John Roe and Coarse Geometry

Part I

John Roe, 1959-2018

Curriculum Vitae

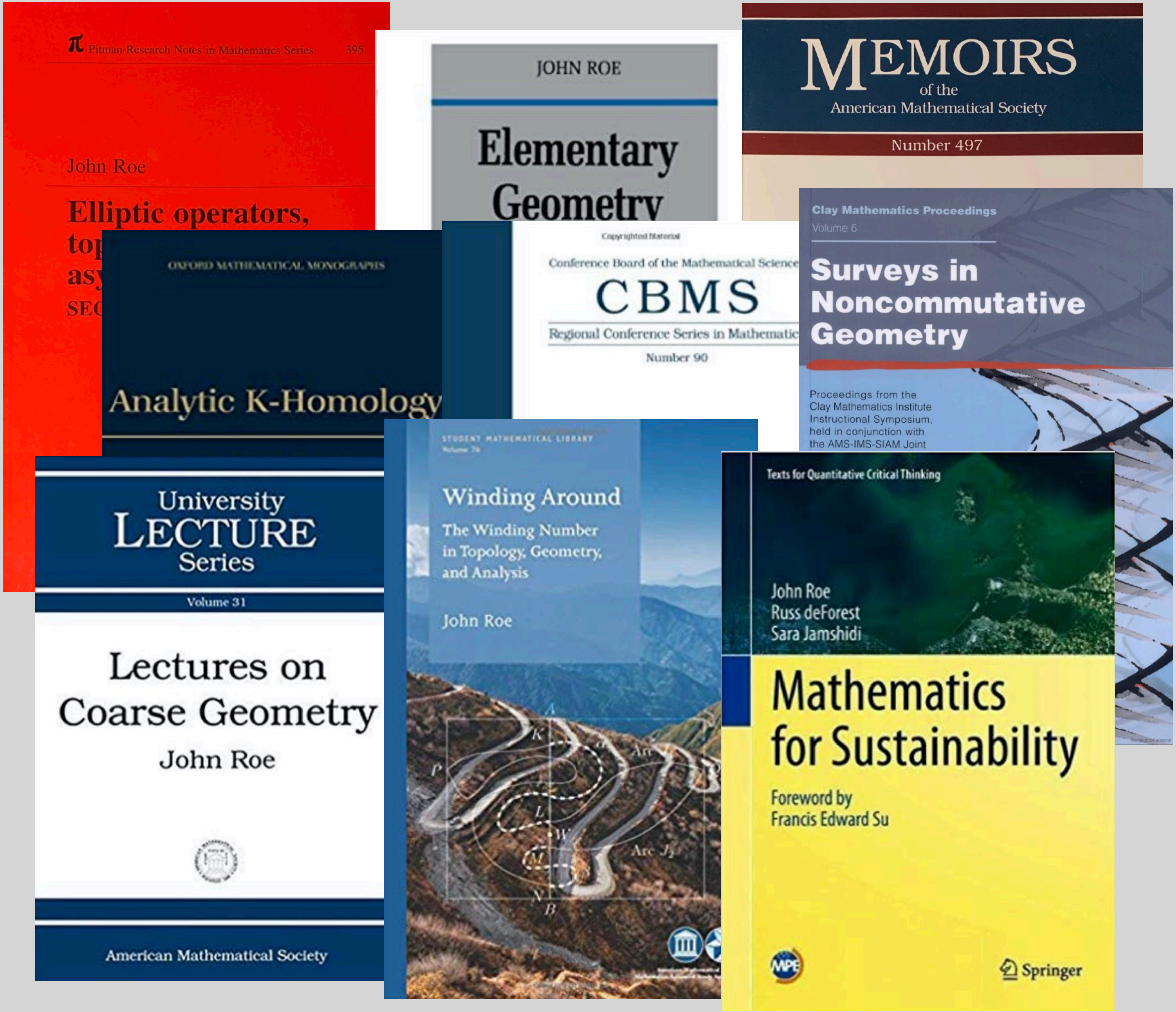
- D.Phil, 1984, Oxford. Supervisor: Michael Atiyah. Thesis title: *Analysis on Manifolds*
- Lecturer in Mathematics, Jesus College, 1986-1998
- Ulam Professor, University of Colorado, 1995
- Professor of Mathematics, 1998-2018, Penn State
- Head of the Department of Mathematics, 2006-2012, Penn State

"[John's work] grew out of a conviction and prayerful reflection that [his] knowledge as a mathematician and an educator could be channeled into wise action on matters that will impact us all."

Francis Su

From Jesus College ...


... to Penn State
PENN STATE
PENN STATE



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John Roe
Elliptic operators,
JOHN ROE
Elementary
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Lectures on
Coarse Geometry
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for Sustainability
Foreword by
Francis Edward Su
Springer


Security

The Mathematical Institute,
24 - 29 St Giles,
Oxford.

30th July 1983

Dear Dr. Hurder,

Thank you for your letter of 1 July, which was waiting for me when I returned to Oxford a few days ago.

I'm afraid that I have nothing which is in a sufficiently final form for me to send to you, but I will try to describe the idea of my work. The 'grand design' is to mimic Connes procedure for certain non compact Riemannian manifolds instead of compact manifolds. The manifolds I consider are those with "bounded geometry" which means a lower bound on the injectivity radius and an upper bound on the curvature tensor and its covariant derivatives. Such manifolds, for example, occur as the leaves of foliations of compact manifolds - in fact it seems to be an open question whether they all do. Now to such a Riemannian manifold (M,g) I propose to associate a C^* algebra $C^*(M,g)$ in such a way that the corresponding K-theory group $K_0(C^*(M,g))$ will contain the indices of the 'natural' elliptic operators on M such as the de Rham operator $d+d^*$. Next, some manifolds (in fact exactly those manifolds "closed" at infinity" in the sense of Sullivan, Inventiones 36) admit certain functionals which I call 'invariant means'; these play the role of transverse measures in the Connes theory, giving rise to ~~x~~ dimension functions $K_0(C^*(M,g)) \to \mathbb{R}$. There seems to be a Gauss Bonnet formula ~~xxxx~~ which expresses the real valued index of the de Rham operator in terms of curvature - modulo some analysis which is proving rather recalcitrant at present. Finally, I hope to be able to compute $K_0(C^*(M,g))$ for nice M - e.g. symmetric spaces or deformations thereof - by means of representation theory.

