WEYL, HEISENBERG AND LEVINSON THEOREMS, AFTER KODAIRA

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ABSTRACT. These are notes from two 2020 Noncommutative Geometry Seminars at Penn State. The goal in the seminars was to describe and prove theorems of Weyl, Heisenberg and Levinson about Sturm-Liouville operators on a half-line, at least in simple special cases, following an approach due to Kodaira.

1. INTRODUCTION AND STATEMENT OF THE MAIN THEOREMS

I'm going to be considering a linear differential operator on $[0, \infty)$ of the form

$$H = -\frac{d^2}{dx^2} + q.$$

To keep things simple, I'll assume that q is a real-valued, smooth and *compactly supported* function on $[0, \infty)$, as in Fig. 1. This is too

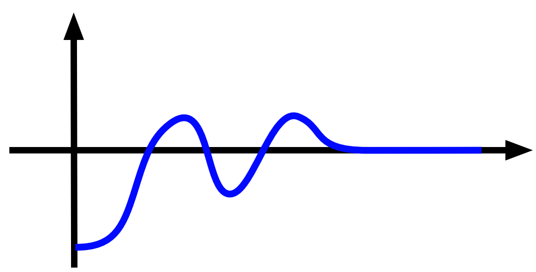

FIGURE 1. A graph of the potential function $q(x)$. For simplicity I'll be examining smooth and compactly supported potentials on $[0, \infty)$. Kodaira's arguments cover much more general potentials, but at the cost of some added complications, and somewhat reduced conclusions.

simplistic for applications. More realistic would be something like the "Coulomb potential"

$$q(x) = \frac{\ell(\ell+1)}{x^2} - \frac{\alpha}{x}.$$

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This arises in the study of the Schrödinger equation for the hydrogen atom (for an explanation, see Higson and Subag (2019)). It's called a "long-range potential," which means that the component α/x doesn't converge to zero fast enough as $x \to \infty$ to avoid analytic troubles. See for instance Yafaev (2010, Chapter 4) for much more information about long-range versus short-range.

Throughout most of these notes I'll be following a beautiful paper of Kodaira (1949), which doesn't shrink from the challenge of handling the difficulties attendant to at least some long-range potentials. I, on the other hand, will take pains to avoid such troubles.

Anyway, of interest here will be the eigenvalue problem

$$F_\lambda = \lambda F_\lambda, \tag{1.1}$$

subject to suitable boundary conditions, which I shall fix in these notes to be

$$F_\lambda(0) = 0 \quad \text{and} \quad F'_\lambda(0) = 1. \tag{1.2}$$

For each $\lambda \in \mathbb{C}$ there is a unique solution.

Thanks to the assumption that q is compactly supported, one can say with confidence that this solution has the form

$$F_\lambda(x) = c(\lambda)\exp(i\sqrt{\lambda}x) + \overline{c(\lambda)}\exp(-i\sqrt{\lambda}x) \qquad \text{for } x \gg 0, \tag{1.3}$$

where $c(\lambda)$ is independent of x but holomorphically dependent on λ. The simple fact expressed by is what makes dealing with compactly supported potentials so much easier than dealing with long range potentials.

The first theorem that I want to discuss is due to Weyl (1910). It explains how to reconstruct a general function on $(0,\infty)$ from its inner products with the eigenfunctions F_λ. Qijun Tan and I studied Weyl's theorem from a certain geometric point of view in Higson and Tan (2020). But in these notes I shall follow the quite different approach of Kodaira.

Theorem (Weyl 1910)**.** *If f is a smooth and compactly supported function on $(0,\infty)$, then*

$$f(x) = \sum_{\lambda<0} \frac{\langle F_\lambda, f\rangle_{L^2}}{\langle F_\lambda, F_\lambda\rangle_{L^2}} F_\lambda(x) + \frac{1}{4\pi}\int_0^\infty \langle F_\lambda, f\rangle_{L^2} F_\lambda(x)\,\frac{1}{|c(\lambda)|^2}\,\frac{d\lambda}{\sqrt{\lambda}},$$

where the sum is over those $\lambda < 0$ for which F_λ is square-integrable.

