Integral Equations from Hilbert and Weyl to Connes and Beyond

Kemeny Lectures, Dartmouth College

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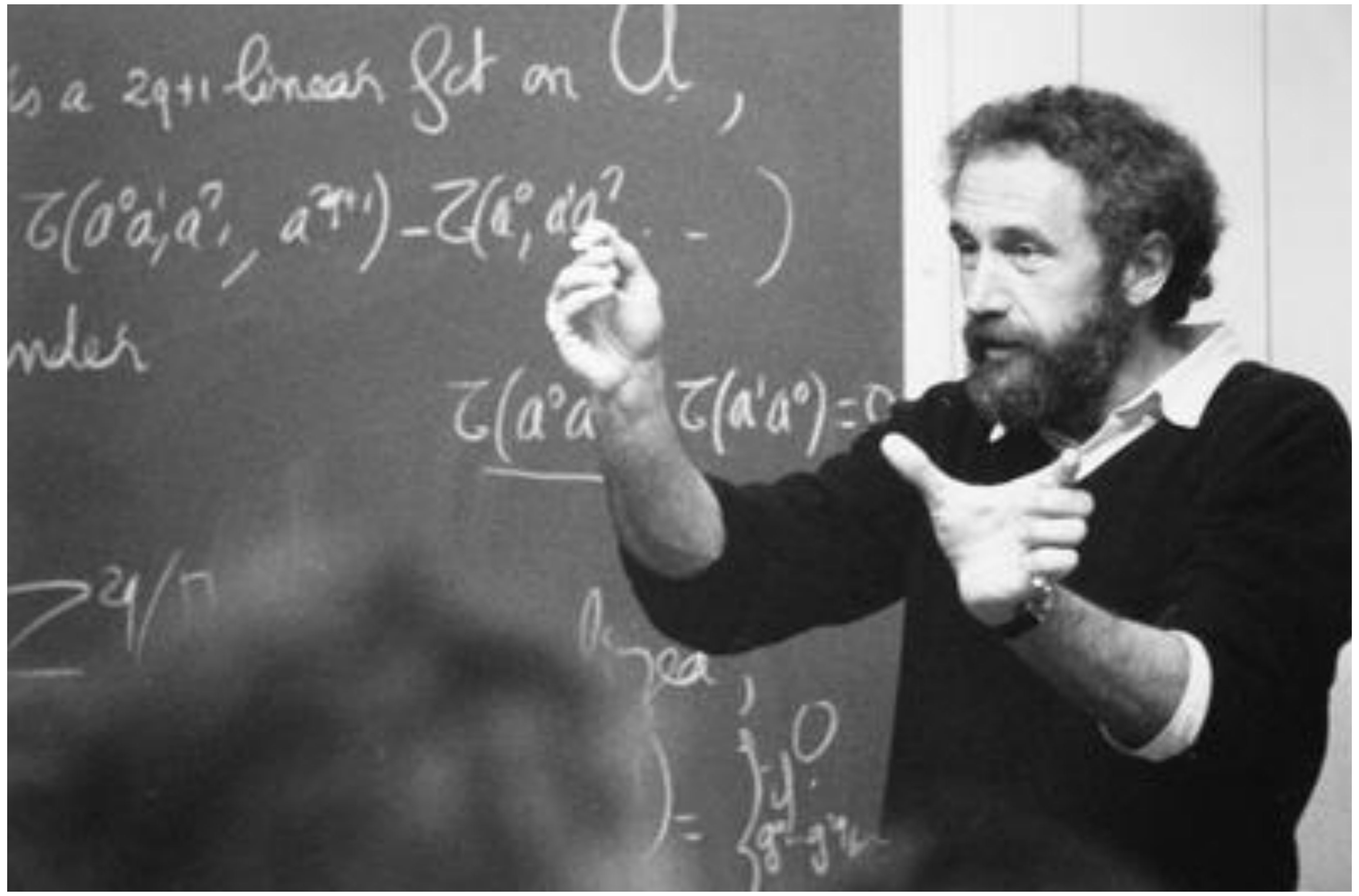
Noncommutative geometry

I shall give a view of Alain Connes' noncommutative geometry through the lens of David Hilbert's work on integral equations from more than 100 years ago.

Connes' theory started in the late 1970’s from simple elaborations of the theory of integral equations that Hilbert created. It has grown immensely since then, and as a result this introduction will be a bit one-sided. But there is still plenty to see from this perspective.

After discussing some background ideas in spectral geometry, I shall discuss Connes’ work with foliations, before returning to ideas very close to those of Hilbert, but now with geometry and not just spectral theory in sight.



Oberwolfach collection

Weyl on Hilbert

IAS collection

A great master of mathematics passed away when David Hilbert died in Göttingen on February the 14th, 1943, at the age of eighty-one. In retrospect, it seems to us that the era on which he impressed the seal of his spirit and which is now sinking below the horizon achieved a more perfect balance than prevailed before and after, between the mastering of single concrete problems and the formulation of general, abstract concepts.

Hermann Weyl
David Hilbert and his mathematical work

Hilbert's spectral theorem for integral operators

In the winter of 1900-1901 the Swedish mathematician E. Holmgren reported in Hilbert's seminar on Fredholm's first publications on integral equations, and it seems that Hilbert caught fire at once...

Hilbert saw with fresh eyes that:

1. The equation $\Delta f = \lambda f$ may be changed into an integral equation

$$f(x) = \lambda \int k(x, y) f(y)\, dy$$

with self-adjoint integral kernel $k(x, y) = \overline{k(y, x)}$, and

2. The problem of determining the solutions is essentially the problem of determining the eigenvalues and eigenvectors of a self-adjoint matrix $[k_{ij}]$.

Self-adjoint compact operators

If others saw the same, Hilbert saw it at least that much more clearly that he bent all his energy on problem that proposition…

By now, the proof of Hilbert's spectral theorem has been polished to a fine shine …

1. A Hilbert space operator is *compact* if it maps the closed unit ball into a compact set. It follows from the Arzela-Ascoli theorem that every integral operator with continuous, compactly supported integral kernel is compact.

2. If K is a compact operator, then the scalar function $\langle Kf, f \rangle$ on the closed unit ball is continuous, even if the ball is equipped with the weak topology in which it is compact.

