal bundle of the boundary N). Therefore $\widehat{A}(TN)$ is the restriction of $\widehat{A}(TM)$ to the boundary N. Then by Stokes's Theorem,
$$\int_N \widehat{A}(TN) = \int_{\partial M} \widehat{A}(TM) = \int_M d\widehat{A}(TM) = 0$$
   because $\widehat{A}(TN)$ is a closed form. □

Any topological invariant with properties (1), (2), (3) is called a *genus*. A priori, the $\widehat{A}$-genus is a rational number. There are examples of manifolds for which $\widehat{A}(M)$ is not an integer. But it was known, before the work of Atiyah and Singer, that the $\widehat{A}$-genus of a *spin* manifold is always an integer. The fact that the $\widehat{A}$-genus of a spin manifold can be identified with the index of the Dirac operator "explains" in some sense why it is an integer. Understanding the integrality of the $\widehat{A}$-genus of spin manifolds was Atiyah's original motivation for defining the Dirac operator.

6.2 Other versions of the Atiyah-Singer theorem

Recall: If M is a closed complex manifold (to be more precise, a Kähler manifold), then we obtain a Spin^c structure by

$$\mathrm{Spin}^c : \begin{cases} c : T_p^{1,0}M = T_pM \longrightarrow \mathrm{End}(\mathbb{S}_p), \text{ where } \mathbb{S} = \bigwedge^0 T^{1,0}M \\ c(v)\alpha = v \wedge \alpha - v \llcorner \alpha \qquad v \in T_p^{1,0}(M), \ \alpha \in \mathbb{S}_p \\ c(v) = -\|v\|^2 I \end{cases}$$

The Dirac operator for this Spin^c structure is $D = \overline{\partial} + \overline{\partial}^*$, where $\overline{\partial}$ is the Dolbeault operator, and $\overline{\partial}^*$ its adjoint. The Dolbeault operator is the $\frac{\partial}{\partial \bar{z}}$ version of the de Rham operator d. The index of D is the Euler characteristic $\chi(\overline{\partial})$ of the Dolbeault complex, which is called the *arithmentic genus* of M. This number plays an important role in algebraic geometry.

Theorem 6.2 (Todd, 1937; Hirzebruch ~ 1954)**.** *Considering* D *as above*

$$\mathrm{Index} D = \int_M \mathrm{Td}(T^{1,0}M),$$

where $\mathrm{Td}(T^{1,0}M)$ *is the Todd class of the complex vector bundle* $T^{1,0}M$*.*

The Todd class that appears in this formula is a characteristic class of $\mathbb{C}$ vector bundles, defined by Hirzebruch using the formalism of Chern classes. However, the Todd class can be defined for every Spin^c manifold M as

$$\mathrm{Td}^c(TM) = \widehat{A}(TM)\exp(c_1(L)/2)$$

Here L is the so-called Spin^c line bundle, a complex line bundle on M associated, in a natural way, to the Spin^c structure of M. Recall that complex line bundles on M are in one-to-one correspondence with elements in $H^2(M, \mathbb{Z})$. The class $c_1(L) \in H^2(M, \mathbb{Z})$ denotes the 2-cocycle associated to L (the first Chern class).

On a spin manifold, the line bundle L is trivial, and we have

$$\mathrm{Td}^c(TM) = \widehat{A}(TM)$$

On a complex manifold (of complex dimension r), the Spin^c line bundle L is just the determinant line bundle $L = \Lambda^r T^{1,0}M$, and it is an easy calculation to see that in this case

$$\mathrm{Td}^c(TM) = \mathrm{Td}(T^{1,0}M)$$

The two versions of the Atiyah-Singer formula discussed so far–the first for spin manifolds, the second for complex manifolds–can be generalized to,

Theorem 6.3. *If* M *is an even dimensional* Spin^c *manifold, then*

$$\mathrm{Index} D = \int_M \mathrm{Td}^c(TM).$$

Twisting A last generalization of the theorem that we need to consider involves the construction referred to as "twisting" the Dirac operator by a complex vector bundle.

M Spin^c $\rightsquigarrow$ D =Dirac operator, as above

$E \longrightarrow M$, a $\mathbb{C}$ vector bundle on M

The symbol of the Dirac operator D is the map

$$c : T_p M \longrightarrow \mathrm{End}(\mathbb{S}_p)$$

that is part of the Spin^c-structure. We twist c by E to obtain

$$c \otimes \mathrm{Id}_E : T_p M \longrightarrow \mathrm{End}(\mathbb{S}_p \otimes \mathbb{E}_p)$$

where $\mathrm{Id}_E : E_p \longrightarrow E_p$ is the identity. Then the twisted Dirac operator D_E is a first order elliptic differential operator with symbol $c \otimes \mathrm{Id}_E$,

$$D_E : C^\infty(\mathbb{S}^+ \otimes E) \to C^\infty(\mathbb{S}^- \otimes E)$$

With a suitable choice of lower order terms the twisted operator D_E (like D itself) is an essentially self-adjoint elliptic operator.

Theorem 6.4. *If M is an even dimensional* Spin^c *manifold and $E \longrightarrow M$ is a $\mathbb{C}$ vector bundle on M, then for the twisted Dirac operator*

$$\mathrm{Index} D_E = \int_M \mathrm{ch}(E) \wedge \mathrm{Td}^c(TM),$$

The expression $\mathrm{ch}(E)$ in the formula refers to the Chern character of E. It can be defined as

$$\mathrm{ch}(E) = \mathrm{trace}\left(\exp\left(-\frac{\Omega}{2\pi i}\right)\right) \in H^{\mathrm{even}}_{dR}(M)$$

where Ω is the curvature of a (random choice of) connection on E.

Remark. There is an even more general version of the Atiyah-singer formula which is valid for *all* elliptic operators. In this more general formula of the theorem there appears a Todd class. However, the Todd class in the most general formula is $\mathrm{Td}(TM \otimes \mathbb{C})$—i.e., the Todd class of the *complexified* tangent space. This is not to be confused with $\mathrm{Td}(T^{1,0}M$, which only makes sense if M is a complex manifold. To make matters more confusing, this class can also be written as $\mathrm{Td}(TM \otimes \mathbb{C}) = \widehat{A}(TM)^2$.

proof of above theorem. Strategy of topological proof. Reduce $(M, E) \rightsquigarrow (S^{2n}, F)$ and then apply Bott periodicity. □

The Bott Generator in K-theory

Theorem 6.5 (Bott, 1959). *For the even integers* $k = 0, 2, 4, \ldots, 2n-2$ *we have* $\pi_k(GL(n, \mathbb{C})) = 0$. *For the odd integers* $k = 1, 3, 5, \ldots, 2n-1$ *we have* $\pi_k(GL(n, \mathbb{C})) \cong \mathbb{Z}$.

Stated more simply, if we let $GL = \lim_{k\to\infty} GL(k, \mathbb{C})$ then $\pi_k(GL) = 0$ if k even, and $\pi_k(GL) \cong \mathbb{Z}$ if k odd. In this section we give various (equivalent) descriptions of the "Bott generator" of the cyclic group $\pi_k(GL)$ for odd k. This element plays a central role in index theory.

(6.4)

1. At the very beginning we defined the $\mathbb{R}$ linear map

$$c : \mathbb{R}^{2n} \longrightarrow \mathrm{End}(\mathbb{C}^{2n}) \qquad c(v) = \sum_{j=1}^{2n} v_j E_j$$

   for $v = (v_1, \ldots v_{2n}) \in \mathbb{R}^{2n}$. We had a split $\mathbb{C}^{2n} = W = W^+ \oplus W^-$ into positive and negative spinors, and $c(v)$ maps $W^+ \to W^-$

$$c : \mathbb{R}^{2n} \longrightarrow \mathrm{Hom}(W^+, W^-)$$

   From $c(v)^2 = -\|v\|^2 \cdot I$, we see that $c(v)$ is an isomorphism of vector spaces $W^+ \cong W^-$ as long as $v \neq 0$. Thus the pair of (trivial) vector bundles $\mathbb{R}^{2n} \times W^+$ and $\mathbb{R}^{2n} \times W^-$ together with the maps $c(v)$ defined at every point $v \in \mathbb{R}^{2n}$ defines a class in the K-theory with compact supports,

$$[c] = \left[c, \mathbb{R}^n \times W^+, \mathbb{R}^n \times W^-\right] \in K^0\left(\mathbb{R}^{2n}\right) \cong \mathbb{Z}$$

   This group is $\mathbb{Z}$ by Bott periodicity in K-theory. The class $[c]$ is a generator of this group $K^0(\mathbb{R}^{2n})$.

2. If $v \in \mathbb{R}^{2n}$ is a unit vector, then $c(v)^2 = -I$ implies that $c(v)$ is invertible. Thus, the map $c(v) : W^+ \to W^-$ it is an isomorphism (in fact, it is a unitary). If we identify $W^+ = W^- = \mathbb{C}^{2^{n-1}}$, we obtain a map

$$c : S^{2n-1} \to GL(2^{n-1}, \mathbb{C})$$

   Thus, c defines an element in the homotopy group

$$[c] \in \pi_{2n-1}(GL) \cong \mathbb{Z}$$

   This homotopy element $[c]$ is a generator of the group $\pi_{2n-1}(GL)$.

3. We have the isomorphism

$$K^0\left(\mathbb{R}^{2n}\right) \cong K^0\left(S^{2n}, \bullet\right)$$

Under this isomorphism, the Bott generator [c] of $K^0(\mathbb{R}^{2n})$ maps to a vector bundle β on the sphere,

$$\left[c, \mathbb{R}^{2n} \times W^+, \mathbb{R}^{2n} \times W^-\right] \longmapsto [\beta] - [S^{2n} \times \mathbb{C}^{2^{n-1}}]$$

The vector bundle β on S^{2n} is obtained as follows. On the upper hemisphere S^{2n}_+ we have the trivial vector bundle $S^{2n}_+ \times W^+$, and on the lower hemisphere we have $S^{2n}_- \times W^-$. The equator can be identified with the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$. Now use the map

$$c : S^{2n-1} \to \mathrm{Iso}(W^+, W^-)$$

to clutch the trivial bundle with fiber W^+ on the upper hemisphere to the trivial bundle with fiber W^- on the lower hemisphere. The resulting bundle is the *Bott generator vector bundle*. We denote it by β.

4. The chern character gives an isomorphism

$$K^0(S^{2n}, \bullet) \cong H^{2n}(S^{2n}, \mathbb{Z}) \cong \mathbb{Z}$$

by

$$\beta \longmapsto \mathrm{ch}(\beta) \to \int_{S^{2n}} \mathrm{ch}(\beta) \in \mathbb{Z}$$

Exercise: With β as defined above, $\int \mathrm{ch}(\beta) = -1$.

7 13 September 2016, Nigel Higson

We're going to continue our examination of zeta functions

$$s \longmapsto \mathrm{Trace}(T\Delta^{-s}) \tag{7.1}$$

from the last lecture. Our aim is to show that the residues of these zeta functions can be calculated, at least in principle, by purely local means, from the coefficients of T and Δ and their partial derivatives. And we'll show in a bit more detail what this has to do with index theory.

7.1 The Index as a Zeta Value st Zero

Let M be a closed manifold of dimension n. Let T be a differential operator of order q. Let

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

be a (first order, for simplicity) formally self adjoint elliptic partial operator on M and let $\Delta = I + D^2$. Finally, let

$$\gamma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

be the grading operator. We noted in an earlier lecture that

$$\operatorname{Index}(D_+) \equiv \operatorname{Trace}(\gamma \Delta^{-s})$$

(that is, the zeta function is in fact constant, and is equal to the index everywhere). Atiyah and Bott suggested that this observation might open a route toward the index theorem because they knew from work of Seeley and others that the value of any zeta function at $s = 0$ can in principle be computed from the coefficients the operators involved. In this section we shall begin to explain this fact by proving that every zeta function

$$s \longmapsto \operatorname{Trace}(T\Delta^{-s})$$

is regular at zero (there is no pole).

Definition 7.1. The *Residue Trace* on the algebra of differential operators is the functional

$$\operatorname{ResTr}(T) := \operatorname{Res}_{s=0}(\operatorname{Trace}(T\Delta^{-s})).$$

Remark 7.2. The same formula can be defined for a much wider class of operators than the differential operators, namely the pseudodifferential operators, where it is known as the *noncommutative residue*. This is a very interesting quantity; in contrast, our aim here is to show that the residue trace is zero on the algebra of differential operators.