Remark. Except for this comment, I won't discuss the what type of converge is involved, beyond saying that one can use convergence in $L^2(0,\infty)$. But it's also true that the sum and integral are absolutely convergent, uniformly so on compact subsets of $(0,\infty)$.

To formulate our other theorems, I want to look at the eigenfunctions F_λ just a little differently. For $\lambda > 0$ they can be written as

$$F_\lambda(x) = \text{constant}_\lambda \cdot \sin(kx + \delta(k)) \qquad \text{for } x \gg 0,$$

where the constant in front is positive, and where k is the positive square root of λ. This characterizes $\delta(k) \in \mathbb{R}$ as a function of $k>0$, modulo multiples of 2π. The function δ is called the *scattering phase shift* and it is important (or so I understand) from the physics perspective because it is *measurable*. The objective of *scattering theory*, invented by Wheeler and Heisenberg, is to determine other features of the equations (1.1) and 1.2 from the scattering phase shift.

With that superficially-described background in mind, here is the next result, which Kodaira attributes to Heisenberg (1943a,b, 1944).

Theorem (Heisenberg 1944)**.** *The function*

$$s(k) = \exp(2i\delta(k))$$

defined initially for $k > 0$, has a meromorphic continuation to $\mathbb{C}$ with only simple poles in the upper half-plane. The eigenfunction F_λ is square-integrable if and only if $\lambda < 0$ and $i\sqrt{-\lambda}$ is a pole of the function s.

Remark. I won't do so, but one can say more, for instance about the L^2-norm of F_λ. See Kodaira (1949, Theorem 6.1).

Remark. In more physically realistic contexts (beyond our toy model short range potential) the conclusions tend to be a bit weaker, in that the function $s(k)$ has a meromorphic continuation to $\mathbb{C} \setminus [0, \infty)$, but not necessarily to all of $\mathbb{C}$. Moreover the conclusion of the theorem about the poles of s can sometimes fail: there can be poles of s that do not correspond to eigenvalues. See the commentary and references in Kodaira (1949).

Before stating the final theorem, I need a preliminary result:

Theorem. *The number of negative eigenvalues of H as a self-adjoint operator on $L^2(0, \infty)$, or equivalently the number of square-integrable F_λ, is finite.*

Now let n be the number of negative eigenvalues of H (incidentally, each eigenvalues has multiplicity one).

Theorem (Levinson 1949)**.** *If the eigenfunction F_0 is a bounded function of $x \in [0, \infty)$, then*

$$n\pi = \delta(0) - \delta(\infty).$$

If F_0 is unbounded, then

$$(n + \tfrac{1}{2})\pi = \delta(0) - \delta(\infty).$$

The theorem requires a little explanation, since the scattering phase shift $\delta(k)$ is only defined modulo 2π, and it is not defined at all for $k = 0$ or $k = \infty$. Initially the scattering phase shift is defined as a smooth function $\delta\colon (0,\infty) \to \mathbb{R}/2\pi\mathbb{Z}$. But it may be lifted to a smooth function we can choose δ to be a smooth function $\delta\colon (0,\infty) \to \mathbb{R}$, and then

$$s'(k)/s(k) = 2i\delta'(k),$$

The right hand sides of the formulas in the theorem should be computed using

$$\begin{aligned}\delta(0) - \delta(\infty) &:= \lim_{\varepsilon\to 0}\lim_{R\to\infty}\big(\delta(\varepsilon) - \delta(R)\big)\\ &= \lim_{\varepsilon\to 0}\lim_{R\to\infty}\frac{i}{2}\int_{\varepsilon}^{R} s'(k)/s(k)\,dk.\end{aligned}$$

We shall prove below that the limits exist.