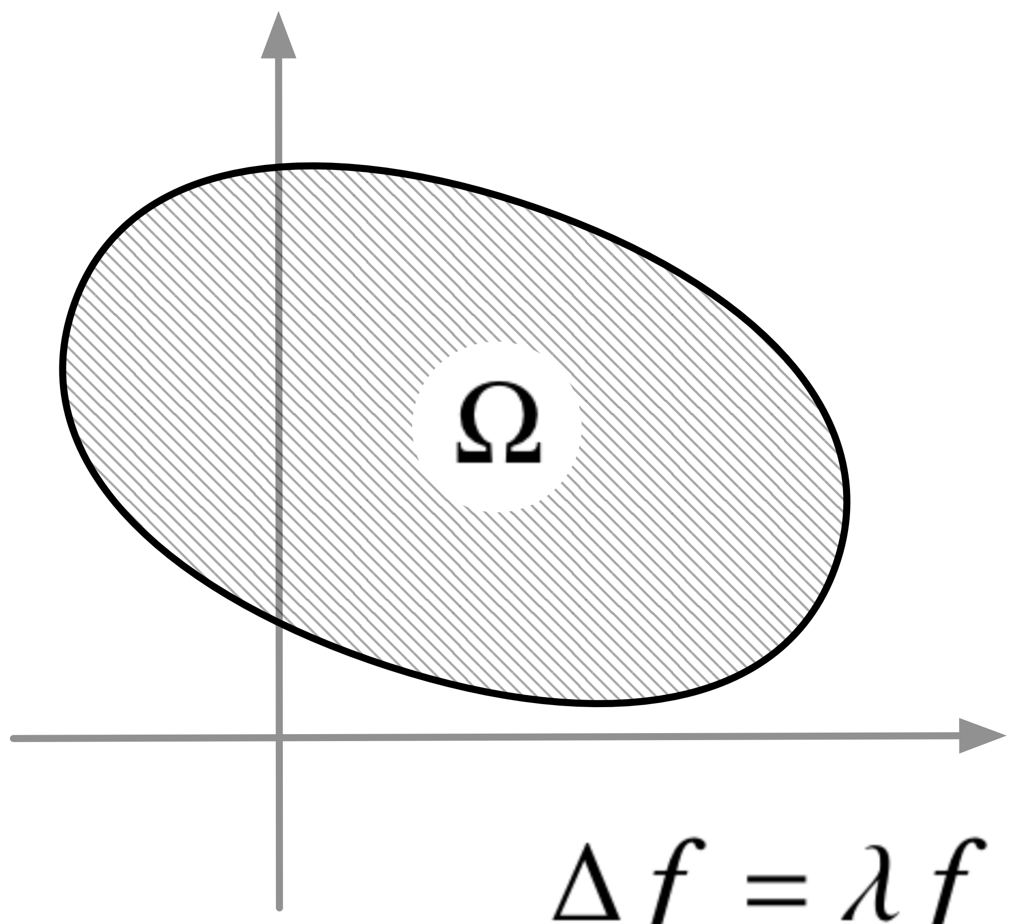
3. So there exists f in the unit ball at which $\langle Kf, f \rangle$ takes an extreme value. And it may be readily checked that if K is self-adjoint, then this f is an eigenvector.

Weyl’s asymptotic law

… in the terrain of analysis a rich vein of gold had been struck, comparatively easy to exploit and not soon to be exhausted.

Weyl lists singular integral kernels, the Riemann-Hilbert problem, the Birkhoff decomposition, representations of compact groups, Hodge theory and more.

Here is a famous application due to Weyl himself.



$$\Delta f = \lambda f$$
$$f\big|_{\partial\Omega} = 0$$

Weyl’s Asymptotic Law. *Write the eigenvalues for the Helmholtz problem in increasing order:*

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

Then

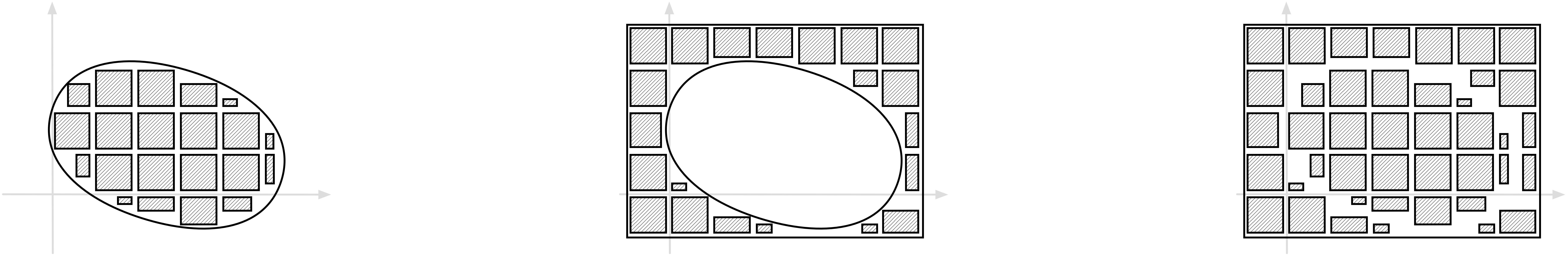
$$\lambda_k \sim \frac{4\pi k}{\text{Area}(\Omega)}$$

Weyl's proof of the asymptotic law

For a pupil of Hilbert around 1910 it was natural to visualize the question as one concerning integral equations …

Hermann Weyl
Ramifications of the eigenvalue problem, old and new

Weyl's proof hinges on two things. First, the formula can be checked directly for a rectangle. Second, if $\Omega_1 \subseteq \Omega_2$ then $\lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$. This is a consequence of the operator inequality $\Delta_{\Omega_1}^{-1} \leq \Delta_{\Omega_2}^{-1}$. The rest of the proof is by pictures.



Foliations and algebras of integral operators

The story would be dramatic enough had it ended there, but then a sort of miracle happened …

Weyl is referring to quantum mechanics here, but the words apply equally well to developments spearheaded by Alain Connes, starting in the late 1970's.

Connes was awarded the Fields Medal for his work on von Neumann algebras, and began to examine foliations in connection with that.