Lemma 7.3. *The residue trace is a trace on the algebra of differential operators*

Proof. Let S and T be differential operators. We calculate, using the trace property of the ordinary trace, that

$$\begin{aligned} \operatorname{ResTr}(ST) &= \operatorname{Res}|_{s=0}(\operatorname{Trace}(ST\Delta^{-s})) \\ &= \operatorname{Res}|_{s=0}(\operatorname{Trace}(T\Delta^{-s}S)) \\ &= \operatorname{Res}|_{s=0}(\operatorname{Trace}(TS\Delta^{-s})) + \operatorname{Res}|_{s=0}(\operatorname{Trace}(T[\Delta^{-s}, S]). \end{aligned}$$

But

$$\begin{aligned} T[\Delta^{-s}, S] &= \pm\binom{s}{1} T[\Delta, S]\Delta^{-s-1} \pm \binom{s}{2} T[\Delta[\Delta, S]]\Delta^{-s-2} \pm \cdots \\ &= s \times R_s, \end{aligned}$$

where R_s is a combination of the families that we have been considering all along (whose traces are meromorpohicm functions on $\mathbb{C}$ with only simple poles). Thanks to the factor of s we find that

$$\operatorname{Res}|_{s=0}(\operatorname{Trace}(T[\Delta^{-s}, S]) = 0,$$

and so the residue trace is indeed a trace, as required. $\square$

Corollary 7.4. *If* T *is any differential operator, then the meromorphic function*

$$s \longmapsto \mathrm{Trace}(T\Delta^{-s})$$

is regular at $s = 0$.

Proof. We have already shown that there are no nonzero traces on the algebra of differential operators. □

We find that for any T the zeta function $\mathrm{Trace}(T\Delta^{-s})$ is regular at $s = 0$, and so we can study its value there. In fact the value is computable as a residue, as we shall show next.

7.2 Zeta Values at Zero as Residues

In this section we shall realize the zeta value

$$\mathrm{Trace}(T\Delta^{-s})|_{s=0}$$

(which we've just seen is extremely interesting from the point of view on index theory) as a residue of a zeta function.

We'll consider the simple case where $T = \gamma$, since this is all we actually need. If we examine again the proof of the M-S theorem, then we see that in the identity

$$(n-2s)\gamma\Delta^{-s} = \sum [B_i\gamma\Delta^{-s}, A_i] + R_s$$

the remainder R_s, which is

$$-B_i[\gamma\Delta^{-s}, A_i] - 2s\Delta^{-s} = -B_i\gamma[\Delta^{-s}, A_i] - 2s\Delta^{-s}$$

is s times a combination F_s of families of the type $T\Delta^{-s-k}$ (which we could write down explicitly). So

$$(n-2s)\,\mathrm{Trace}(\gamma\Delta^{-s}) = s\,\mathrm{Trace}(F_s)$$

As a result

$$n \cdot \mathrm{Trace}(\gamma\Delta^{-s})|_{s=0} = (s\,\mathrm{Trace}(F_s))\,|_{s=0} = \mathrm{Res}\,|_{s=0}\,\mathrm{Trace}(F_s),$$

as required.

7.3 Zeta Values and Residues as Distributions

Recall that if T is a differential operator of order q on a closed manifold M, and if Δ is a postive, invertible, elliptic operator of order 2 with scalar symbol, then the zeta function (7.1) is defined as an ordinary trace for $\mathrm{Re}(s) > (n+q)/2$, and is a holomorphic function there. And, by the Minakshisundaram-Pleijel

Theorem (5.2), the zeta function extends to a meromorphic function on $\mathbb{C}$ with only simple poles, at

$$\frac{n+q}{2}, \frac{n+q-1}{2}, \frac{n+q-2}{2}, \dots$$

(of course, some of these singularites might be removeable, and indeed we shall see precisely this in a little while).

It will be convenient to enlarge our analytical perspective a bit. We shall no longer require M to be compact. Instead, let $U \subseteq M$ and let D be a formally self-adjoint, first-order differential operator on M that is elliptic over U. Assume that D is essentially self-adjoint (for instance, this will be so if D is compactly supported). Let $\Delta = I + D^2$. Let f be a smooth function on M whose support is a compact subset of U. If T is a differential operator on M of order q, then the zeta function

$$s \longmapsto \operatorname{Trace}(f \cdot T\Delta^{-s}), \tag{7.2}$$

is defined and analytic on $\operatorname{Real}(s) > (n+q)/2$. This uses the facts about the maximal and minimal domains of elliptic operators on nocompact manifolds that we mentioned in an earlier lecture.

Think of the zeta function (7.2) is associating to each complex scalar s a distribution on U:

$$f \longmapsto \operatorname{Trace}(fT\Delta^{-s}).$$

The argument from last Friday shows that this is a meromorphic function on $\mathbb{C}$ with values in distributions on U. The following argument shows that the residue distributions

$$f \longmapsto \operatorname{Res}|_{s=s_0} \operatorname{Trace}(fT\Delta^{-s}) \tag{7.3}$$

depend only on the restriction of T and D to U:

Lemma 7.5. *If T_1, D_1 are equal to T, D, respectively on U, then the difference of zeta functions*

$$\operatorname{Trace}(f \cdot T_1\Delta_1^{-s}) - \operatorname{Trace}(f \cdot T\Delta^{-s})$$

extends to an entire function.

Proof. Recall

$$\Delta^{-s} = \frac{1}{2\pi i}\int_L \lambda^{-s}(\lambda - \Delta)^{-1}\, d\lambda,$$

where L is a downwards oriented vertical line in the plane separating 0 from the spectrum of Δ. Choose L so that it separates 0 fromthe spectrum of Δ_1, too. Then

$$\begin{aligned} f \cdot \Delta^{-s} - f \cdot \Delta_1^{-s} &= \frac{1}{2\pi i}\int_L \lambda^{-s} f \cdot \left((\lambda - \Delta)^{-1} - (\lambda - \Delta_1)^{-1}\right) d\lambda \\ &= \frac{1}{2\pi i}\int_L \lambda^{-s} f \cdot (\lambda - \Delta_1)^{-1}(\Delta - \Delta_1)(\lambda - \Delta)^{-1}\, d\lambda, \end{aligned}$$

where $(\Delta - \Delta_1)$ is supported away from $\mathrm{supp}(f)$.

Choose a smooth function h so that $h \equiv 1$ whereever $D - D_1$ is nonzero, yet $f \cdot h = 0$. Then

$$f \cdot \Delta^{-s} - f \cdot \Delta_1^{-s} = \frac{1}{2\pi i} \int_L \lambda^{-s} f \cdot (\lambda - \Delta_1)^{-1} \cdot h \cdot (\Delta - \Delta_1)(\lambda - \Delta)^{-1} \, d\lambda, \tag{7.4}$$

and now we calculate that

$$\begin{aligned} f \cdot (\lambda - \Delta_1)^{-1} \cdot h &= f \cdot [(\lambda - \Delta_1)^{-1}, h] \\ &= f(\lambda - \Delta_1)^{-1} [\Delta_1, h] (\lambda - \Delta_1)^{-1} \\ &= f(\lambda - \Delta_1)^{-1} [\Delta_1, [\Delta_1, h]] (\lambda - \Delta_1)^{-2} \\ &\vdots \\ &= f(\lambda - \Delta_1)^{-1} \cdot \underbrace{h^{(k)}}_{\text{order } k} \cdot \underbrace{(\lambda - \Delta_1)^{-k}}_{\text{order } -2k}, \end{aligned}$$

where $h^{(k)} = [\Delta_1, h^{(k-1)}]$ (more or less as before). We find that for any s the integrand in (7.4) is trace class with rapidly decaying trace norm as a function of λ. $\square$

Remark 7.6. We can say a little bit more: not only is the difference of traces an entire function, but before traces, the quantity

$$f T \Delta^{-s} - f T_1 \Delta_1^{-s}$$

is an entire function with values in the trace class operators, and indeed, if we multiply by a compactly supported function on the right, with values in those trace class operators represented by smooth kernels $k(x, y)$.

We are going to study the residue distributions. We shall see that on a closed manifold the index of an elliptic (first order) operator is representable as an (in principle) explicit combination of residue zeta functions. We shall also see that each residue distribution is in fact a smooth function on M (so that the index is in principle computable as a combination of integrals over M of these smooth functions). Finally we shall see that each of these smooth functions is (in principle) computable as a function of the coefficients of T and D and their derivatives. This gives an (in principle) explicit local solution of the index problem.

7.4 Residue Trace Distributions as Smooth Functions

In this section we shall prove that the residue distribution (7.3) is (integration against) a smooth function.

To do this, let's return one more time to the argument that proved the M-S theorem. The argument in the previous section shows that it suffices to prove the M-S theorem on $\mathbb{R}^n$. In this context we can set

$$A_i = x_i \quad \text{and} \quad B_i = \frac{\partial}{\partial x_i},$$

which have the property that $[B_i, A_i] = I$, of course, and for any differential operator, T, of order q,

$$(n+q)T = \sum [B_i T, A_i] + R,$$

where R is differential operator of order less than q. Recall that we proved the M-S theorem by computing the same for families of operators with asymptotic expansions

$$T_1 \Delta^{-z-k_1} + T_2 \Delta^{-z-k_2} + \cdots$$

in place of T and R. The terms here should be of decreasing order $q_i - 2(z+k_i)$.

Sobolev theory tells us that for any $r = 1, 2, 3, \ldots$, if $\mathrm{Real}(z) > r + (n+q)/2$, then $fT\Delta^{-z}$ is represented by a kernel $k_z(x, y)$ which is r-times continuously differentiable in x and y.

Now, if the kernel of $fT\Delta^{-z}$ is $k_z(x, y)$, as above, then the kernel of the operator $\sum [B_i fT\Delta^{-z}, A_i]$ is

$$(\partial_i k_z(x, y))(x_i - y_i)$$

where the partial derivative operator ∂_i is applied to the x-slot. *This kernel vanishes on the diagonal* $x = y$. So in the formula

$$(n+q-2z) fT\Delta^{-z} = \sum [B_i fT\Delta^{-z}, A_i] + R_z,$$

the kernels of $fT\Delta^{-z}$, times $(n+q-2z)$, is exactly equal to the kernel of R_z on the diagonal. We find that the restriction of the kernel for $fT\Delta^{-z}$ to the diagonal is not only a C^r-function but a C^∞-function (away from poles). Hence:

Theorem 7.7. *There is a meromorphic function on* $\mathbb{C}$ *with values in* $C^\infty(M)$*,*

$$z \longmapsto k_z(x, x),$$

such that

$$\mathrm{Trace}(fT\Delta^{-z}) = \int_M k_z(x, x)\, dx$$

away from poles and

$$\mathrm{Res}\,|_{s=s_0} \mathrm{Trace}(fT\Delta^{-z}) = \int_M \mathrm{Res}\,|_{s=s_0} k_z(x, x)\, dx$$

at poles.

In what follows we'll write this function as

$$k_z(x,x) = \mathrm{trace}_x(fT\Delta^{-z})$$

to indicate as clearly as possible the relation between the kernel and the operator.

In the reverse direction we'll write

$$\mathrm{Op}(k)\colon f \longmapsto \Big[x \mapsto \int k(x,y)f(y)\,dy\Big]$$

for the integral operator associated to a kernel k.

Remark 7.8. It follows easily from the explicit formulas that the local residues $\mathrm{Res}|_{s=s_0} k_z(x,x)$ (which are our main interest) are *continuous* functions of the coefficients of the operators T and Δ near x. We shall use this in the final section of this lecture.

7.5 Computation of Residues

Our last topic is the computation of the residues

$$\mathrm{Res}|_{s=s_0} \mathrm{Trace}(fT\Delta^{-s}).$$

As for where the residues are, we have seen that the poles of our zeta function are at

$$s_0 = \frac{n+q}{2}, \frac{n+q-1}{2}, \frac{n+q-2}{2} \dots$$

We'll call $\frac{n+q}{2}$ the *leading residue* and we shall indicate how to compute it in the next section. The purpose of this section is to point out that every other residue of our zeta function can be identified with an explicit (but complicated) combination of leading residues of different zeta functions.