2. The Wronskian

The classical *Wronskian* of a pair of smooth functions of one real variable is the function

$$\mathrm{Wr}(F,G)(x) = F'(x)G(x) - G'(x)F(x).$$

2.1. **Lemma.** *If F and G are eigenfunctions of H for the same eigenvalue, then $\mathrm{Wr}(F,G)$ is a constant function of x.* □

2.2. **Lemma.** *For every $\lambda \in \mathbb{C}$ the Wronskian is a nondegenerate alternating bilinear form on the 2-dimensional space of λ-eigenfunctions for H on $(0,\infty)$.* □

The Wronskian will have several roles to play in what follows. Perhaps most important among them is the following simple fact:

2.3. **Lemma.** *If U_+ and U_- are linearly independent eigenfunctions of H for the same eigenvalue λ, then the morphism*

$$U \longmapsto \big(\mathrm{Wr}(U_+,F), \mathrm{Wr}(U_-,F)\big)$$

is an isomorphism from the space of all λ-eigenfunctions into $\mathbb{C}^2$. □

3. Weyl's Spectral Theorem

The Wronksian also has a small role to play in the following result:

3.1. **Theorem.** *The operator H is essentially self-adjoint on the domain of smooth, compactly supported functions on $[0,\infty)$ that vanish at 0.*

Sketch of the proof. First, if f is smooth and compactly supported on $[0,\infty)$, then for any other smooth function on $[0,\infty)$,

$$\langle Hf, g\rangle - \langle f, Hg\rangle = -\operatorname{Wr}(f,g)(0).$$

Now if H *failed* to be essentially self-adjoint, then according to the theory worked out by von Neumann, there would be a function g in the domain of H^* with $H^*g = \pm ig$, and in particular $Hg = \pm ig$ in the sense of distributions. Such a function, being an eigenfunction would necessarily be smooth on $[0,\infty)$. The display above, and the definition of H^*, would imply that the linear functional

$$f \longmapsto -\operatorname{Wr}(f,g)$$

would be L^2-continuous on the domain of H. But

$$\operatorname{Wr}(f,g) = f'(0)g(0)$$

on the domain of H and continuity would imply $g(0) = 0$, and there is no square-integrable $\pm i$-eigenfunction satisfying this boundary condition. ☐

According the von Neumann's spectral theorem, there is a projection-valued measure on the spectrum of H such that

$$H = \int_{\operatorname{Spec}(T)} \lambda \, dP(\lambda).$$

The following result is a reformulation of Weyl's theorem:

3.2. **Theorem.** *If f is a smooth, compactly supported function on $(0,\infty)$, and if $b>a>0$, then*

$$(P_{[a,b]}f)(x) = \frac{1}{4\pi}\int_a^b \langle F_\lambda, f\rangle_{L^2} F_\lambda(x)\,\frac{1}{|c(\lambda)|^2}\frac{d\lambda}{\sqrt{\lambda}}.$$

To prove this, we start with the formula

$$P_{[a,b]} = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} (\lambda - H)^{-1}\, d\lambda \tag{3.1}$$

from the holomorphic functional calculus, where Γ_ε is the contour in Fig. 2. Initially (3.1) is valid only for operators H for which a and b are excluded from the spectrum, but the resulting formula

$$P_{[a,b]} = \lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_a^b (H-\lambda-i\varepsilon)^{-1} - (H-\lambda+i\varepsilon)^{-1}\, d\lambda \tag{3.2}$$

is actually valid as long as a and b are not eigenvalues. The limit exists in the norm topology, even.

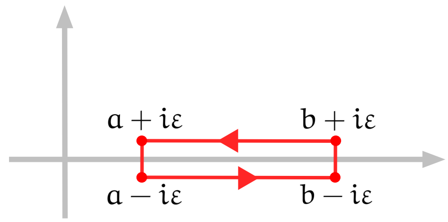

FIGURE 2. The contour Γ_ε for the integral in the Weyl-Kodaira formula.

Theorem 3.2 will be proved by computing the limit in (3.2) when $b > a > 0$. The main step is to introduce certain new λ-eigenfunctions, which will also see duty throughout the rest of the paper.

3.3. **Definition.** For $k \in \mathbb{C}$, denote by U_k the solution to $HU_k = k^2 U_k$ for which

$$U_k(x) = \exp(ikx) \qquad (x \gg 0).$$

Now let $\lambda \in \mathbb{C}$. As long as $\lambda \neq 0$, we obtain from Definition 3.3 two linearly independent λ-eigenfunctions for H from the two square roots k of λ. We shall use the following notation throughout the rest of the notes:

3.4. **Definition.** For $k \in \mathbb{C}$ we shall write

$$a(k) = \mathrm{Wr}(F_\lambda, U_k)$$

where $\lambda = k^2$. This is an entire function of k.