His simple idea: build *foliation algebras* of integral operators from integral kernels defined on pairs (x, y) for which x and y belong to the same leaf using

$$(k_1 \star k_2)(x, y) = \int_L k_1(x, z) k_2(z, y)\, dz \qquad (x, y \in L)$$

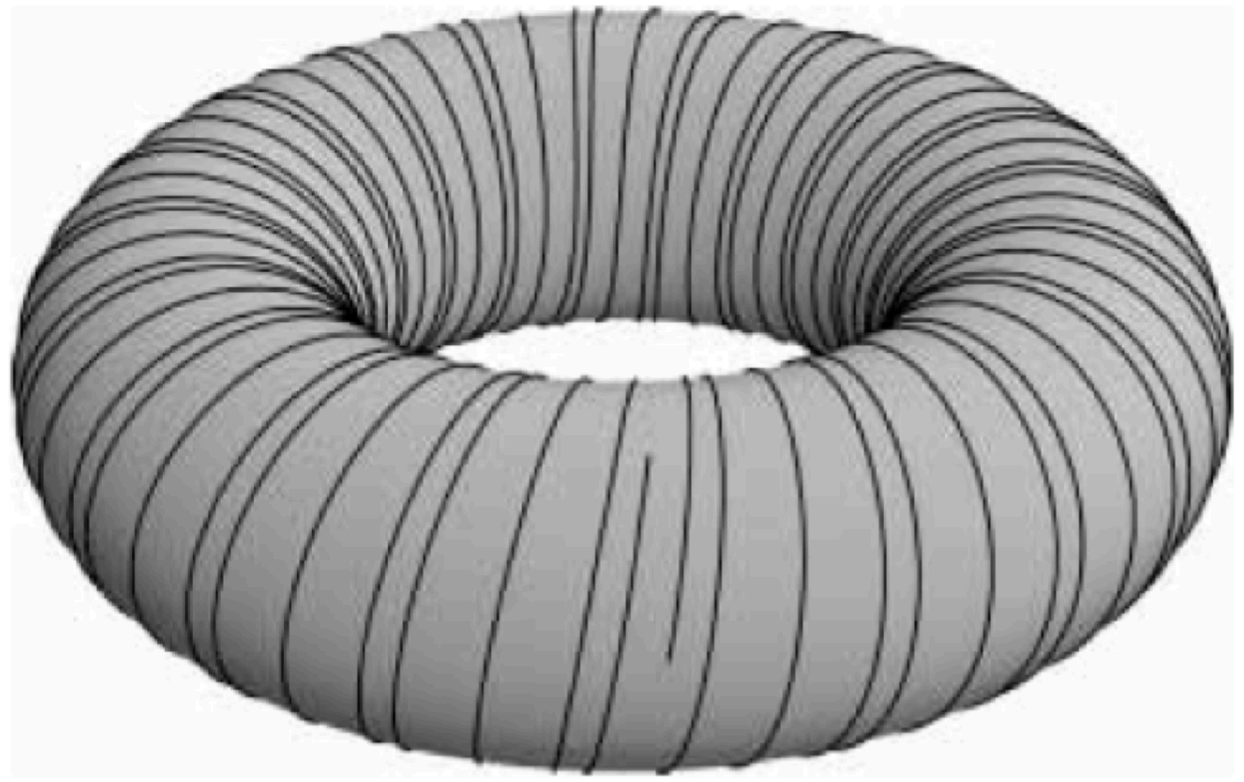


Image credit: Walter van Suijlekom

Foliation algebras: a few more details

Of course, the formula

$$(k_1 \star k_2)(x,y) = \int_L k_1(x,z)k_2(z,y)\,dz \qquad (x,y \in L)$$

is modeled on the composition formula for Hilbert's integral operators. But where do the new "operators" act?

Often that question is of secondary importance, since often the primary object of interest is the foliation algebra itself

But there are operators on the L^2-Hilbert spaces of each of the leaves. Using them, the foliation algebra can be completed to obtain a C*-algebra or a von Neumann algebra.

Foliations from group actions

The Kronecker foliation in the picture comes from Lie theory. It is the foliation of the torus T, which is a Lie group, by cosets of a one-parameter subgroup H.

If a pair (x, y) of points belong to the same leaf, then $(x, y) = (ht, t)$ for some unique $t \in T$ and $h \in H$.

So an integral kernel $k(x, y)$ defined only one pairs (x, y) of points in the same leaf is the same thing as a function of pairs $(h, t) \in H \times T$.

This makes it easy to precisely define continuity for $k(x, y)$, compact support, and so on.

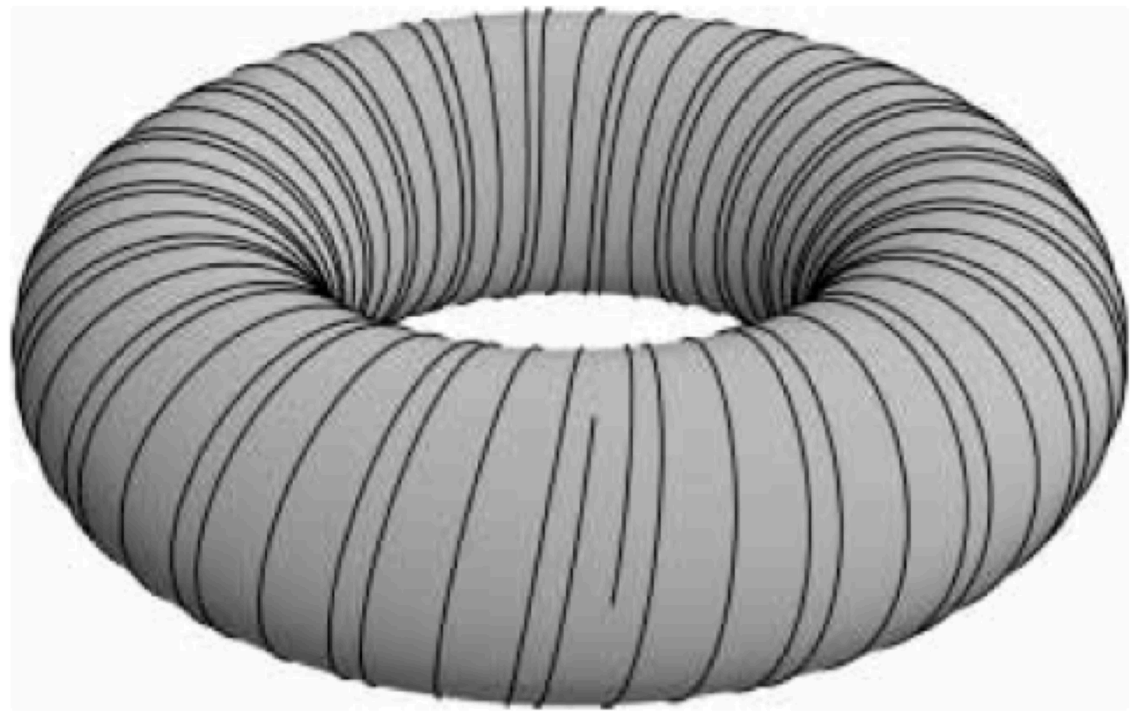


Image credit: Walter van Suijlekom

Integral kernels associated to a group action

Suppose that a Lie group G acts on a manifold X.