It follows that the *value* of our zeta function at can also be identified with an explicit but complicated combination of leading residues of other zeta functions, since we have already seen how to identify the value at $s=0$ with an explicit but complicated combination of (non-leading) residues.

Let's consider the problem of computing the next-to-leading residue of our zeta function, at

$$s_0 = \frac{n+q-1}{2}.$$

Let's invoke our Minakshisundaram-Pleijel formula

$$(n+q-2s)fT\Delta^{-s} = \sum [B_i fT\Delta^{-s}, A_i] + R_s,$$

one last time. The remainder R_s is given by the formula

$$R_s = (q-2s)fT\Delta^{-s} - \sum B_i[fT\Delta^{-s}, A_i] \tag{7.5}$$

and

$$B_i[fT\Delta^{-s}, A_i] = B_i[fT, A_i]\Delta^{-s} + B_i fT[\Delta^{-s}, A_i],$$

or, better,

$$B_i[fT\Delta^{-s}, A_i] = B_i[fT, A_i]\Delta^{-s} + [B_i, fT][\Delta^{-s}, A_i] + fTB_i[\Delta^{-s}, A_i] \tag{7.6}$$

Let's analyze the terms on the right-hand side.

(i) The first term on the right of (7.6), summed over all i, yields

$$qfT\Delta^{-s} + S\Delta^{-s},$$

where S is a differential operator of order less than q.

(ii) For each value of i the second term on the right of (7.6) has the form

$$T_i[\Delta^{-s}, A_i]$$

where T_i is a differential operator of order no more than q. And the commutator can be expanded as

$$[\Delta^{-s}, A_i] = \binom{-s}{1}[\Delta, A_i]\Delta^{-s-1} + \binom{-s-1}{1}[\Delta, [\Delta, A_i]]\Delta^{-s-2} + \cdots$$

The first term on the right has order $-2s-1$; the next has order $-2s-2$, and so on. So overall, the second term on the right of (7.6) has an expansion as a sum of terms

$$X_k\Delta^{-s-k}$$

for $k = 1, 2, \ldots$, with X_k a differential operator, and with order $q-2s-k$. There are infinitely many terms, but only finitely many will contribute to the residue that we are interested in.

(iii) The third term is

$$\begin{aligned} &fTB_i[\Delta^{-s}, A_i] \\ &= -sfTB_i[\Delta, A_i]\Delta^{-s-1} + \binom{-s-1}{2} fTB_i[\Delta, [\Delta, A_i]]\Delta^{-s-2} + \cdots \end{aligned} \tag{7.7}$$

Summing over i we get

$$-sfT\sum B_i[\Delta, A_i]\Delta^{-s-1} = -2s\Delta\Delta^{-s-1} + Y\Delta^{-s-1}$$

where Y has order 1 or less. The remaining terms in (7.7) are of the form $X_k\Delta^{-s-k}$ as in item (ii) above.

Putting all of this mess together (and it is a mess, although we can in principle write down all the terms involved with great precision), and returning to (7.5), we find that there is an "asymptotic expansion"

$$\mathrm{Trace}(T\Delta^{-s}) \sim \sum_{k\geq 1} \mathrm{Trace}(Z_k \Delta^{-s-k}),$$

where the order of Z_k is no more than $q+k$, and where the meaning of the term "asymptotic expansion" is that for any given right-half plane Real $s > -N$ in $\mathbb{C}$ the difference between the left-hand side and the sum of any sufficently large (but finite) number of terms from the right is holomorphic in that half-plane (so the other terms will not contribute to residues in our half-plane). We get

$$\mathrm{Res}_{s=(n+q-1)/2}(\mathrm{Trace}(T\Delta^{-s})) = \sum_{k\geq 1} \mathrm{Res}_{s=(n+q-1)/2}(\mathrm{Trace}(Z_k \Delta^{-s-k}))$$

and the sum is actually finite. Moreover, *all the residues on the right-hand side are leading residues* or zero.

If we want to consider the next-to-next-to-leading order residue, then the above argument identifies it with a combination of next-to-leading order residues, which we can then reduce to a combination of leading order residues. And so on.

7.6 Calculation of the Leading Residue

In this final section we shall explain how to calculate the leading residue.[3] From everything that went before, we can write

$$\mathrm{Res}\,|_{s=(n+q)/2}\,\mathrm{Trace}(fT\Delta^{-s}) = \int_M \mathrm{Res}\,|_{s=s_0}\,\mathrm{trace}_x(fT\Delta^{-s})\,dx.$$

We shall show how to compute integrand

$$\mathrm{Res}\,|_{s=(n+q)/2}\,\mathrm{trace}_x(fT\Delta^{-s}).$$

Since we are computing at a single point we can assume that

$$M = \mathbb{R}^n,$$

that $x = 0$. We are going to use a rescaling method that we shall see again in the next lecture.

Define operators

$$U_\varepsilon : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

for $\varepsilon > 0$ by

$$(U_\varepsilon f)(x) = \varepsilon^{n/2} f(\varepsilon x).$$

[3] Actually, the final part of the computation is given at the beginning of Lecture 9.

The scalar factor $\varepsilon^{n/2}$ is not very important, but it makes U_ε into a unitary operator.

Given a differential operator

$$T = \sum_{|\alpha| \leqslant q} a_\alpha(x) \partial^\alpha,$$

we compute that

$$U_\varepsilon T U_\varepsilon^* = \varepsilon^{-q} \sum_{|\alpha| \leqslant q} a_\alpha(\varepsilon x) \varepsilon^{q-|\alpha|} \partial^\alpha.$$

Let's write this as

$$U_\varepsilon T U_\varepsilon^* = \varepsilon^{-q} T_\varepsilon,$$

and note that the operator

$$T_\epsilon = \sum_{|\alpha| \leqslant q} a_\alpha(\varepsilon x) \varepsilon^{q-|\alpha|} \partial^\alpha$$

can be defined for negative ε and for $\varepsilon = 0$, where we obtain a constant coefficient operator of homogeneous degree q (in effect, it is the symbol of T at $x = 0$), and that the coefficients of T_ε vary smoothly with $\varepsilon \in \mathbb{R}$.

Suppose now that $k(x, y)$ is a kernel function, and that $\mathrm{Op}(k)$ is the associated integral operator. Then

$$U_\varepsilon \mathrm{Op}(k) U_\varepsilon^* = \varepsilon^n \mathrm{Op}(k_\varepsilon),$$

where $k_\varepsilon(x, y) = k(\varepsilon x, \varepsilon y)$ (the factor ε^n comes from the change of variables formula

$$\varepsilon^n \int h(\varepsilon x) dx = \int h(x) dx$$

for the integral on $\mathbb{R}^n$). As a result,

$$\mathrm{trace}_0\big(U_\varepsilon \mathrm{Op}(k) U_\varepsilon^*\big) = \varepsilon^n k(0,0) = \varepsilon^n \mathrm{trace}_0\big(\mathrm{Op}(k)\big).$$

Putting these things together, we find that

$$\begin{aligned} \mathrm{trace}_0(T_\varepsilon \Delta_\varepsilon^{-s}) &= \varepsilon^{q-2s} \mathrm{trace}_0\big(U_\varepsilon T \Delta^{-s} U_\varepsilon^*\big) \\ &= \varepsilon^{n+q-2s} \mathrm{trace}_0(T\Delta^{-s}) \end{aligned}$$

and therefore

$$\mathrm{Res}_{n+q-2s=0}(\mathrm{trace}_0(T_\varepsilon \Delta_\varepsilon^{-s})) = \mathrm{Res}_{n+q-2s=0}(\mathrm{trace}_0(T\Delta^{-s}))$$

for all $\varepsilon > 0$ since the function ε^{n+q-2s} is entire and nowhere vanishing.

We would now like to take a limit as $\varepsilon \to 0$ and conclude that

$$\mathrm{Res}_{n+q-2s=0}(\mathrm{trace}_0(T_0 \Delta_0^{-s})) = \mathrm{Res}_{n+q-2s=0}(\mathrm{trace}_0(T\Delta^{-s}))$$

but this doesn't quite make sense because Δ_0 is not invertible. Instead we first note that

$$T_\varepsilon \Delta_\varepsilon^{-s} - T_\varepsilon (I + \Delta_\varepsilon)^{-s} = \text{trace-class operator}$$

for $n + q - 2s \geq 0$ and all $\varepsilon > 0$, and so

$$\mathrm{Res}_{n+q-2s=0}\big(\mathrm{trace}_0(T_\varepsilon \Delta_\varepsilon^{-s})\big) = \mathrm{Res}_{n+q-2s=0}\big(\mathrm{trace}_0(T_\varepsilon (I + \Delta_\varepsilon)^{-s})\big)$$

Now we can use the continuity of the residue as a function of the coefficients of T and Δ and conclude that

$$\mathrm{Res}_{n+q-2s=0}\big(\mathrm{trace}_0(T\Delta^{-s})\big) = \mathrm{Res}_{n+q-2s=0}\big(\mathrm{trace}_0(T_0(I + \Delta_0)^{-s})\big)$$

But the residue on the right, which involves only constant coefficient operators on $\mathbb{R}^n$, is easily computed explicitly using the Fourier transform. More on this in Lecture 9.

8 14 September 2016, Erik van Erp

Recalling the setting of Theorem (6.4). Let M be an even dimensional Spin^c manifold and $E \longrightarrow M$ complex vector bundle.

$$\mathrm{Index} D_E = \int_M \mathrm{ch}(E)\mathrm{Td}^c(TM). \tag{8.1}$$

Here D_E was the twisted Dirac operator. We note that this formula is exactly correct, i.e., there are not hidden constants or signs. This is due to the fact that we have defined the characteritic classes in the right hand side terms with the appropriate factors of 2π or $2\pi i$.

Also, recall that Td^c denotes the Todd class of a Spin^c vectr bundle, which is defined in such a way that

$$\mathrm{Td}^c(TM) = \begin{cases} \widehat{A}(TM) & M \text{ spin} \\ \mathrm{Td}(T^{1,0}M) & M \text{ complex} \end{cases}$$

Proof. Idea: To reduce via two moves to a sphere, then use Bott periodicity.
Step 1: Prove the above formula if $M = S^{2n}$ with its standard spin structure.

Bott periodicity [Theorem of Raoul Bott, 1959]

$$\pi_k(GL(n,\mathbb{C})) = \begin{cases} 0 & k \text{ even} \\ \mathbb{Z} & k \text{ odd} \end{cases}$$

if $k = 1, 2, \ldots, 2n-1$.

This theorem tells us exactly what the vector bundles on S^{2n} are, up to stable isomorphism. Every vector bundle on S^{2n}, when restricted to the upper or lower hemisphere, can be trivialized. Thus, the isomorphism class of the vector bundle is determined by the transition function, which is a map $S^{2n-1} \to \pi_k(GL(k,\mathbb{C}))$. The equator of S^{2n} is S^{2n-1}, and k is the rank of the vector bundle. The isomorphism class of the vector bundle only depends on the homotopy type of the transition function, which is an element in $\pi_{2n-1}(GL(k,\mathbb{C}))$. If we only want to classify vector bundle up to stable isomorphism, we may assume that k is large enough so that Bott's theorem applies. Therefore, the reduced K-theory of S^{2n} is $K^0(S^{2n},\bullet) = \mathbb{Z}\beta$. The non-reduced K-theory also includes trivial vector bundles. So,

$$K^0(S^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z}\beta$$

Here β is the Bott generator vector bundle on S^{2n}.

Remark. Rcall the Clifford map c and that we viewed $\mathbb{C}^{2^n}$ as $W = W^+ \oplus W^-$. We then have

$$\begin{tikzcd} \mathbb{R}^{2n} \arrow[rr, "c"] & & \mathrm{End}(W^+, W^-) \\ S^{2n-1} \arrow[u, "\subseteq"] \arrow[r] & GL(2^{n-1},\mathbb{C}) \cong \mathrm{Iso}(W^+, W^-) & \end{tikzcd}$$

This defines the generator of $\pi_{2n-1}(GL(2^{n-1},\mathbb{C}))$.

Lemma 8.1. $\mathrm{Index} D_e = \int_{S^{2n}} \mathrm{ch}(E)$ *on* S^{2n}.

Proof. First establish that

- $\mathrm{Index} D_E = 0$ (which will be proven later)
- $\mathrm{Index} D_\beta = -1 = \int_{S^{2n}} \mathrm{ch}(\beta)$.