3.5. **Lemma.** *If λ is any nonzero complex number, and if $k^2 = \lambda$, then*

$$F_\lambda = \frac{1}{2ik}\big(a(-k)U_k - a(k)U_{-k}\big).$$

Proof. Certainly we can write

$$F_\lambda = \alpha(k)U_k - \beta(k)U_{-k} \tag{3.3}$$

for some $\alpha(k)$ and some $\beta(k)$. Taking the Wronskian of both sides of (3.3) with U_k and U_{-k} gives

$$\begin{cases} a(k) = -\beta(k)\,\mathrm{Wr}(U_{-k}, U_k) \\ a(-k) = \alpha(k)\,\mathrm{Wr}(U_k, U_{-k}) \end{cases}$$

But by computing for $x \gg 0$ we find immediately that

$$\mathrm{Wr}(U_k, U_{-k}) = 2ik,$$

from which the result follows. $\square$

The lemma implies that

$$F_\lambda(x) = c(\lambda)\exp(i\sqrt{\lambda}x) + \overline{c(\lambda)}\exp(-i\sqrt{\lambda}x) \qquad \text{for } x \gg 0 \tag{3.4}$$

where

$$c(\lambda) = \frac{a(-\sqrt{\lambda})}{2i\sqrt{\lambda}}. \tag{3.5}$$

The following lemma is the crucial step towards the proof of Weyl's theorem:

3.6. **Lemma.** *Assume that* $\nu \notin \operatorname{Spec}(H)$, *that* $\nu = k^2$, *and that* $\operatorname{Im}(k) > 0$. *If* f *is any smooth and compactly supported function on* $(0,\infty)$, *then*

$$\begin{aligned} a(k)\cdot\big((H-\nu)^{-1}f\big)(x) \\ = F_\nu(r)\int_r^\infty U_k(s)f(s)\,ds + U_k(r)\int_0^r F_\nu(s)f(s)\,ds. \end{aligned}$$

Proof. If we denote by $(K_\nu f)(r)$ the right-hand side of the formula in the statement of the lemma, then the formula defines a linear operator K_ν from smooth compactly supported functions on $(0,\infty)$ to the vector space of smooth functions in $\operatorname{dom}(H)$ (here we use the exponential decay of U_k, and hence the assumption $\operatorname{Im}(k) > 0$). We compute directly from the definition of the Wronskian that

$$(H-\nu)K_\nu f = \operatorname{Wr}(F_\nu, U_k)f,$$

The lemma follows. □

Now we apply the lemma with $\nu = \lambda \pm i\varepsilon$, with $\lambda > 0$ and $\varepsilon > 0$. We are going to take a limit as $\varepsilon \to 0+$ of the operator

$$(H-\lambda-i\varepsilon)^{-1} - (H-\lambda+i\varepsilon)^{-1},$$

and since by the previous lemma the expression for this operator involves $a(k)^{-1}$, it will be helpful to know the following fact:

3.7. **Lemma.** *If* $k \in \mathbb{R}$ *and* $k \neq 0$, *then* $a(k) \neq 0$.

Proof. Examine the formula in the Lemma 3.5. If $k \in \mathbb{R}$ and if $\lambda = k^2$, then F_λ is a real-valued function, and as a result $\overline{a(k)} = a(-k)$. So if $a(k)$ was zero, then F_λ would be zero, too, which it isn't. □

We can begin our computation of the limit. If f is smooth and compactly supported on $(0,\infty)$, then

$$\lim_{\varepsilon\to 0+}\big((H-\lambda-i\varepsilon)^{-1}f - (H-\lambda+i\varepsilon)^{-1}f\big)(x)$$

$$= F_\lambda(x)\int_x^\infty \tfrac{1}{a(k)}U_k(y)f(y)\,dy + \tfrac{1}{a(k)}U_k(x)\int_0^x F_\lambda(y)f(y)\,dy$$
$$+ F_\lambda(x)\int_x^\infty \tfrac{1}{a(-k)}U_{-k}(y)f(y)\,dy + \tfrac{1}{a(-k)}U_{-k}(x)\int_0^x F_\lambda(y)f(y)\,dy,$$

where k is the positive square root of λ. Taking into account Lemma 3.5 we obtain

$$\lim_{\varepsilon\to 0+}\big((H-\lambda-i\varepsilon)^{-1}f-(H-\lambda+i\varepsilon)^{-1}f\big)(x) = \frac{2i\sqrt{\lambda}}{|a(\sqrt{\lambda})|^2}F_\lambda(x)\int_0^\infty F_\lambda(y)f(y)\,dy$$