If the action is faithful, then the group element g can be dropped from the notation

Form the space of triples

$$\{(x, g, y) : g \cdot y = x\}$$

If it is also transitive, then x and y are independent

The formula

$$(k_1 \star k_2)(x, g, z) = \int_{g_1 g_2 = g} k_1(x, g_1, y) k_2(y, g_2, z)$$

gives a convolution law for continuous and compactly supported integral kernels.

So every Lie group action gives a new algebra of integral operators. Even the trivial action on a point is extremely interesting from this point of view …

The noncommutative torus

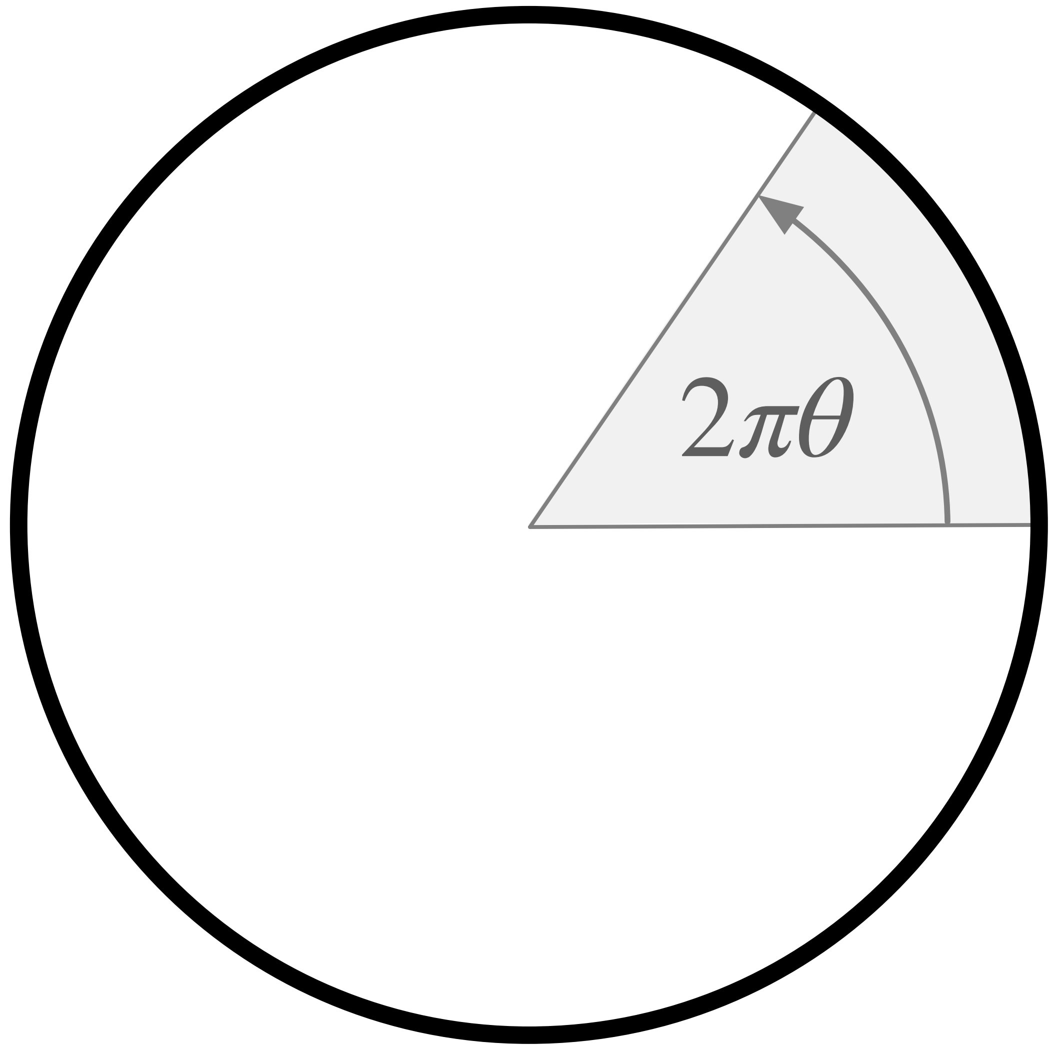
This is the algebra A_θ of integral operators associated to the action of the integers on the circle by rotation through multiples of a fixed multiple θ of 2π.

Generators U and V are defined by

$$u(x, n, y) = \begin{cases} 1 & n = 1 \\ 0 & \text{else} \end{cases}$$

and

$$v(x, n, y) = \begin{cases} e^{2\pi i x} & n = 0 \\ 0 & \text{else} \end{cases}$$



They obey the relation

$$UV = e^{2\pi i\theta} VU$$

The noncommutative torus has a long history …

Dimensions in the noncommutative torus

Following a theme in the previous lecture, consider the shape of the space of all operators in A_θ with spectrum $\{0,1\}$.

For the algebra of $n \times n$ matrices, the components of this space are indexed by the dimension $1,2,\ldots,n-1$ of the 0-eigenspace. So it's reasonable to use the term dimensions for the set of components.

The dimensions of A_θ are (labeled by) those positive real numbers between 0 and 1 that are of the form $m + n\theta$ with m and n integral, possibly negative.

This is a difficult result! At one point it was conjectured that there weren't any operators in A_θ with spectrum $\{0,1\}$.

Spectrum in the noncommutative torus

The complexity of the dimensions for A_θ is reflected in the complexity of the spectra of its elements, because gaps in the spectrum can be labeled by dimensions.