The first item follows from bordism invariance of the index (to be discusses later), and the fact that the spin structure on S^{2n} is the boundary of the spin structure on the unit ball in R^{2n+1}. The second equality is obtained by direct calculation.

Now Bott periodicity, and the knowledge it gives us of the group $K^0(S^{2n})$, implies that the lemma holds for all vector bundle E on S^{2n}. ☐

Remark 8.2. The right hand side of the above equation is indeed the Atiyah-Singer formula, because $\widehat{A}(TS^{2n}) = 1$. This is why the $\widehat{A}$-class does not appear in the formula. Indeed,

$TS^{2n} \oplus \underline{\mathbb{R}} = \underline{\mathbb{R}}^{2n+1}$, where $\underline{\mathbb{R}}$ denotes the trivial bundle with fiber $\mathbb{R}$. Now, recalling the genus Properties (6.3), we have

$$\widehat{A}(TS^{2n})\widehat{A}(\underline{\mathbb{R}}) = \widehat{A}(\underline{\mathbb{R}}^{2n+1}),$$

and $\widehat{A}(\underline{\mathbb{R}}) = \widehat{A}(\underline{\mathbb{R}}^{2n+1}) = 1$ implies that $\widehat{A}(TS^{2n}) = 1$.

We now consider pairs (M, E), where M is a closed Spin^c manifold and E is a complex vector bundle. To every such pair we assign two numbers.

$$\begin{array}{c} (M, E) \\ \big\downarrow \\ \text{Index} D_E \in \mathbb{Z} \text{ (Analytic index)} \end{array}$$

$$\begin{array}{c} (M, E) \\ \big\downarrow \\ \int_M \text{ch}(E)\text{Td}^c(TM) \in \mathbb{Q} \text{ (Topological index)} \end{array}$$

We must prove that for every pair (M, E) these two numbers—the analytic and topological index—are equal. To achieve this we will modify the pair in 2 moves

$$(M, E) \longrightarrow (\cdot, \cdot) \longrightarrow (S^{2n}, F)$$

and we will show that the both the analytic index and the topologically defined number are invariant under each of the two moves. We have already proved (above) that the two numbers are equal for the pair (S^{2n}, F). So it will follow that the two numbers must be equal for the pair (M, E). This is how we will prove the index formula.

<u>The first move</u> we will cover is the move $(\cdot, \cdot) \longrightarrow (S^{2n}, F)$. This involves the notion of bordism.

Definition 8.3. Two closed manifolds are *bordant* if $\exists$ compact W such that the boundary of W is the disjoint union of M with N.

$$\partial W = M \sqcup N.$$

<u>Oriented bordism:</u> M, N, W are oriented then

$$\partial W = M \sqcup (-N),$$

where $(-N)$ denotes the manifold N with its orientation reversed. □

<u>Spin^c bordism:</u> Let M, N, W be Spin^c, in which oriented is implicit by Spin^c. Consider $\dim(W) = 2r + 1$ and $\dim(\mathbb{S}_p) = 2^r$. Note that $\dim(\partial W) = 2r$.

W is Spin^c : $\mathbb{S} \longrightarrow W$ and $c : T_pW \longrightarrow \text{End}(\mathbb{S}_p)$. Restrict both the spinor bundle $\mathbb{S}$ and the map c to the boundary to get the Spin^c structure on the boundary ∂W.

We will say that a pair (M, E_0) is *bordant* to a pair (N, E_1) if M, N are Spin^c bordant by W, and $\exists E \longrightarrow W$ such that $E|_M \cong E_0$ and $E|_N \cong N$.

To see that the analytic index is invariant under this move is hard. But to see that the topological index is invariant under this move is an easy application of Stokes's theorem $\int_{\partial W} = \int_W d$.

Next, we cover the first move We first cover the notion of a Thom Isomorphism from K-theory. We recall the following Setting (6.4).

We had a split $\mathbb{C}^{2n} = W = W^+ \oplus W^-$ into positive and negative spinors, and $c(v)$ maps $W^+ \to W^-$

$$c : \mathbb{R}^{2n} \longrightarrow \mathrm{Hom}(W^+, W^-)$$

and we had

$$[c] = \left[c, \mathbb{R}^n \times W^+, \mathbb{R}^n \times W^-\right] \in K^0\left(\mathbb{R}^{2n}\right) \cong \mathbb{Z}$$

Recall that $K^0(\mathbb{R}^{2n}) \xrightarrow{\cong} K^0(S^{2n}, \bullet) = \mathbb{Z}$ by Bott periodicity in K-theory. The class $[c]$ is a generator of this group $K^0(\mathbb{R}^{2n})$.

Vector bundle version: Consider $F \xrightarrow{\pi} M \rightsquigarrow \mathbb{R}$ vector bundle with fibers $F_p \approx \mathbb{R}^{2n}$.

A Thom class for F is a class in $K^0(F)$ that is a "Bott generator in each fiber".

Explicitly, this means that we have two complex vector bundles $E^0, E^1 \longrightarrow M$. If $\pi : F \to M$ denotest the projection onto the base point, let π^*E^0, π^*E^1 be the pull back of the bundles E^j to the total space of F. A Thom class for F consists of two such bundles together with a vector bundle map τ,

$$\begin{tikzcd} \pi^*E^0 \arrow[rr, "\tau"] \arrow[dr] & & \pi^*E^1 \arrow[dl] \\ & F & \end{tikzcd}$$

which restricted to each fiber $F_p \approx \mathbb{R}^{2n}$ is a Bott element. This means that the lmap

$$\tau_p : F_p \to \mathrm{Hom}(E^0_p, E^1_p)$$

defined by

$$\tau_p(v) = \tau(p, v) : E^0 \longrightarrow E^1 \qquad v \in F_p, p \in M$$

is isomorphic to our familiar map

$$c : \mathbb{R}^{2n} \to \mathrm{Hom}(W^+, W^-)$$

Notet that $[\tau, \pi^*E^0, \pi^*E^1]$ is an element in the compactly supported K-theory $K^0(F)$.

Theorem 8.4 (Thom isomorphism in K-theory)**.** *If $\exists$ Thom class for F, then $K^0(F) \cong K^0(M)$*

The Thom isomorphism generalizes the Bott isomorphism $K^0(\mathbb{R}^{2n}) \cong \mathbb{Z}$ to vector bundles.

We now discuss how the Thom isomorphism is used as one of the two moves that gets us from an arbitrary pair (M, E) to (S^{2n}, F).

Given: $F \xrightarrow{\pi} M$ a vector bundle F on M with a Thom class. The fiber of this vector bundle $F_p \approx \mathbb{R}^{2n}$. If we compactify each fiber we get $F_p^+ \approx S^{2n}$. The resulting space is denoted ΣF, and it is a fiber bundle over M whose fibers are spheres. Alternatively, we may identify ΣF with

$$\underbrace{B(F) \bigsqcup_{S(F)} B(F)}_{\Sigma F \longrightarrow M}$$

in which $\Sigma F \longrightarrow M$ is fiber bundle with fibers $\approx S^{2n}$ and $B(F)$ denotes the ball bundle of F and $S(F)$ is the sphere bundle.

The Thom class τ_F of F corresponds to a vector bundle $\beta_F \longrightarrow \Sigma F$. Recall how the Bott vector bundle $\beta \to S^{2n}$ was obtained by clutching the Bott element of $K^0(\mathbb{R}^{2n})$ to get an element in $K^0(S^{2n}, \bullet)$. The same construction, performed in each fiber, turns the Thom class τ_F of F into the vector bundle β_F on the bundle of spheres ΣF. Put differently, $\beta_F \to \Sigma F$ is a vector bundle that, when restricted to a fiber $(\Sigma F)_p \approx S^{2n}$, is isomorphic to the Bott vector bundle $\beta \to S^{2n}$.

We then have

$$K^0(\Sigma F) = K^0(M) \oplus K^0(M)\beta_F$$

where the first copy of $K^0(M)$ on the RHS corresponds to vector bundles on ΣF obtained by pulling back vector bundles from M to ΣF, while the second copy is obtained by tensoring such vector bundles from M by β_F. This generalizes $K^0(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z}\beta$. It is the "compactified" version of the Thom isomorphism in K-theory.

Remark. A real vector bundle with a Thom class is called a (Spin^c vector bundle). We have discussed here only the case of vector bundles of even rank, but one can also define the Thom isomorphism for real vector bundles of odd rank. Note that M is a spin^c manifold precisely if the tangent space TM is a spin^c vector bundle, i.e., if it has a Thom class.

"Move 2": Compactified Thom isomorphism.

Start with a pair (M, E) of a closed spin^c manifold M with a complex vector bundle $E \to M$. Given a real vector bundle $F \longrightarrow M$ with even dimensional fibers and a Thom class τ_F. We may then replace the pair (M, E) with the pair $(\Sigma F, \beta_F \otimes \pi^* E)$. It has to be shown that both the analytic index and the topological index are invariant under this move.

Now that we know what the two moves are, how to get from M to S^{2n}?

$$(M, E) \xrightarrow[\text{Thom Iso}]{} (\cdot, \cdot) \xrightarrow[\text{bordism}]{} (S^{2n}, F)$$

This is done as follows. Embed M in $\mathbb{R}^{2N}$, for large enough N. The normal bundle ν of M in $\mathbb{R}^{2N}$ is the quotient bundle of $M \times \mathbb{R}^{2N}$ by TM. M is Spin^c means that there is a Thom class for TM. There is a Thom class for $M \times \mathbb{R}^{2N}$, because this is a trivial bundle and we can just take the Bott element of $\mathbb{R}^{2N}$ and place it in each fiber. Then by the "2-out-of-3" principle for Thom classes, there is also a Thom class for the normal bundle ν.

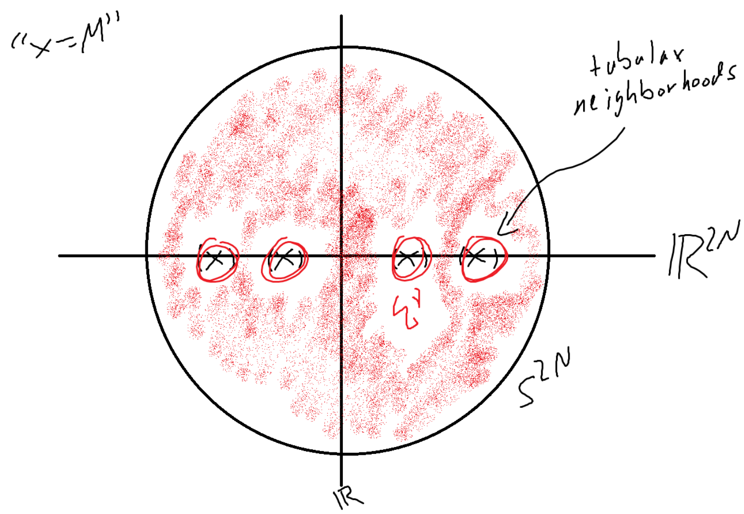
$$\begin{tikzcd}
0 \arrow[r] & TM \arrow[r] & M \times \mathbb{R}^{2N} \arrow[r] & \nu \arrow[r] \arrow[d] & 0 \\
& \text{Spin}^c \text{ Thom class} \arrow[u] & \text{Thom class} \arrow[u] \arrow[r, Rightarrow] & \text{Thom class} &
\end{tikzcd}$$

Step 1. Using the Thom isomorphism for the normal bundle $\nu \to M$, replace the pair (M, E) by $(\Sigma\nu, \pi^*\nu \otimes \beta_\nu)$.