We find that

$$(P_{[\alpha,\beta]}f)(x) = \frac{1}{\pi}\int_\alpha^\beta\int_0^\infty F_\lambda(x)F_\lambda(y)f(y)\,dy\,\frac{\sqrt{\lambda}}{|a(\sqrt{\lambda})|^2}d\lambda$$

or equivalently

$$(P_{[\alpha,\beta]}f)(x) = \frac{1}{4\pi}\int_\alpha^\beta\int_0^\infty F_\lambda(x)F_\lambda(y)f(y)\,dy\,\frac{1}{|c(\lambda)|^2}\frac{d\lambda}{\sqrt{\lambda}}$$

as required.

4. Heisenberg's Theorem

In this section I shall take advantage of the fact that a is an entire function of k (although a weaker hypothesis would suffice, as in Kodaira (1949)).

4.1. **Lemma.** *If* $\mathrm{Im}(k)>0$*, then* $a(k)=0$ *if and only if* k *is purely imaginary and* $\lambda = k^2$ *is a (negative) eigenvalue of* H.

Proof. The proof will use the formula

$$F_\lambda(x) = \frac{1}{2ik}\big(a(-k)\exp(ikx) - a(k)\exp(-ikx)\big)$$

for $\lambda = k^2$, plus the facts that $\exp(ikx)$ is exponentially decreasing when $\mathrm{Im}(k)>0$, while $\exp(-ikx)$ is exponentially increasing. It follows from the formula that for $\mathrm{Im}(k)>0$,

$$a(k) = 0 \quad\Leftrightarrow\quad F_\lambda \in \mathrm{dom}(H)$$

(I mean here the domain of the closure of H, on which H is self-adoint). Now the only eigenvalues of H are real numbers, and the only eigenvalues with positive imaginary square roots area negative real numbers. The lemma follows. □

4.2. **Lemma.** *Every zero of* $a(k)$ *in the upper half-plane is a simple zero.*

Proof. This follows from the formula

$$\big((H-\nu)^{-1}f\big)(x) = a(k)^{-1}\Big(F_\nu(r)\int_r^\infty U_k(s)f(s)\,ds + U_k(r)\int_0^r F_\nu(s)f(s)\,ds\Big)$$

and the fact that $\nu \mapsto (H-\nu)^{-1}$ has a simple pole at the (isolated) negative eigenvalues of H. Now the only eigenvalues of H (if there are any at all) are negative real numbers. □

4.3. **Lemma.** *If $k \neq 0$, and if $a(k)=0$, then $a(-k) \neq 0$.*

Proof. It follows from the formula

$$F_\lambda(x) = \frac{1}{2ik}\big(a(-k)\exp(ikx) - a(k)\exp(-ikx)\big)$$

that $a(k)$ and $a(-k)$ cannot simultaneously be zero. □

4.4. **Lemma.** $a(\overline{k}) = \overline{a(-k)}$.

Proof. If k is real and nonzero, and if $\lambda = k^2$, then from

$$F_\lambda(x) = \frac{1}{2ik}\big(a(-k)\exp(ikx) - a(k)\exp(-ikx)\big)$$

and the fact that F_λ is real-valued, it follows that $a(k) = \overline{a(-k)}$. In other words

$$k \in \mathbb{R} \quad \Rightarrow \quad a(k) = \overline{a(-\overline{k})}.$$

The lemma follows by analytic continuation of this formula from $\mathbb{R}$ to $\mathbb{C}$. □

4.5. **Theorem.** *The function $s(k) = \exp(2i\delta(k))$ has a meromorphic continuation from $(0,\infty)$ to $\mathbb{C}$. If $\mathrm{Im}(k) > 0$, then k is a pole of s if and only if k is purely imaginary and k^2 is a (negative) eigenvalue of H.*