For example, consider the element $U + V + U^* + V^*$, which is called Harper's operator.

The spectrum of Harper's operator was famously studied by Hofstadter, who numerically plotted the Hofstadter butterfly, illustrating the spectrum for varying θ.

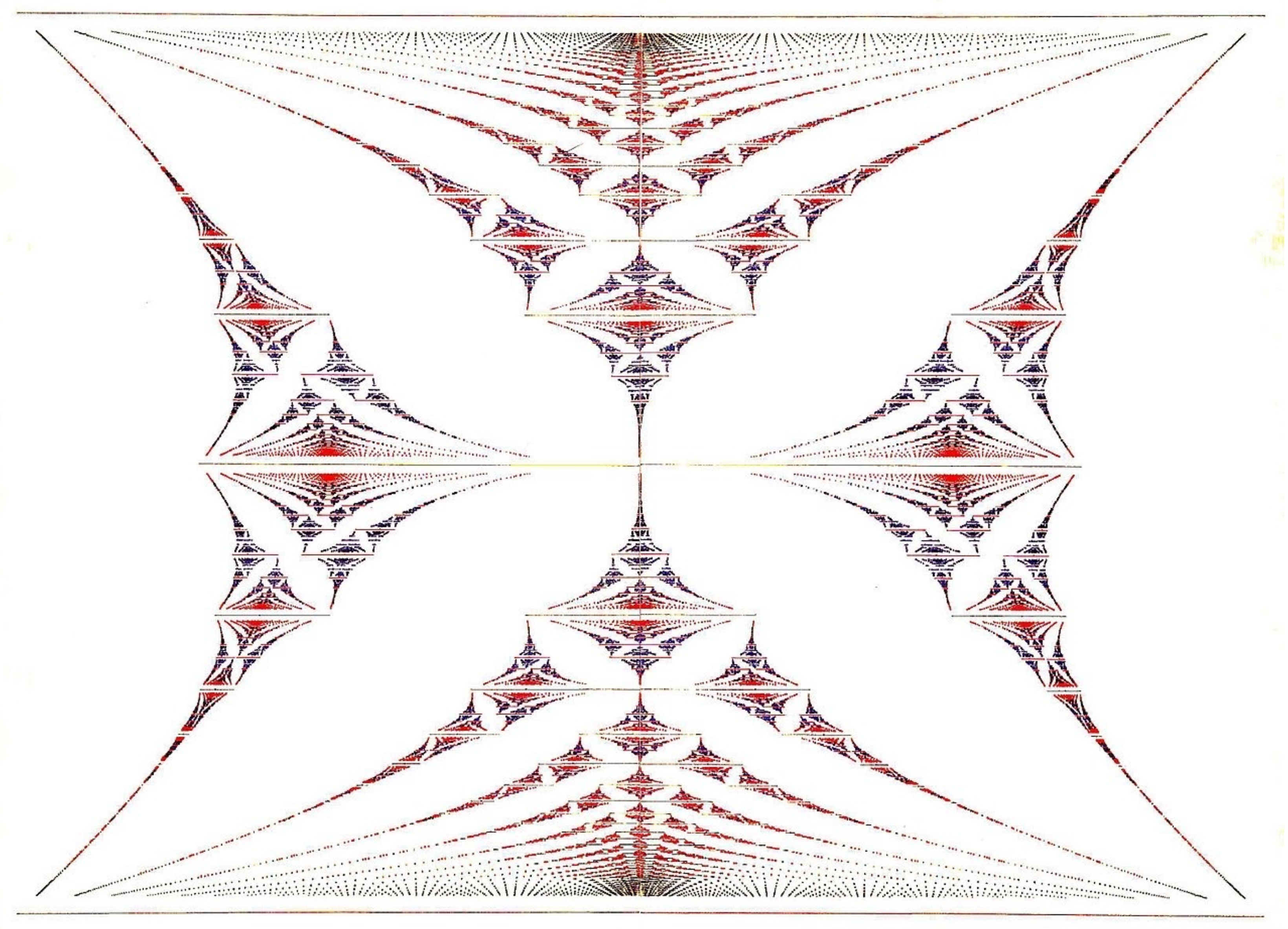


Image credit: Wikipedia

Triples from group actions and groupoids

The convolution integral

$$(k_1 \star k_2)(x, g, z) = \int_{g_1 g_2 = g} k_1(x, g_1, y) k_2(y, g_2, z)$$

reflects an obvious "multiplication law" on the space of triples

$$\{(x, g, y) : g \cdot y = x\},$$

namely

$$(x, g_1, y) \cdot (y, g_2, z) = (x, g_1 g_2, z).$$

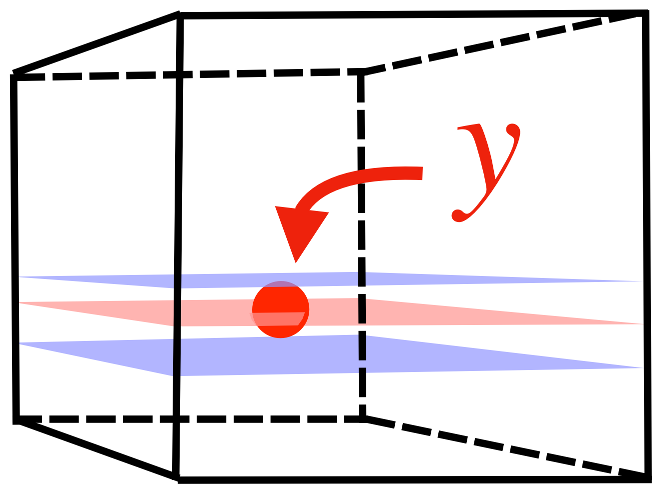
This gives the space of triples the structure of a Lie groupoid. There is a convolution algebra associated to each Lie groupoid.