Step 2. $\Sigma\nu$ is bordant to S^{2N}. To see this identify

$$\Sigma F = S(\nu \oplus \underline{\mathbb{R}})$$

where $S(\nu \oplus \underline{\mathbb{R}}) \to M$ is the sphere bundle of the vector bundle $\nu \oplus \underline{\mathbb{R}} \to M$. This shows that we may identify the sphere bundle $\Sigma\nu$ with the boundary of a tubular neighborhood of M in the larger vector space $\mathbb{R}^{2N+1} = \mathbb{R}^{2N} \times \mathbb{R}$. This boundary of the tubular neighborhood is bordant to S^{2N}, as we can see by considering the following picture,



The red region in the picture is the manifold W whose boundary is $\partial W = S^{2N} \sqcup (-\Sigma\nu)$. Thus, ignoring the vector bundle E for the moment, our two steps are a Thom isomorphism followed by a bordism,

$$\begin{tikzcd} M \arrow[r, rightsquigarrow, "\text{Thom}"] & \Sigma\nu \arrow[r, rightsquigarrow, "\text{bordism}"] \arrow[d, equal] & S^{2N} \\ & S(\nu \oplus \underline{\mathbb{R}}) & \end{tikzcd}$$

Remark. The vector bundle $\pi^*E \otimes \beta_\nu$ may not extend from the sphere bundle $\Sigma\nu$ to the red colored region in the picture. This is an obstruction to obtain a bordism from the pair $(\Sigma\nu, \pi^*E \otimes \beta_\nu)$ to a pair (S^{2N}, F). However, this can be fixed by an easy Mayer-Vietoris argument in K-theory. Before you extend the vector bundle $\pi^*E \otimes \beta_\nu$, you may need to add a vector bundle on $\Sigma\nu$ that is obtained as the pull-back of a vector bundle from M. It follows easily from bordism invariance that this modification does not affect the analytic or toplogical index. This is a minor point in the proof, and I will leave out the details.

We have sketched how the verification of the index formula can be reduced, in two moves, to the problem on a sphere. On S^{2N} we verified the formula by direct calculation. The crux of the proof is therefore to verify that both sides of the index formula—the analytic index on the left hand side, and the topological index on the right hand side—are preserved under the two moves. In other words, there are four things to prove. The proofs of these four facts are entirely independent.

$\mathrm{Index} D_E$	$\int_M \mathrm{ch}(E)\mathrm{Td}^c(TM)$	
Hard	Stokes's Theorem	bordsim
Easy	Todd class	Thom

Note that the index formula on the spin manifold S^{2N} (to which we reduce) contains the Chern character $\mathrm{ch}(E)$, but reveals nothing about the Todd class $\mathrm{Td}^c(TM)$. Next time I will discuss the details of the proof of the invariance of the topological index under the Thom isomorphism (the bottom right corner in the above diagram). It is in this part of the proof that the formula for the Todd class (and therefore also the $\widehat{A}$-class) is calculated.

Of the other three corners in the diagram, two are easy (bordism invariance of the topological index, and invariance of the analytic index under the Thom isomorphism). However, the bordism invariance of the analytic index is the deepest fact of the four. Like Bott Periodicity, it is a result of independent interest that is a key ingredient of the topological proof as I outlined it here.

9 15 September 2016, Nigel Higson

In this final lecture on the local index theorem we shall (more or less) reach our goal of computing the index of the Dirac operator in purely local terms, arriving at the $\widehat{A}$-genus. But first we shall say more about the leading residue that we were considering in the last lecture.

9.1 Completion of the Leading Order Residue Computation

Let M be a closed manifol of dimension n. As usual, let Δ be a positive invertible operator of order 2, and let T be a compactly supported differential operator of order q. There is a smooth function

$$x \mapsto \operatorname{trace}_x(T\Delta^{-s})$$

on M, varying meromorphically with $s \in \mathbb{C}$, such that

$$\operatorname{Trace}(T\Delta^{-s}) = \int_M \operatorname{trace}_x(T\Delta^{-s})dx.$$

Our function is analytic in half-plane $\operatorname{Re}(s) > \frac{n+q}{2}$, and $T\Delta^{-s}$ is trace-class there. The pointwise residue

$$\operatorname{Res}_{s=\frac{n+q}{2}} \operatorname{trace}_x(T\Delta^{-s})$$

is a smooth function of x, and

$$\operatorname{Res}_{s=(n+q)/2}\operatorname{Trace}(T\Delta^{-s}) = \int_M \operatorname{Res}_{s=(n+q)/2}\operatorname{trace}_x(T\Delta^{-s})\,dx.$$

Finally, the pointwise residue at x depends only on the germs of T and Δ at x.

Because of the last point, it suffices to compute residues in the case where $M = \mathbb{R}^n$ and $x = 0 \in \mathbb{R}^n$. We proved that

$$\operatorname{Res}_{\frac{n+q}{2}} \operatorname{trace}_0(T\Delta^{-s}) = \operatorname{Res}_{\frac{n+q}{2}} \operatorname{trace}_0(T_0(I+\Delta_0)^{-s}), \tag{9.1}$$

where "0" means we freeze the coefficients at $0 \in \mathbb{R}^n$ and drop the lower order terms. Thus if

$$T = \sum_{|\alpha|\leq q} a_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha},$$

then

$$T_0 = \sum_{|\alpha|=q} a_\alpha(0)\frac{\partial^\alpha}{\partial x^\alpha}.$$

We shall review this reduction-to-constant-coefficients argument in a little while, but for now let us continue with an analysis of the right-hand side in (9.1). We calculate, using Fourier analysis, that if $\operatorname{Re}(s) > (n+q)/2$, then

$$\operatorname{trace}_0(T_0(I+\Delta_0)^{-s}) = \int_{\mathbb{R}^n} \text{total-symbol}(T_0)\,\text{total-symbol}(I+\Delta_0)^{-s}\,d\xi. \tag{9.2}$$

Here if $D_0 = \sum a_\alpha \partial^\alpha$ is any constant coefficient operator, then the total symbol is the function

$$\text{total-symbol}(D_0) = \sum a_\alpha (i\xi)^\alpha .$$

In contrast to the symbol discussed earlier, we have inserted powers of $i = \sqrt{-1}$. We have also *not* dropped the lower terms of D_0. Note also that the total symbol of $I + \Delta_0$ a positive function, so that raising it to a complex power is unproblematic.

A remark or two about the proof of (9.2). Under the Fourier isomorphism

$$L^2(\mathbb{R}^n) \xrightarrow{\cong} L^2(\widehat{\mathbb{R}^n})$$

the constant coefficient operators T_0 and $I + \Delta_0$ correspond to multiplication operators, the multipliers being the total symbols. So $T_0(I+\Delta_0)^{-s}$ corresponds to multiplication by the integrand in (9.2). By Fourier theory, the operator $T_0(I + \Delta_0)^{-s}$ itself is convolution by the inverse Fourier transform of the integrand. And by more Fourier theory the value of this inverse Fourier transform at zero (which is the local trace we are seeking) is the integral in (9.2).

The behaviour of the integral (9.2) as s converges to $(n + q)/2$ is easy to analyze by changing to polar coordinates, and we find that

$$\text{(9.3)} \quad \operatorname{Res}_{(n+q)/2} \operatorname{trace}_x(T_0 \Delta_0^{-s}) = \text{constant}_n \int_{S^{n-1}} \text{symbol}(T_0)\, \text{symbol}(\Delta_0)^{-\frac{n+q}{2}}\, d\xi,$$

where the constant in front depends on the dimension n alone (and since we are now dealing with the homogeneous operators there is essentially no difference between the symbol considered before and the total symbol, since we can absorb the powers of $\sqrt{-1}$ into the constant).

Remark 9.1. We've tacitly assumed that our operators are acting on scalar functions. If that's not the case, we need to insert traces at various points in the formulas above. For instance if T_0 and Δ_0 act on vector-valued functions rather than scalar functions, so that the coefficients of these operators are matrices, then (9.3) becomes

$$\operatorname{Res}_{(n+q)/2} \operatorname{trace}_x(T_0 \Delta_0^{-s}) = \text{constant}_n \int_{S^{n-1}} \operatorname{trace} \left[\text{princ-symbol}(T_0)\, \text{princ-symbol}(\Delta_0)^{-\frac{n+q}{2}} \right] d\xi,$$

where the trace within the integral is the standard trace on matrices.

9.2 Weyl's Theorem on Eigenvalue Asymptotics

The computation has a famous consequence, which it's worth pausing to enjoy. Let's go back to a closed manifold M. Assume for simplicity that Δ acts on scalar functions. There is an orthonomal basis of $L^2(M)$ consisting of eigenfunctions for Δ, say

$$\Delta f_k = \lambda_k f_k,$$

and

$$\mathrm{Trace}(\Delta^{-s}) = \sum_k \lambda^{-s}.$$

Therefore

$$\mathrm{Res}_{n/2}\,\mathrm{Trace}(\Delta^{-s}) = \lim_{s \searrow n/2} (s - n/2) \sum_k \lambda_k^{-s}$$

Let us now apply the following Tauberian theorem (see for example Hardy's book *Divergent Series*):

Theorem 9.2. *If* μ_k *are positive numbers with*

$$\lim_{s \searrow 1} \sum_k \mu_k^s = M$$

then

$$\lim_{\mu \to 0} \mu \cdot \#\{\, \mu_k > \mu \,\} = M.$$

We obtain Weyl's famous asymptotic law for the the eigenvalues λ_k:

Theorem 9.3. *If* Δ *is the Laplace-Beltrami operator on a closed Riemannian manifold* M *of dimension* n*, and if*

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

is the eigenvalue sequence of Δ (*with eigenvalues listed possibly multiple times, according to their multiplicity*)*, then*

$$\lim_{k \to \infty} \lambda_k \cdot k^{-\frac{2}{n}} = \mathrm{constant}_n \cdot \mathrm{vol}(M),$$

where the constant depends only on n*.*

This uses the fact that that, thanks to our explicit computation, the leading residue is proportional to the volume of M, with the constant of proportionality depending only on the dimension of M.

9.3 Review of the Rescaling Argument

In this section we shall review the rescaling argument that we used last time to reduce the computation of leading order residues to the constant coefficient case. We want to study whether or not we can adapt it to the direct computation other residues, and, most importantly, to zeta values at zero (of course we already know that we can identify these with complicated combinations of leading order residues by a laborious inductive argument involving the operators A_i and B_i from the M-S theorem; but we want to study whether or not we can find a more direct route).

We defined a rescaling unitary isomorphism on $L^2(\mathbb{R}^n)$ by

$$U_\varepsilon : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

defined by

$$(U_\varepsilon f)(x) = \varepsilon^{n/2} f(\varepsilon x)$$

(the fact that this operator is unitary is not important, and we could drop the $\varepsilon^{n/2}$ factor without really affecting anything, but we'll keep it in here). Our analysis of the leading residue using these scaling operators had the following two parts (which for clarity we'll formulate for Δ^{-s} alone, on a compact M, rather than a more general family $T\Delta^{-s}$):

(i) If we define

$$U_\varepsilon \Delta U_\varepsilon^{-1} = \varepsilon^{-2} \Delta_\varepsilon$$

then the coefficients of differential operator Δ_ε depends smoothly on ε, and the family of differential operators so obtained extends smoothly through $\varepsilon = 0$, where we obtain an operator with constant coefficients, giving us a smooth family of operators parametrized by $\varepsilon \in \mathbb{R}$.

(ii) If k is any smooth kernel function, then

$$U_\varepsilon \mathrm{Op}(k) U_\varepsilon^{-1} = \varepsilon^n \mathrm{Op}(k_\varepsilon)$$

where $k_\varepsilon(x, y) = k(\varepsilon x, \varepsilon y)$. As a result

$$\varepsilon^{2s} \mathrm{trace}_0(\Delta_\varepsilon^{-s}) = \varepsilon^n \mathrm{trace}_0(\Delta^{-s})$$

Taking residues at $s = n/2$, it follows from the second formula in (ii) that

$$\varepsilon^n \mathrm{Res}_{n/2} \mathrm{trace}_0(\Delta_\varepsilon^{-s}) = \varepsilon^n \mathrm{Res}_{n/2} \mathrm{trace}_0(\Delta^{-s})$$

Cancelling the ε^n-factors and integrating over a compact M gives

$$\mathrm{Res}_{n/2} \mathrm{Trace}(\Delta_\varepsilon^{-s}) = \mathrm{Res}_{n/2} \mathrm{Trace}(\Delta^{-s}),$$

That is, the leading residue is unchanged under rescaling.

It is at this point that we invoke item (i) above to complete our calculation. The residue on the left-hand side varies continuously with ε,and not only with $\varepsilon > 0$ but with $\varepsilon \in \mathbb{R}$ (all we need is that the operators Δ_ε vary smoothly with ε and are elliptic, positive and invertible). So we obtain

$$\mathrm{Res}_{n/2} \mathrm{Trace}(\Delta_0^{-s}) = \mathrm{Res}_{n/2} \mathrm{Trace}(\Delta^{-s}),$$

and we can complete the leading order residue computation as we did earlier in this lecture.