Proof. It follows from Lemma 4.4 that if $k > 0$, then

$$a(-k)/a(k) = \exp(2i\delta(k)).$$

The left-hand side of this identity provides the meromorphic continuation. The remaining assertions in the theorem follow from Lemmas 4.1, 4.2 and 4.3. □

5. Levinson's Theorem

Here is our starting point:

5.1. **Theorem.** *The number of negative eigenvalues of H as a self-adjoint operator on $L^2(0,\infty)$, or equivalently the number of square-integrable F_λ, is finite.*

Proof. We saw that λ is a negative eigenvalue of H if and only if $\lambda = k^2$, with k positive-imaginary and $a(k) = 0$. We also saw that the negative eigenvalues are bounded from below, since H is bounded from below. If there were an infinite number of negative eigenvalues, there would be an infinite number of zeros of the nonzero entire function a on a bounded interval on the positive imaginary axis. This is impossible. $\square$

With that, here is our objective, repeated from the Introduction:

5.2. **Theorem.** *Let n be the number of negative eigenvalues of H. If the eigenfunction F_0 is a bounded function of $x \in [0, \infty)$, then*

$$n\pi = \delta(0) - \delta(\infty).$$

If F_0 is unbounded, then

$$(n + \tfrac{1}{2})\pi = \delta(0) - \delta(\infty).$$

Recall that I am writing

$$\delta(\infty) - \delta(0) := \lim_{\varepsilon \to 0+} \lim_{R \to \infty} \frac{1}{2i} \int_{\varepsilon}^{R} s'(k)/s(k)\, dk.$$

To prove Levison's theorem, we are going to count the number of zeros of $a(k)$ in the upper half-plane, and hence the number of negative eigenvalues of H, using complex analysis techniques.

Consider a contour like the one in Fig. 3 that includes all the zeros of a. Since all the poles are simple, and since there are no zeros of s within the contour, it follows from the argument principle that

$$\text{number of negative eigenvalues} = \frac{-1}{2\pi i} \int_{\Gamma} a'(k)/a(k)\, dk. \tag{5.1}$$

We shall compute the contour integral in three stages.

Horizontal Components. If the small arc in Fig. 3 has radius ε and the large arc has radius R, then the contribution to the integral in (5.1) from the part of the contour along the x-axis is

$$-\frac{1}{2\pi i} \int_{-R}^{-\varepsilon} a'(k)/a(k)\, dk - \frac{1}{2\pi i} \int_{\varepsilon}^{R} a'(k)/a(k)\, dk$$

or in other words

$$-\frac{1}{2\pi i} \int_{\varepsilon}^{R} a'(k)/a(k) - a'(-k)/a(-k)\, dk$$

Now for k real,

$$s(k) = a(-k)/a(k),$$

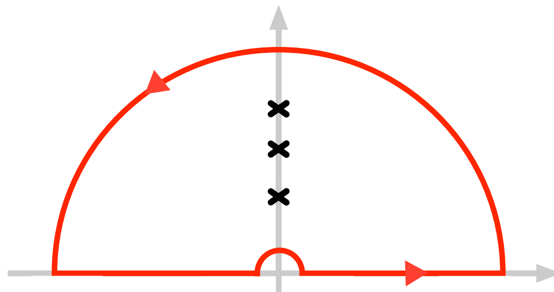

FIGURE 3. The contour Γ for the proof of Levinson's theorem. The radius of the outer arc is very big, and the radius of the inner arc is very small.

and so by a little calculus,

$$s'(k)/s(k) = -a'(-k)/a(-k) - a'(k)/a(k),$$

and therefore the contribution to the right-hand side in (5.1) from the part of the contour along the x-axis is

$$-\frac{1}{2\pi i}\int_{\varepsilon}^{R} s'(-k)/s(k)\,dk,$$

which, according to our conventions, is $\frac{1}{\pi}(\delta(\varepsilon) - \delta(R))$.
Small Semicircular Arc. It follows from the above that the number of negative eigenvalues is

$$\frac{1}{\pi}(\delta(\varepsilon) - \delta(R)) \\ - \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} a'(k)/a(k)\,dk - \frac{1}{2\pi i}\int_{\Gamma_R} a'(k)/a(k)\,dk, \quad (5.2)$$

where Γ_ε and Γ_R are the small and large semicircular arcs in the contour Γ, respectively. We'll now tackle the integral over Γ_ε.