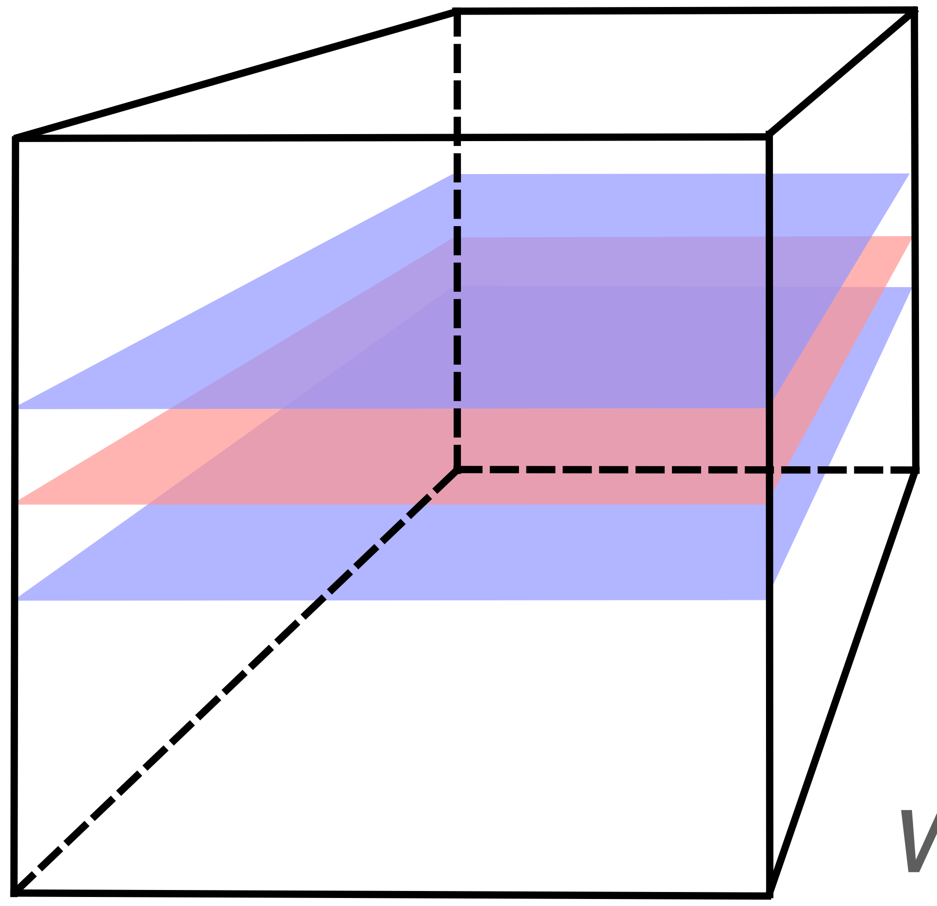
The groupoid of a foliation

The groupoid perspective solves a problem with foliations.

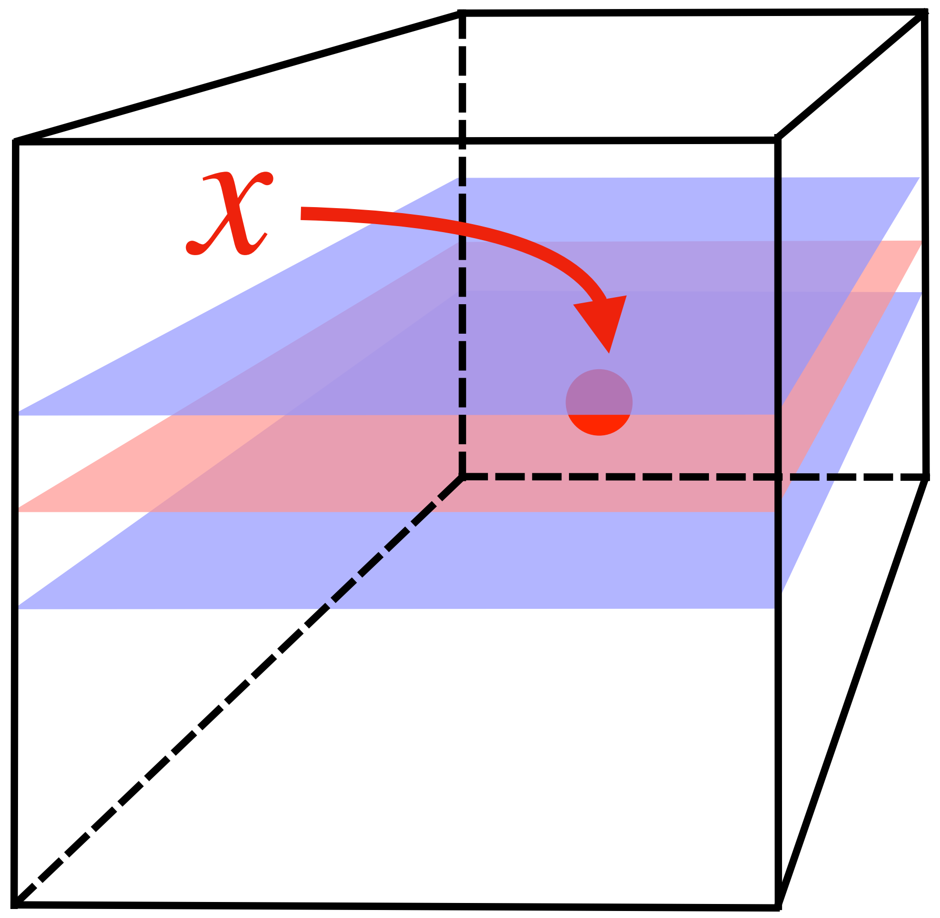
Each foliation chart may include (infinitely) many parts (called plaques) of the same leaf. How do you correspond the parts in one chart with those in another?

This is necessary to assemble all of Connes' pairs (x, y) into a manifold.