For the purposes of index theory we're interested not in the leading residue but in the zeta value at $s = 0$. Can we somehow adapt the the argument just given to cover zeta values at zero?

It is not evident at all that this is possible. For instance if we repeat the steps just taken without any serious chance, then we obtain

$$\varepsilon^n \operatorname{trace}_0(\gamma\Delta^{-s}) = \varepsilon^{2s}\operatorname{trace}_0(\gamma\Delta_\varepsilon^{-s}) \tag{9.4}$$

(here γ is the grading operator), and so, evaluating at $s = 0$,

$$\varepsilon^n \operatorname{trace}_0(\gamma\Delta^{-s})|_{s=0} = \operatorname{Trace}_0(\gamma\Delta_\varepsilon^{-s})|_{s=0}. \tag{9.5}$$

Unfortunately if we now take the limit as $\varepsilon \to 0$ we obtain

$$0 \cdot \operatorname{trace}_0(\gamma\Delta^{-s})|_{s=0} = \operatorname{Trace}_0(\gamma\Delta_0^{-s})|_{s=0},$$

which tells us nothing of interest since we have multiplitied the quantity of importance to us by zero.

In order to repair the argument we need to somehow get rid of the term ε^n that appears in (9.4). There is a remarkable trick that allows us to do this for (the squares of) Dirac operators, using information about the Clifford algebra, and specifically its relation to exterior algebra. We shall discuss this in just a moment, and so conclude the proof of the index theorem. But first we shall pause to disucss the relation between zeta functions and heat kernel traces.

9.4 Zeta Functions and Heat Traces

Our zeta functions are related to traces of heat kernels by the Mellin transform formula

$$\Gamma(s)\lambda^{-s} = \int_0^\infty e^{-\lambda t} t^s \, \frac{dt}{t},$$

which implies that

$$\Gamma(s) \operatorname{Trace}(\Delta^{-s}) = \int_0^\infty \operatorname{Trace}(e^{-t\Delta}) t^s \frac{dt}{t}.$$

The formula requires some care. Since we need to be sure that both sides make sense (the traces exist and the integrals converge) in at least some region of the s-plane. But suffice it to say that the material we've covered is sufficient to prove trace-ability for all $t > 0$ and convergence for all s with $\operatorname{Real}(s) > n/2$.

There is an inverse Mellin transform formula and it implies that

$$\operatorname{Trace}(e^{-t\Delta}) = \frac{1}{2\pi i} \int_L \Gamma(s) t^{-s} \operatorname{Trace}(\Delta^{-s}) ds$$

where the contour of integration is any upwards-traversed) vertical line in the plane to the right of all the poles of $\Gamma(s)\,\mathrm{Trace}(\Delta^{-s})$. Once again, a bit of care is needed, but the integrand is in fact a rapidly decreasing function on any vertical line away from the poles, which is what we need to make sense of (and to prove) the formula.

Further information can be gathered by considering the integral of the same function around the contour C in Figure 1. The right vertical line is to the right

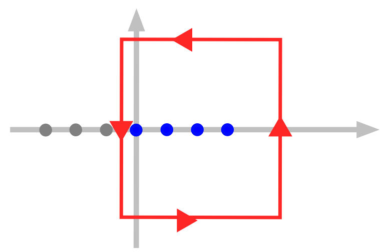


Figure 1: The contour C.

of all the poles; the left vertical line passes between the poles (coming from the Gamma function) at 0 and -1; the top and bottom horizontal lines are very far above and very far below the x-axis, respectively: soon we are going to take a limit as these distances tend to infinity.

The residue formula says that

$$
\begin{aligned}
\frac{1}{2\pi i}\int_C \Gamma(s)t^{-s}\,\mathrm{Trace}(\Delta^{-s})ds& \\
&= \sum_p \mathrm{Res}_p\,\Gamma(s)\,\mathrm{Trace}(\Delta^{-s}) \\
&= \mathrm{Trace}(\Delta^{-s})|_{s=0} + \sum_{k=1}^{2n}\Gamma(k/2)t^{-k/2}\,\mathrm{Res}_{k/2}\,\mathrm{Trace}(\Delta^{-s}).
\end{aligned}
$$

On the other hand the contour integral can be analyzed as follows:

(i) In the limit as the height and depth of the rectangular contour C converge to infinity, the contributions to the integral from the horizontal parts of the contour are zero.

(ii) In the same limit, the contibution from the left vertical part of the contour is $\mathrm{Trace}(e^{-t\Delta})$.

(iii) Whether or not we take a limit, the left vectical part of the contour is $o(t)$.

We find that

$$
\mathrm{Trace}(e^{-t\Delta}) = o(t^{-1}) + \mathrm{Trace}(\Delta^{-s})|_{s=0} + \sum_{k=1}^{2n}\Gamma(k/2)t^{-k/2}\,\mathrm{Res}_{k/2}\,\mathrm{Trace}(\Delta^{-s}),
$$

and so we have obtained the asymptotic expansion for the heat kernel (and determined the coefficients of the asymptotic expansion).

9.5 Getzler's Rescaling and the Index Theorem

Let's start with the ingredients we're given. The first is the Dirac operator D on a closed Riemannian spin manifold. We're interested in its square Δ, and if we compute it in geodesic coordinates (and the associated trivialization of the spinor bundle), then what we find is that

$$\Delta = -\sum_i \left(\partial_i + \frac{1}{4} R_{ij} x_j\right)^2 + \text{small error} \tag{9.6}$$

where R_{ij} entry at 0 in Riemann's curvature matrix of 2-forms, and where we'll explain what we mean by the "small error" in due course.

The formula (9.6) requires still further explanation, beyond a discussion of the small error term. First, we haven't discussed in these lectures what we mean by "the" Dirac operator, only the Dirac symbol. But using ideas from Riemannian goemetry, in particular the Levi-Civita connection, we can define a single preferred (by geometers, and by us here) operator whose symbol is the Dirac operator. It has the important and rather beautiful property that

$$\Delta = \nabla^* \nabla + \frac{\kappa}{4},$$

where ∇ is the Levi-Civita connection on the spinor bundle, and κ is the scalar curvature function on the underlying Riemannian manifold. This is called the Lichnerowicz identity (although it was discovered earlier by Schrodinger).

Next, we said that R_{ij} is a two-form at 0, but it appears in (9.6) as a coefficient of an operator that acts on spinors. We are using the standard isomorphism of vector spaces

$$\mathrm{Cliff}(V) \cong \wedge^{\bullet} V$$

that corresponds

$$e_{i_1} \ldots e_{i_p} \in \mathrm{Cliff}(V)$$

to

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \in \bigwedge{}^{\bullet} V$$

whenever $e_{i_1}, \ldots, e_{i_p}$ are distinct and orthornormal.

Before we go on, let's imagine for a moment that there was no "small error" term in (9.6). The value of $\mathrm{trace}_0(\gamma\Delta^{-s})$ at $s = 0$ would then necessarily be a function of the terms R_{ij} alone, because these would be all the information from the manifold M, and from its spin structure, that would be present in the formula for Δ.

In fact, by re-examining our clumsy-but-in-principle-explicit formula for the value of $\mathrm{trace}_0(\gamma\Delta^{-s})$ at $s = 0$, we would find that this function would be a *polynomial* in the R_{ij}, and indeed a *symmetric polynomial* in them. So we would be able to conclude that the local index of D is some universal polynomial in the R_{ij}. Compare the form of the actual (global) index formula, given in Lecture 4.

Remember that, in contrast, what we know up to now is that the local index is the integral of a—possibly immensely complicated—function of all partial derivatives, up to order n, or perhaps it's $2n$, of the coefficients of the Dirac operator. *But if we can disregard the "small error" terms, then we can guarantee that no higher partial derivatives of any sort are involved in the local formula; we will have verified the miraculous cancellation phenomenon, or most of it*.[4].

This observation should make it obvious what we ought to do: find a rescaling that shrinks the error part of (9.6) to zero, while preserving the local index, that is, the value of $\mathrm{trace}_0(\gamma\Delta^{-s})$ at $s = 0$.

To accomplish this we shall use Getzler's method of rescaling not only the underlying space, as we did to compute the leading residue, but also the Clifford algebra (or, if you like, the noncommutative space underlying the Clifford algebra).

The method is a bit awkward to describe because it involves not only traces on operators, but also traces on algebras. That is, rather than think of Δ as an operator acting on some Hilbert space, we shall need to think of Δ (or rather the semigroup of complex powers Δ^s for $\mathrm{Real}(s) \gg 0$) as lying in an appropriate algebra with a trace. We shall then rescale the algebra and its trace, along with everything else.

Let's start by describing how to rescale the Clifford algebra. Recall that

$$\mathrm{Cliff}(\mathbb{R}^n) = \left\langle\, e_j \,:\, e_i e_j + e_j e_i = 0 \;\&\; e_j^2 = -1 \,\right\rangle$$

(the angle-brackets mean the algebra generated by the indicated elements subject to the indicated relations) and form the following variant:

$$\mathrm{Cliff}_\varepsilon(\mathbb{R}^n) = \left\langle\, e_j \,:\, e_i e_j + e_j e_i = 0 \;\&\; e_j^2 = -\varepsilon^2 1 \,\right\rangle$$

When $\varepsilon \neq 0$ this is not really different from the original Clifford algebra, because the correspondence

$$e_j \longmapsto \varepsilon^{-1} e_j \tag{9.7}$$

induces an isomorphism of algebras

$$\mathrm{Cliff}(\mathbb{R}^n) \longrightarrow \mathrm{Cliff}_\varepsilon(\mathbb{R}^n). \tag{9.8}$$

On the other hand when $\varepsilon = 0$ we get

$$\mathrm{Cliff}_0(\mathbb{R}^n) = \left\langle\, e_j \,:\, e_i e_j + e_j e_i = 0 \;\&\; e_j^2 = 0 \,\right\rangle.$$

[4]Following this line of reasoning, it would remain to determine which symmetric polynomial gives the local index. This is a relatively simple explicit computation based on the form of the operator in (9.6)

This is of course the exterior algebra on the indicated generators. Altogether, we obtain a smooth bundle of algebras parametrized by $\varepsilon \in \mathbb{R}$. The sections $e_{i_1} \cdots e_{i_p}$ for $i_1 < \cdots < i_p$ give a global frame and so trivialize it as a smooth vector bundle.

We shall need to use the so-called *supertraces* or (fancier) *ferminonic integrals* on these algebras. These are the linear forms on $\mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$ that are defined as follows:

$$\mathrm{str}\colon \sum a_{i_1 \ldots i_p} e_{i_1} \cdots e_{i_p} \longmapsto a_{12\ldots n}.$$

Thus all the supertraces on all the $\mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$ are given by the same formula.

We're assuming throughout that n is even, and so if $\varepsilon \neq 0$, then there there is an isomorphism

$$\mathrm{Cliff}_\varepsilon(\mathbb{R}^n) \xrightarrow{\cong} \mathrm{End}(\mathbb{S}_\varepsilon), \tag{9.9}$$

where $\mathbb{S}_\varepsilon$ is the spinor representation of the Clifford algebra—its unique irreducible representation. Under this algebra isomorphism, the supertrace corresponds to a rescaling of the usual operator trace, but with the grading operator γ inserted:

$$\begin{tikzcd} \mathrm{Cliff}_\varepsilon(\mathbb{R}^n) \arrow[r, "x \longmapsto \varepsilon^n (-2i)^{n/2}\, \mathrm{str}(x)"] \arrow[d, "\cong"'] & \mathbb{C} \arrow[d, equal] \\ \mathrm{End}(\mathbb{S}_\varepsilon) \arrow[r, "X \mapsto \mathrm{trace}(\gamma T)"'] & \mathbb{C} \end{tikzcd} \tag{9.10}$$

On the other hand the supertrace at $\varepsilon = 0$ obviously calls to mind the integral of differential forms.