5.3. **Lemma.** *The eigenfunction F_0 is bounded if and only if $a(0) = 0$.*

Proof. If $a(0) = 0$, then from

$$U_0 = a(0)G_0 - b(0)F_0$$

we find that F_0 is a multiple of U_0, and is hence bounded. If F_0 is bounded, then G_0 cannot be bounded, since the space of 0-eigenfunctions is 1-dimensional. So it follows from

$$a(0)G_0 = U_0 + b(0)F_0$$

that necessarily $a(0) = 0$. ◻

So 0 is a pole if F_0 is bounded, and is a regular point otherwise. By a standard complex analysis computation,

$$\lim_{\varepsilon\to 0+} \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} a'(k)/a(k)\, dk = \begin{cases} \frac{1}{2} & \text{if } F_0 \text{ is bounded} \\ 0 & \text{if } F_0 \text{ is unbounded} \end{cases}$$

Large Semicircular Arc. The remaining task is to show that the integral over the large arc contributes nothing (in the limit as $R \to \infty$) to the formula (5.2). This is a consequence of the following result:

5.4. **Proposition.** *Within the region* $\mathrm{Im}(k) \geq 0$ *we have*

$$\lim_{k\to\infty} a(k) = 1.$$

I believe that from a physics point of view this is supposed to be clear. High energy waves will be scattered little to not at all by the potential q, so that the potential might as well be zero. But when $q = 0$, we have

$$\begin{aligned} F_\lambda(x) &= 1/\sqrt{\lambda}\sin(\sqrt{\lambda}x) \\ &= \frac{1}{2ik}\exp(ikx) - \frac{1}{2ik}\exp(-ikx), \end{aligned}$$

where $k^2 = \lambda$. Now compare with Lemma 3.5.

In any case, it follows directly from the proposition that

$$\frac{1}{2\pi i}\int_{\Gamma_R} a'(k)/a(k)\, dk = 0$$

since we can write $a'(k)/a(k) = d/dk \log(a(k))$ for the principal branch of log and then use the antiderivative $\log(a(k))$ to evaluate the integral. Putting everything together, we obtain a proof of Theorem 5.2.

To give a mathematical proof of Proposition 5.4 we shall use the following interesting formula for $a(k)$:

$$a(k) = 1 + \int_0^\infty \exp(iky)q(y)F_{k^2}(y)\, dy \tag{5.3}$$

This comes from the following computation:

5.5. **Lemma.** *If* $\lambda \neq 0$ *and* $k^2 = \lambda$, *then*

$$F_\lambda(x) = \frac{1}{k}\sin(kx) - \frac{1}{k}\int_0^x \sin(k(x-y))q(y)F_\lambda(y)\, dy$$

for every $x \in [0, \infty)$.

Proof. This formula is obtained by the method of "variation of constants." But no matter how it is obtained, it can be checked as follows: the difference between the left-hand side and the right-hand side has value 0 and derivative 0 at $x=0$, and the second derivative of the difference is zero everywhere. □

As for the formula (5.3), it follows from the lemma and the computation

$$\begin{aligned}\frac{1}{k}\int_0^x \sin(k(x-y))q(y)F_\lambda(y)\,dy \\ = \frac{1}{2ik}\exp(ikx)\int_0^x \exp(-iky)q(y)F_\lambda(y)\,dy \\ + \frac{1}{2ik}\exp(-ikx)\int_0^x \exp(iky)q(y)F_\lambda(y)\,dy.\end{aligned}$$

Note that the integrals on the right hand side are independent of $x \gg 0$, since q is compactly supported.

Finally, Proposition 5.4 is proved using (5.3) and the following estimate for $F_\lambda(x)$ whose prove I'll omit, except to say that it simply uses the definition of F_λ as an H-eigenfunction: if $k^2 = \lambda$, with $\mathrm{Im}(k) \geq 0$, then

$$|F_\lambda(x)| \leq \frac{K}{|\lambda|}|\exp(-ik)|.$$

The constant is independent of λ in the region $|\lambda| \geq 1$. See Levinson (1949, Lemma 2.0).