A foliation chart with two plaques from the same leaf



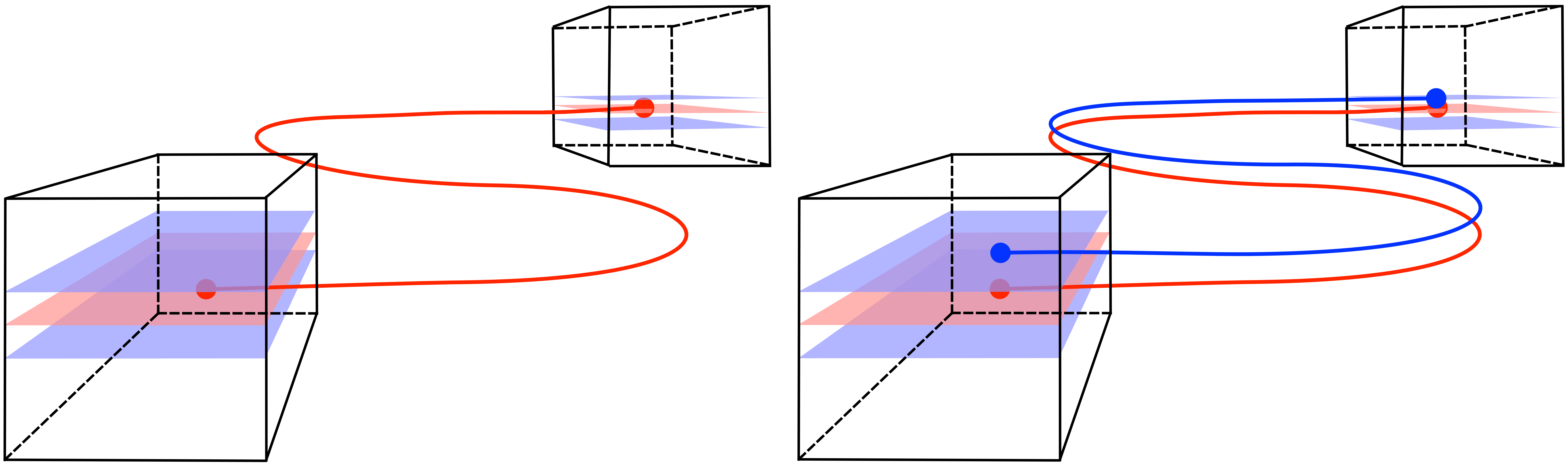
What pairs (x', y') are near the indicated pair (x, y)?

The groupoid of a foliation

How do you correspond the plaques in one chart with those in another?

By specifying a leafwise path γ from x to y (up to equivalence).

The foliation groupoid, a Lie groupoid, is the space of all triples (x, γ, y).



Self-adjoint operators and the von Neumann symbol

I shall now examine not integral operators, but (linear partial) differential operators.

The symmetry condition $\langle Dv, w\rangle = \langle v, Dw\rangle$ is not enough to prove a spectral theorem.

But von Neumann showed that the existence of the resolvent operators $(D \pm iI)^{-1}$ is both necessary and sufficient for spectral theory. (This is automatic in finite dimensions, for compact operators, etc.)

Self-adjointness in von Neumann's sense gives rise to a von Neumann symbol morphism

$$C_0(\mathbb{R}) \longrightarrow B(L^2(M))$$

that sends the resolvent *functions* $(x \pm i)^{-1}$ to the resolvent *operators* $(D \pm iI)^{-1}$.

Elliptic operators

Making an operator self-adjoint in von Neumann's sense can be a tricky business. But elliptic operators, like Δ, on compact manifolds are easy to handle.

Moreover for these the von Neumann symbol has the form

$$C_0(\mathbb{R}) \longrightarrow K(L^2(M))$$

where the target is the compact operators.

This was Hilbert's observation about the conversion of the Helmholtz eigenvalue problem into an integral operator eigenvalue problem.

The principal symbol of an partial differential operator

In local coordinates the differential operator D will have the form

$$D = \sum_{|\alpha| \leq q} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$$

In partial differential equations one associates to D its principal symbol function

$$\sigma(x, \xi) = \sum_{|\alpha| = q} a_\alpha(x)(i\xi)^\alpha$$

in which each derivative ∂_k in D is replaced by (the square root of minus one times) a new variable ξ_k. It may be invariantly defined (in a coordinate-free way) on T^*M.

The principal symbol of an elliptic partial differential operator

Definition. A partial differential operator D on a compact manifold is elliptic if its symbol function

$$\sigma\colon T^*M \longrightarrow \mathbb{C}$$

is a proper function (the inverse image of every compact set is compact).

If D is also formally self-adjoint, then σ is real-valued, and there is then an algebra homomorphism

$$C_0(\mathbb{R}) \longrightarrow C_0(T^*M), \qquad f \longmapsto f \circ \sigma$$

This obviously resembles von Neumann's symbol homomorphism

$$C_0(\mathbb{R}) \longrightarrow K(L^2(M))$$

Rescaling integral operators

Connes has given a beautiful explanation of the resemblance within his framework.

For now, assume that the additive Lie group $V = \mathbb{R}^n$ acts simply and transitively on M (which is therefore not compact). The algebra associated to this action is isomorphic to Hilbert's algebra of classical integral operators.

Now V carries the rescaling operation of scalar multiplication, and for each $t \neq 0$ the formula

$$v \cdot_t x = tv + x$$

defines a *new* simply transitive action, and hence a *new* algebra of integral operators which is *also* isomorphic to the classical algebra of integral operators.

We obtain, then, a one-parameter family of copies of the algebra of classical integral operators.

The t=0 limit

When $t = 0$ the group action is obviously trivial, but the algebra that Connes constructs is not. The product is given by the convolution formula

$$(k_1 \star_0 k_2)(x, v, x) = \int_V k_1(x, w, x) k_2(x, v - w, x)\, dw$$

This is ordinary commutative convolution in the middle (group) variable.

So our family of copies of the algebra of Hilbert's integral operators limits to a commutative algebra.

Taking C*-algebra completions, we obtain a continuous field of C*-algebras

$$\begin{cases} K(L^2(M)) & t \neq 0 \\ C^*(TM) & t = 0 \end{cases}$$

A remarkable invariance property

Vector space structure (addition, scalar multiplication) is used throughout the construction. BUT the finished product does NOT involve this structure:

$$\begin{cases} K(L^2(M)) & t \neq 0 \\ C^*(TM) & t = 0 \end{cases}$$

And the group of diffeomorphisms of M acts on all the algebras.

Theorem. *The continuous field is invariant under the action of all diffeomorphisms.*

This makes it possible (using charts) to extend the construction to any manifold.

A continuous family of symbols

A second remarkable property: if D is a self-adjoint and elliptic on M of order q, and if $f \in C_0(\mathbb{R})$, then the family

$$\begin{cases} f(t^q D) & t \neq 0 \\ f(\sigma(D)) & t = 0 \end{cases}$$

is a continuous section of the continuous field. Since $C^*(TM) \cong C_0(T^*M)$, this gives

$$C_0(\mathbb{R}) \longrightarrow \begin{cases} K(L^2(M)) & t \neq 0 \\ C_0(T^*M) & t = 0 \end{cases}$$

interpolating between the von Neumann and principal symbol homomorphisms.