Now let's return to the Dirac operator on a closed Riemannian spin manifold, or rather its square, Δ. As usual we shall study Δ, its complex powers, and so on, near one point in M, and introducing coordinates as we did at the beginning of this section we can consider Δ as an operator

$$\Delta\colon \mathcal{C}^\infty(U, \mathbb{S}) \longrightarrow \mathcal{C}^\infty(U, \mathbb{S}),$$

where U is an open ball around $0 \in \mathbb{R}^n$ and $\mathbb{S}$ is the spinor representation of $\mathrm{Cliff}(\mathbb{R}^n)$ as above. This is a second order partial differential operator whose coefficients are smooth $\mathrm{End}(\mathbb{S})$-valued functions. Let's also write

$$U_\varepsilon \Delta U_\varepsilon^{-1} = \varepsilon^{-2} \Delta_\varepsilon,$$

exactly as we did in our treatment of the leading residue. As we saw before, the coefficients of Δ_ε are smooth $\mathrm{End}(\mathbb{S})$-valued functions that vary smoothly with ε. There is a smooth extension to the value $\varepsilon = 0$ (where the coefficients are constant), but this is not what we are interested in. Instead we are first going to modify Δ_ε, or at least modify the way we look at Δ_ε (so to speak, we are going to change gauge), and only then will we extend to $\varepsilon = 0$.

The modification is as follows. Denote by $\Delta_{\varepsilon,\varepsilon}$ the second order differential operator on U whose coefficient functions are smooth, $\mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$-valued functions, which is obtained by applying the isomorphism defined by the diagram

$$\begin{tikzcd} \mathrm{End}(\mathbb{S}) \arrow[r, dotted] & \mathrm{Cliff}_\varepsilon(\mathbb{R}^n) \\ \mathrm{Cliff}(\mathbb{R}^n) \arrow[u, "\cong"] \arrow[ur, "\cong\ (9.8)"'] & \end{tikzcd}$$

to the coefficients of Δ.

What sort of an object is this? We could think of $\Delta_{\varepsilon,\varepsilon}$ as an actual differential operator, acting on the space $C^\infty(U)\otimes\mathbb{S}_\varepsilon$, where $\mathbb{S}_\varepsilon$ is the spinor representation of $\mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$. But this is *not* what we are going to do, and the reason is that the spinor representations of $\mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$ do not have a good limit as $\varepsilon \to 0$. Instead we shall simply view $\Delta_{\varepsilon,\varepsilon}$ as an element of the algebra

$$\mathcal{D}(U) \otimes \mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$$

of differential operators with coefficients in the ε-Clifford algebra. (If you really want $\Delta_{\varepsilon,\varepsilon}$ to act somewhere, you can think of is as acting on the algebra $\mathcal{D}(U)\otimes \mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$ by left multiplication.)

The justification for this point of view is as follows.

Lemma 9.4. *The operators $\Delta_{\varepsilon,\varepsilon}$ vary smoothly with ε and extend smoothly to $\varepsilon = 0$, where*

$$\Delta_{0,0} = -\sum_i \left(\partial_i + \frac{1}{4}R_{ij}x_j\right)^2, \tag{9.11}$$

with R_{ij} the (i,j)-entry at 0 of Riemann's curvature matrix of 2-forms. □

Remark 9.5. To make the lemma precise, it is helpful to note that the differential operators of order 2 or less (or of any order q or less) form a locally free and finitely generated sheaf of modules over the sheaf of smooth functions, or in otherwords can be identified with the sheaf of sections of a smooth vector bundle over U. The family $\{\Delta_{\varepsilon,\varepsilon}\}_{\varepsilon\in\mathbb{R}}$ discussed in the lemma is a smooth section of the tensor product bundle, over $U \times \mathbb{R}$, of this bundle with the bundle of algebras $\mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$.

Let's now think of the operators $\Delta_{\varepsilon,\varepsilon}$ as acting on the Hilbert spaces

$$L^2(\mathbb{R}^n, \mathrm{Cliff}_\varepsilon(\mathbb{R}^n)) = L^2(\mathbb{R}^n)\otimes \mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$$

(the coefficient functions act by left multiplication), where we make $\mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$ into a finite-dimensional Hilbert space by decreeing the monomials $e_{i_1}\cdots e_{i_p}$

to be an orthonormal basis. They are elliptic.[5] The various operators such as $fT\Delta^{-s}$ that we have considered commute with right-multiplication by elements of $\mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n) \otimes \mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$, and when they are trace class they lie in

$$\mathcal{L}^1\big(L^2(\mathbb{R}^n)\big) \otimes \mathrm{Cliff}_\varepsilon(\mathbb{R}^n) \subseteq \mathcal{L}^1\big(L^2(\mathbb{R}^n) \otimes \mathrm{Cliff}_\varepsilon(\mathbb{R}^n)\big) \tag{9.12}$$

(the left-hand copy of the Clifford algebra acts by left multiplication on the Hilbert space). Thus

$$fT\Delta^{-s}_{\varepsilon,\varepsilon} \in \mathcal{L}^1\big(L^2(\mathbb{R}^n)\big) \otimes \mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$$

for $\mathrm{Real}(s) \gg 0$. If $\mathrm{Real}(s)$ is larger still, then the tensor product on the left can be replaced by r-times continuously differentible kernels valued in $\mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$, and moreover the restriction to the diagonal is a smooth function, meromorphic in $s \in \mathbb{C}$. Let's denote by

$$s \longmapsto \mathrm{supertrace}_0(fT\Delta^{-s}_{\varepsilon,\varepsilon})$$

the supertrace (for $\mathrm{Cliff}_\varepsilon(\mathbb{R}^n)$) of the diagonal value at $0 \in \mathbb{R}^n$.

At this point, let's pause to consider $\varepsilon = 1$. Thanks to (9.10), the value of the function

$$s \longmapsto \mathrm{trace}_0\big(\gamma\Delta^{-s} \colon L^2(\mathbb{R}^n, \mathbb{S}) \to L^2(\mathbb{R}^n, \mathbb{S})\big)$$

at $s = 0$ that we need to calculate (at $s = 0$) for index theory is equal to the value of

$$s \longmapsto (-2i)^{n/2}\mathrm{supertrace}_0(fT\Delta^{-s}_{1,1})$$

at $s = 0$. Now think of this this as the value of

$$(-2i)^{n/2}\mathrm{supertrace}_0(fT\Delta^{-s}_{\varepsilon,\varepsilon})$$

at $s = 0$ and $\varepsilon = 1$, and observe from (9.10) again, and (9.5) that this is *independent of* ε. We find that

$$\mathrm{trace}_0\big(\gamma\Delta^{-s}\big)\Big|_{s=0} = (-2i)^{n/2}\mathrm{supertrace}_0(fT\Delta^{-s}_{0,0})\Big|_{s=0}.$$

This is really our grand conclusion about index theory for Dirac operators (just as our grand conclusion about point-values of leading residues was that they depended only on principal symbols, and so could be computed using constant coefficient operators). Keeping in mind the formula for the supertrace at $\varepsilon = 0$, we see immediately that the index is *some* degree $n/2$ invariant polymnomial in the R_{ij} over M, and so is a characteristic number. To understand *which*

[5] They are not quite formally self-adjoint, but if we cut off on the left and right by a suitable smooth bump function that is supported near $0 \in \mathbb{R}^n$, then they are norm-bounded perturbations of formally self-adjoint and indeed essentially self-adjoint operators, and this is enough for the analysis to proceed.

characteristic number it is, only a modest amount of extra work is needed, involving an explicit calculation parallel to the one we did with Fourier theory at the beginning of this lecture.

Namely to calculate the precise formula for the (local) index, we need to determine the value at zero of the zeta function

$$s \longmapsto \operatorname{supertrace}_0\left(\Delta_{0,0}^{-s}\right)$$

for the operator $\Delta_{0,0}$ in (9.11). Using the result of our discussion on heat traces and zeta functions, this is essentially the same as the following computation, which can be done quickly by appealing to the theory of the quantum harmonic oscillator

$$\Delta_{\text{harmonic}} = -\frac{d^2}{dx^2} + x^2$$

(although time is up and we won't do the calculation here).

Lemma 9.6. *If $\Delta_{0,0}$ is the operator (9.11), then*

$$\operatorname{supertrace}_0(\exp(-t\Delta)) = (4\pi t)^{\frac{n}{2}} \det{}^{1/2}\left(\frac{tR/2}{\sinh(tR/2)}\right)$$

where $R = [R_{ij}]$. □

10 16 September 2016, Erik van Erp

The Dirac operator D of a closed even dimensional spin^c manifold M determines a summable Fredholm module, which gives an element in K-homology $K_0(M)$. A complex vector bundle $E \to M$ represents an element in K-theory $K^0(M)$ (a cohomology theory). The pairing of the K-homology cycle $[D]$ and the K-theory class $[E]$ is the integer $\operatorname{Index}(D_E)$. The Chern character (in homology and cohomology) turns this into a pairing between the homology class $\operatorname{ch}_\bullet(D)$ and the cohomology class $\operatorname{ch}^\bullet(E)$.

$$\begin{tikzcd}
\langle [D],[E]\rangle \in K_0(M)\times K^0(M) \arrow[r, mapsto] \arrow[d, mapsto] & \operatorname{Index}D_E \in \mathbb{Z} \\
\langle \operatorname{ch}_\bullet(D), \operatorname{ch}^\bullet(E)\rangle \in H_{ev}(M,\mathbb{R})\times H^{ev}(M,\mathbb{R}) \arrow[r] & \mathbb{R}
\end{tikzcd}$$

The Chern character of a vector bundle $\operatorname{ch}^\bullet(E)$ is well understood. We can therefore think of the Atiyah-Singer index theorem as giving a topological formula for the homology Chern character $\operatorname{ch}_\bullet(D)$. The formula

$$\operatorname{Index}D_E = \int_M \operatorname{ch}(E)\wedge \operatorname{Td}^c(TM)$$

is equivalent to

Theorem 10.1 (Atiyah-Singer).

$$\mathrm{ch}_\bullet(D) = \int_M - \wedge \mathrm{Td}^c(TM) = \textit{Poincaré Dual of } \mathrm{Td}^c(TM)$$

Topological proof. We have sketched a proof of the Atiyah-Singer formula by reduction to the spin manifold S^{2n}. The formula can be directly verified on S^{2n}, because Bott Periodicity gives us full knowledge of the stable isomorphism classes of vector bundles $E \to S^{2n}$. The reduction of the problem to the calculation on S^{2n} proceeds in two steps, and we must prove invariance of both the analytic index and the topological index in each of these two steps. Thus, the proof of the index formula in this approach depends on Bott Periodicity, and four "invariance" proofs.

	$\mathrm{Index} D_E$	$\int_M \mathrm{ch}(E)\mathrm{Td}^c(TM)$
(co)bordism	Hard	Stokes's theorem
Thom Isom.	Easy	calculation of $\mathrm{Td}^c(TM)$

In this approach, the calulation of the homology Chern character $\mathrm{ch}_\bullet(D)$ takes place in the bottom right corner of the diagram, i.e., it is implicit in the proof of the invariance of the topological index under the Thom isomorphism. In today's lecture I will focus on this calculation. I will first very briefly comment on the other three "invcariances".

The bordism invariance of the analytic index is a difficult analytic fact. There are several proofs available, many of which depend on the analysis of elliptic boundary value problems. An elementary proof that avoids boundary value problems, and only requires some basic knowledge about the resolvent of the Dirac operator on complete manifolds, can be found in a paper by Nigel Higson, *A note on the cobordism invariance of the index*, published in Topology (1991).

By contrast, the bordism invariance of the *topological* index is an easy consequence of Stokes's Theorem.

The invariance of the analytic index under the (compactified) Thom isomorphism is also not difficult to prove. It relies on the fact that if M and N are two even dimensional spin^c manifolds, then the index of the Dirac operator $D_{M\times N}$ of the product manifold $M \times N$ is the product

$$\mathrm{Index}(D_{M\times N}) = \mathrm{Index}(D_M) \cdot \mathrm{Index}(D_N)$$

We know that on the even sphere S^{2n} with Bott generator vector bundle β we have $\mathrm{Index} D_{\bar\beta} = 1$. Therefore on the product $M \times S^{2n}$ the index of the twisted Dirac D_E on M equals the index of the operator $D_{E\otimes\bar\beta}$ on $M \times S^{2n}$. This same argument can be easily adapted to nontrivial sphere bundles $\Sigma F \to M$ because the Dirac operator and Bott generator vector bundle of the even sphere S^{2n} are both equivariant for the structure group of the spin^c vector bundle $F \to M$.