Appendix A. Transmission Through a Barrier

Here are some calculations for Jonathan on the Scrhödinger equation for a potential barrier. For the most part they're from a first course in quantum mechanics (I learned this from Dicke and Wittke (1960)). The point is to determine the sign of δ and its magnitude in relation to the size of the potential.

$$-u_{xx} + Vu = k^2u$$

$$V(x) = bW(x) \qquad (b \in [0,1])$$

$$-u_{xxb} + Wu + bWu_b = k^2u_b$$

$$\begin{cases} -\overline{u}_b u_{xx} + \overline{u}_b V u = k^2\overline{u}_b u \\ -\overline{u}_{xxb}u + W\overline{u}u + V\overline{u}_b u = k^2\overline{u}_b u \end{cases}$$

Subtracting the second equation from the first yields

$$\overline{u}_{xxb}u - \overline{u}_b u_{xx} = V|u|^2,$$

which I can write as

$$(\overline{u}_{xb}u - \overline{u}_b u_x)_x = V|u|^2.$$

It follows that

$$\Big[\overline{u}_{xb}u - \overline{u}_b u_x\Big]_{-N}^{N} = \int_{-N}^{N} V\,|u|^2\,dx$$

$$v(x) = \exp(ikx) \qquad (x \gg 0)$$

$$v(x) = c(k)\exp(ikx) + c(-k)\exp(-ikx) \qquad (x \ll 0)$$

$$u(x) = \begin{cases} \exp(ikx) + \alpha(k)\exp(-ikx) & x \ll 0 \\ \beta(k)\exp(ikx) & x \gg 0 \end{cases}$$

$$u(x) = \begin{cases} \exp(ikx) + \rho\exp(-ik(x+\varepsilon)) & x \ll 0 \\ \tau\exp(ik(x+\delta)) & x \gg 0 \end{cases}$$

$$\begin{aligned} u(x) &= \exp(ikx) + \rho\exp(-ik(x+\varepsilon)) \\ \overline{u}_b(x) &= \rho_b\exp(ik(x+\varepsilon)) + ik\rho\varepsilon_b\exp(ik(x+\varepsilon)) \\ u_x(x) &= ik\exp(ikx) - ik\rho\exp(-ik(x+\varepsilon)) \\ \overline{u}_{bx}(x) &= ik\rho_b\exp(ik(x+\varepsilon)) - k^2\rho\varepsilon_b\exp(ik(x+\varepsilon)) \end{aligned}$$

$$\overline{u}_{xb}(-N)u(-N) - \overline{u}_b(-N)u_x(-N) = 2ik\rho\rho_b - 2k^2\rho^2\varepsilon_b$$

$$\begin{aligned} u(x) &= \tau\exp(ik(x+\delta)) \\ \overline{u}_b(x) &= \tau_b\exp(-ik(x+\delta)) - ik\delta_b\tau\exp(-ik(x+\delta)) \\ u_x(x) &= ik\tau\exp(ik(x+\delta)) \\ \overline{u}_{bx}(x) &= -ik\tau_b\exp(-ik(x+\delta)) - k^2\delta_b\tau\exp(-ik(x+\delta)) \end{aligned}$$

$$\overline{u}_{xb}(N)u(N) - \overline{u}_b(N)u_x(N) = -2ik\tau\tau_b - 2k^2\tau^2\delta_b$$

$$\begin{aligned} \Big[\overline{u}_{xb}u - \overline{u}_b u_x\Big]_{-N}^{N} &= -2ik\tau\tau_b - 2k^2\tau^2\delta_b - (2ik\rho\rho_b - 2k^2\rho^2\varepsilon_b) \\ &= -2ik(\tau\tau_b + \rho\rho_b) - 2k^2(\tau^2\delta_b - \rho^2\varepsilon_b) \end{aligned}$$

$$\tau^2 + \rho^2 = 1$$

$$\tau^2\delta_b - \rho^2\varepsilon_b = -\tfrac{1}{2k^2}\langle u, Vu\rangle$$

Feynman, Leighton, and Sands (1963, Sec. 31) and Wellner (1964) and Dicke and Wittke (1960)

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