A new symbol; a new index theorem

Alain Connes once said he'd been to a hundred talks whose title should have been "Now I too have understood the index theorem."

Graeme Segal
Tribute to Sir Michael Atiyah

In summary, by constructing natural new families of integral operators, we interpolate between the principal symbol, which encodes geometric information about D, and the von Neumann symbol, which encodes the spectral theory of D.

This leads to a new proof of the index theorem of Atiyah and Singer.

But there are plenty of those (see above).

Is this one special? Yes, it leads to new developments that are very hard to reach (to say the least) in any other way.

The Heisenberg group

In three dimensions, the Heisenberg group is the simply-connected Lie group whose Lie algebra is spanned by X, Y and Z, with Z central and $[X, Y] = Z$ (there is a version of the Heisenberg group in any odd dimension).

The Heisenberg group admits a family of rescaling automorphisms, similar to scalar multiplication, that on the Lie algebra act as

$$X \mapsto tX, \quad Y \mapsto tY, \quad \text{and} \quad Z \mapsto t^2 Z$$

Connes' constructions may be repeated without difficulty, leading to a continuous field of C*-algebras

$$\begin{cases} K(L^2(M)) & t \neq 0 \\ C^*(T_H M) & t = 0 \end{cases}$$

Integral operators on a contact manifold

The manifold M on the previous slide is the Heisenberg group with the group structure forgotten.

But the continuous field is invariant under contact diffeomorphisms of M, which are diffeomorphisms that preserve the tangent distribution on M spanned by the Lie algebra generators X and Y.

So the construction extends to arbitrary contact manifolds:

$$\begin{cases} K(L^2(M)) & t \neq 0 \\ C^*(T_H M) & t = 0 \end{cases}$$

Index theorem for contact manifolds

The index problem for contact manifolds had been around for some time, but a key ingredient was missing: the proper understanding of the principal symbol of a contact-elliptic operator.

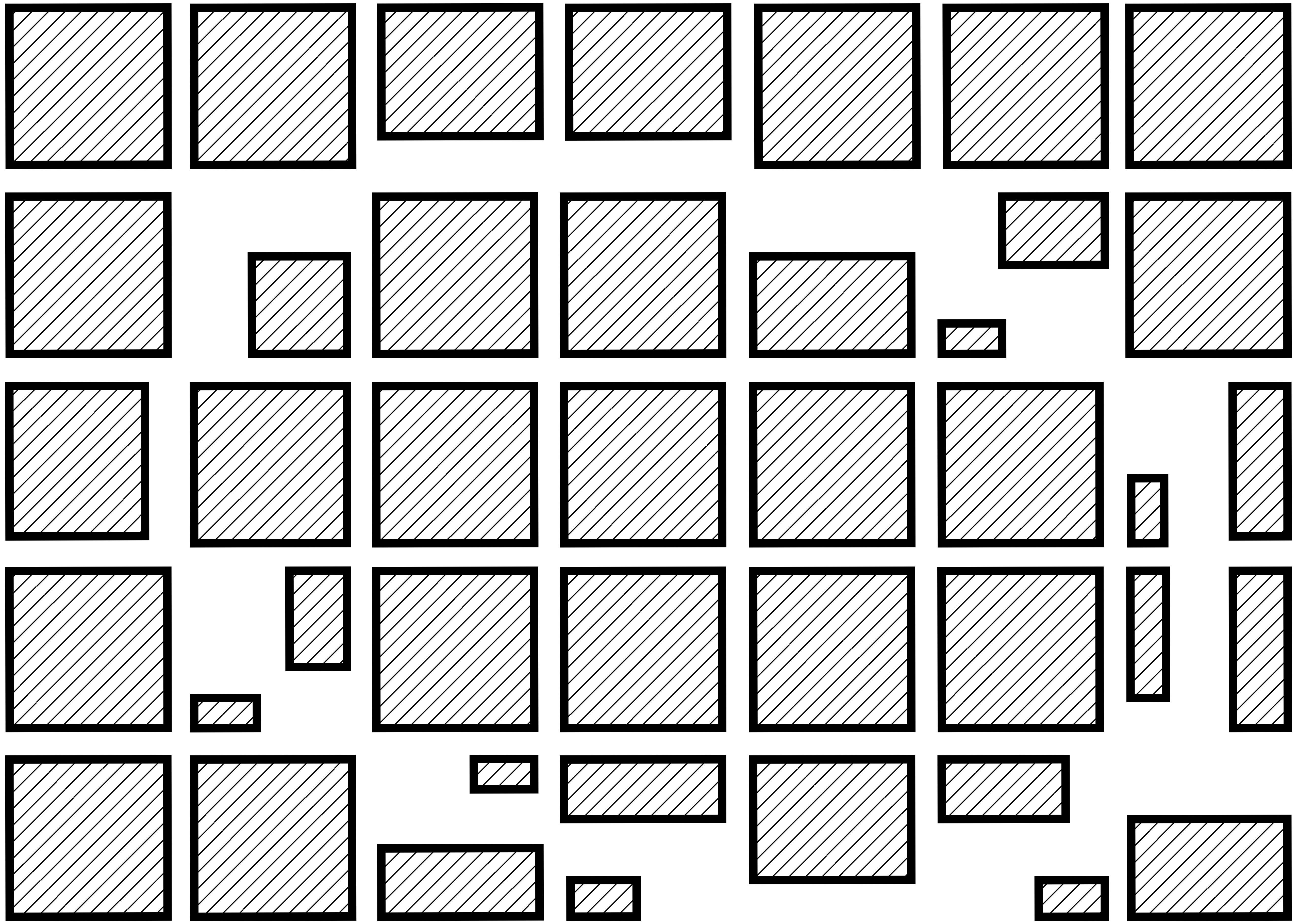
This is now handed directly to us: the symbol should be properly understood as a homomorphism of C*-algebras:

$$C_0(\mathbb{R}) \longrightarrow C^*(T_H M)$$

This and the deformation to the von Neumann symbol homomorphism lead directly to a formulation and proof of the index theorem on contact manifolds, conceptually at least …

This is the work of Erik van Erp. It's a great theorem, and a great advertisement for Connes' theory.

Thank you again!



Thank you again!