Having briefly commented on the other three "invariances", the content of this lecture concerns the bottom right corner of the above table, i.e., the proof of the invariance of the topological index under the Thom isomorphism. As we will see, this amounts to the calulation of the characteristic class $\mathrm{Td}^c(TM)$, and therefore, in essence, this step is where we calculate $\mathrm{ch}_\bullet(D)$.

Invariance of the topological Index under the "Thom Isomorphism". Let M be an even dimensional Spin^c manifold and E be a complex vector bundle. Recall that in our reduction of the index problem to an even sphere S^{2n}, one of our two "moves" was to replace a pair (M, E) by

$$(M, E) \rightsquigarrow (\Sigma F, \pi^* E \otimes \beta_F),$$

Here $\Sigma F \xrightarrow{\pi} M$ is a sphere bundle over M, whose fibers are even dimensional spheres. $F \longrightarrow M$ is a vector bundle with a Thom class $\tau \in K^0(F)$. The fibers of F are even dimensional, $F_p \approx \mathbb{R}^{2n}$. Denote the one-point compactification of F_p by $F_p^+ \approx S^{2n} \approx S(F_p \times \mathbb{R})$. We may identify ΣF with the sphere bundle $S(F \oplus \underline{\mathbb{R}})$.
<u>Need:</u> Before embarking on this proof, one needs to establish the following two properties of the Todd class:

$$\mathrm{Td}(E \oplus F) = \mathrm{Td}(E)\mathrm{Td}(F) \qquad \mathrm{Td}(\text{trivial}) = 1$$

Using these two properties we find that

$$\begin{aligned}\int_M \mathrm{ch}(E)\mathrm{Td}^c(TM) &= \int_{\Sigma F} \mathrm{ch}(\pi^* E)\mathrm{ch}(\beta_F) \underbrace{\mathrm{Td}^c(T(\Sigma F))}_{\mathrm{Td}^c(\pi^* TM)\mathrm{Td}^c(\pi^* F)} \\ &= \int_M \mathrm{ch}(E)\pi_!(\mathrm{ch}(\beta_F))\mathrm{Td}(TM)\mathrm{Td}(F),\end{aligned}$$

where $\pi_! : H_c^\bullet(F) \longrightarrow H^{\bullet - 2n}(M)$ denotes integration in the fiber. Therefore, to prove the equality, and thereby establish invariance of the topological index under the Thom isomorphism, it suffices to prove the following proposition.

Remark: In the next proposition we actually need $\overline{\beta_F}$ instead of β_F, but we will see why this is the case at the end of the lecture.

Proposition 10.2.

$$\pi_!(\mathrm{ch}(\beta_F)) = \frac{1}{\mathrm{Td}^c(F)}$$

Proof. First, it is not hard to prove that we may replace the vector bundle β_F of ΣF by the Thom class τ_F on F. Thus we will prove that,

$$\pi_!(\mathrm{ch}(\tau_F)) = \frac{1}{\mathrm{Td}^c(F)}$$

Facts about Euler class The Euler class is the characteristic class associated to the Pfaffian, i.e., the invariant polynomial $\mathrm{Pf} : \mathfrak{so}(2n) \to \mathbb{R}$ that corresponds to $e = \mathrm{Pf} = \prod x_j$ on skew matrices in normal form (see below).

Let $\kappa : M \to F$ denote the zero section, and κ^* restriction to M of compactly supported differential forms on F,

$$\kappa^* : H_c^\bullet(F) \longrightarrow H^\bullet(M)$$

Then the Euler class $e(F) \in H^\bullet(M)$ has the property that

$$\pi_!(a) \wedge e(F) = \kappa^*(a),$$

where $a \in H_c^\bullet(F) \underset{\pi_!}{\longrightarrow} H^{\bullet-2n}(M)$. With $a = \mathrm{ch}(\tau_F)$ we get

$$\begin{aligned} \pi_!(\mathrm{ch}(\tau_F)) \wedge e(F)) &= \kappa^*(\mathrm{ch}(\tau_F)) \\ &= \mathrm{ch}(\underbrace{\widetilde{\kappa}(\tau_F)}_{[S^+]-[S^-]}) \\ &= \mathrm{ch}(S^+) - \mathrm{ch}(S^-), \end{aligned}$$

where $K^0(F) \xrightarrow{\widetilde{\kappa}} K^0(M)$ is the map in K-theory induced by the zero section $\kappa : M \to F$, and $S^{+/-} \to M$ are the two vector bundles that are part of the Thom class $\tau_F : \pi^*S^+ \longrightarrow \pi^*S^-$ of F. Because M is compact, the restriction map $\widetilde{\kappa}$ "forgets" the map τ_F, and only remembers the vector bundles $S^{+/-}$.

To simplify things, we will now restrict our attention to the case of spin vector bundles. To calculate $\mathrm{Td}^c(TM)$ for spin^c manifolds, we need a little more knowledge about characteristic classes of spin^c vector bundles. Instead, I will discuss the calculation of $\mathrm{Td}^c(TM)$ in the case where M is a *spin* manifold. In other words, I will calculate $\widehat{A}(TM)$. Thus, for the sake of exposition, from now on I will assume that F is a *spin* vector bundle.

To finish the calculation we use the following lemma.

Lemma 10.3. *If F is a $\mathrm{Spin}(2n)$ - vector bundle, then the characteristic class*

$$f(F) = \mathrm{ch}(\mathbb{S}^+) - \mathrm{ch}(\mathbb{S}^-)$$

is associated to the invariant polynomial

$$f : \mathfrak{so}(2n) \longrightarrow \mathbb{R}$$

defined by $f(x_1 x_2 \cdots) = (-1)^n \prod \left(e^{x_j/2} - e^{-x_j/2}\right)$ *for skew matrices in normal form*

$$\begin{bmatrix} 0 & -x_1 & & & \\ x_1 & 0 & & & \\ & & 0 & -x_2 & \\ & & x_2 & 0 & \\ & & & & \ddots \end{bmatrix} \in \mathfrak{so}(2n).$$

This suffices to finish the proof of the proposition 10.2. From

$$\pi_!(\mathrm{ch}(\tau_F)) \wedge e(F) = f(E)$$

we see that the characteristic class $\pi_!(\mathrm{ch}(\tau_F))$ of the spin vector bundle F corresponds to the invariant polynomial defined by

$$(-1)^n \prod_{j=1}^{n} \frac{\left(e^{x_j/2} - e^{-x_j/2}\right)}{x_j}$$

The sign $(-1)^n$ is removed if we replace β_F by the dual bundle $\bar{\beta}_F$. What we get then is precisely the inverse of $\widehat{A}$, which is what we needed to prove. With a minor modification the proof will work for spin^c vector bundles F, in which case we obtain the inverse of the Todd class, as required. $\square$

Proof of Lemma (10.3). A spin vector bundle F is given by transition functions, defined on an open cover of M. For two open subsets $U, V \subset M$, we have

$$U \cap V \to \mathrm{Spin}(2n)$$

If these are the transition functions for F, then the transition functions for the spinors $S = S^+ \oplus S^-$ are obtained as follows,

$$\begin{tikzcd} U \cap V \arrow[r] & \mathrm{Spin}(2n) \arrow[r] \arrow[dr, "\phi"'] & \mathrm{End}(\mathbb{C}^{2n}) \\ & & U(2^n), \arrow[u, "\subseteq"'] \end{tikzcd}$$

The structure group of S is the unitary group $U(2^n)$. In fact, because of the grading $S = S^+ \oplus S^-$, the structure group is really the subgroup $U(2^{n-1}) \times U(2^{n-1}) \subset U(2^n)$.

The curvature of F is a so$(2n)$-valued 2-form, while the curvature of S is a u(2^n)-valued 2-form. To evaluate which characteristic class f of F corresponds to the characteristic class

$$\mathrm{ch}_S(\mathbb{S}) = \mathrm{ch}(\mathbb{S}^+) - \mathrm{ch}(\mathbb{S}^-)$$

of $\mathbb{S}$, we consider the diagram

$$\begin{tikzcd} \mathfrak{so}(2n) \arrow[r, "d\phi"] \arrow[dr, "f"'] & \mathfrak{u}(2^n) \arrow[d, "\mathrm{ch}_S"] \\ & \mathbb{C} \end{tikzcd}$$

We need an explicit formula for the Lie algebra map $d\phi \longrightarrow \mathfrak{u}(2^n)$. We can obtain such a formula from the calculations in the Clifford algebra that we did in an earlier lecture.

Recall,

$$\mathfrak{so}(2n) \xrightarrow{\cong} \Lambda^2\mathbb{R}^{2n} \subset C_{2n}$$

where $\Lambda^2\mathbb{R}^{2n}$ denotes the subset of the Clifford algebra C_{2n} spanned by elements of the form $e_i e_j, i \neq j$. We have shown in an easlier lecture that the element $\frac{1}{2}e_i e_j \in C_n$ corresponds to the skew matrix in $\mathrm{so}(n)$ with a -1 in row i and column j, and $+1$ in row j and column i. We can therefore realize the Lie algebra map $d\phi$ explicitly by means of the $2^n \times 2^n$ complex valued matrices E_i of Equation (2.1). We find, for a skew matrix in normal form,

$$d\phi : \begin{bmatrix} 0 & -x_1 & & & \\ x_1 & 0 & & & \\ & & 0 & -x_2 & \\ & & x_2 & 0 & \\ & & & & \ddots \end{bmatrix} \in \mathfrak{so}(2n) \longmapsto \frac{x_1}{2}E_1E_2 + \frac{x_2}{2}E_3E_4 + \cdots .$$

Let us denote $A = \frac{x_1}{2}E_1E_2 + \frac{x_2}{2}E_3E_4 + \cdots$. To identify the characteristic class f, we need to calculate

$$f(x_1, x_2, \ldots, x_n) = \mathrm{ch}_S(A) = \mathrm{tr}(\gamma \exp(A))$$

where $\gamma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ is the grading operator.

Note that E_iE_j commutes with E_kE_l if i, j, k, l are four distinct indices. Thus,

$$\begin{aligned} \exp(A) &= \prod \exp\left(\frac{x_j}{2}E_{2j-1}E_{2j}\right) \\ &= \prod \left(\cos\left(\frac{x_j}{2}\right) + \sin\left(\frac{x_j}{2}\right)E_{2j-1}E_{2j}\right). \end{aligned}$$

We now need the following fact about the "supertrace" $\mathrm{tr}(\gamma -)$ defined on the Clifford algebra $C_{2n} \to \mathbb{R}$. Exercise: Verify that

$$\mathrm{tr}(\gamma E_{i_1}E_{i_2}\cdots E_{i_p}) = \begin{cases} 0 & \text{, except} \\ (-2i) & \text{, if } E_1E_2\cdots E_{2n}. \end{cases}$$

<u>hint:</u> You need to first verify that the grading operator is given by $\gamma = i^n E_1E_2\cdots E_{2n}$.

With this knowledge about the supertrace on the Clifford algebra, we obtain

$$\begin{aligned} \mathrm{tr}(\gamma \exp(A)) &= (-2i)^n \prod_{j=1}^{n} \sin\left(\frac{x_j}{2}\right) \\ &= (-1)^n \prod \left(e^{ix_j/2} - e^{-ix_j/2}\right) \end{aligned}$$

This should be our expression for the function $f : \mathfrak{so}(2n) \to \mathbb{R}$.

What about the i's in the exponents? This has to do with the fact that $\mathbb{S}$ is a complex vector bundle, and F is a real vector bundle.

conventions. If Ω denotes the curvature of a connection on a vector bundle, then the conventions for characteristic classes of real and complex vector bundles are slightly different.

$\mathbb{R}$ vector bundle	$\mathbb{C}$ vector bundle
$p : \mathfrak{so}(n) \longrightarrow \mathbb{R}$	$p : \mathfrak{u}(N) \longrightarrow \mathbb{C}$
$p(F) = p\left(\frac{\Omega_F}{2\pi}\right)$	$p(S) = p\left(-\frac{\Omega_S}{2\pi i}\right)$.

If we take this into account, the $i's$ in the exponents will go away. Moreover, to get all the signs to work out correctly, we note that in the statement of Proposition (10.2) we really need $\overline{\beta_F}$ instead of β_F. $\square$

Acknowledgements

Konrad Aguilar thanks Sayan Chakraborty and Thomas Williams for finding typos and giving useful suggestions, and Thomas Williams for sharing his well-written notes.