If you think this is interesting let me know and I will try and let you have a copy of anything respectable.

Yours sincerely,

John Roe (Mr)

PS. One interesting aspect of these~~xxxxx~~ results is that they seem to be closely related - at least in the 2 dimensional case - to classical Nevanlinna theory.

The Mathematical Institute,
24-29 St Giles,
Oxford.

30th July 1983

Dear Dr. Hurder,

I'm afraid I have nothing which is in sufficiently final form to send to you, but I will try to describe to you my work.

The "grand design" is to mimic Connes' procedure for certain noncompact Riemannian manifolds ...

... to such a Riemannian manifold (M,g) I propose to associate a C^*-algebra $C^*(M,g)$...

The Roe Algebra

John's $C^*(M,g)$ is what we now call the *Roe algebra* (or one version of it). Nowadays we just write $C^*(M)$.

I want to explain where it came from, what it is, and what it's good for.

I'll focus not so much on John's thesis (which he was describing to Steve Hurder in the letter) but on a later result that is both beautiful and simple. Because it is so beautiful and so simple, it has been widely used and widely imitated.

In the winter of 1900-1901 the Swedish mathematician E. Holmgren reported in Hilbert's seminar on Fredholm's first publications on integral equations, and it seems that Hilbert caught fire at once ...

Hermann Weyl
David Hilbert and his Mathematical Work

Hilbert famously examined integral operators

$$(Kf)(x) = \int k(x, y) f(y) dy$$

with say k continuous and compactly supported, and proved that

$$k(x, y) = \overline{k(y, x)} \quad \Rightarrow \quad \exists \text{ O.N.B. of eigenfunctions}$$

"… in the terrain of analysis a rich vein of gold had been struck, comparatively easy to exploit and not soon to be exhausted."

Weyl lists a range of work:

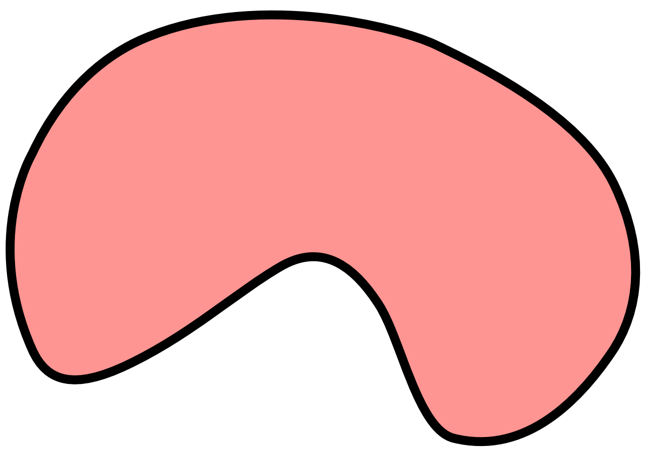
- Simplifications and generalizations (w. Schmidt)
- Singular kernels k(x,y)
- Spectral theorem for bounded self-adjoint operators
- Riemann-Hilbert problem
- Birkhoff decomposition
- Representations of compact groups
- Hodge theory

"It became possible to develop such asymptotic laws for the distribution of eigenvalues as were required by the thermodynamics of radiation"

Weyl is referring to his own work on the asymptotics of eigenvalues for the Laplace operator ...

Dirichlet eigenvalue problem:

$$\Delta f_n = \lambda_n f_n$$
$$f_n|_{\partial\Omega} = 0$$



A bounded planar domain Ω

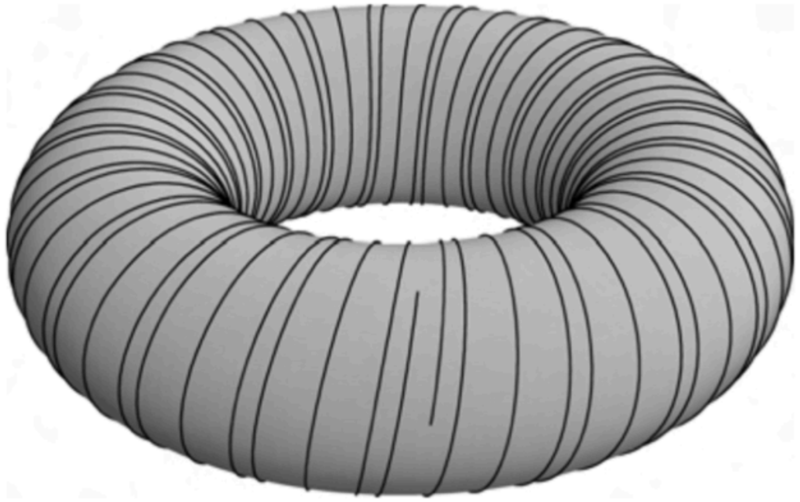
Theorem. $\lambda_n \sim 4\pi n \big/ \mathrm{Area}(\Omega)$

Viewed in retrospect, this is the first theorem of NCG ...

“The story would be dramatic enough had it ended there, but then a sort of miracle happened ...”

I’ll adapt Weyl’s words to my own purposes ...

Given a foliated manifold (M, F), Alain Connes examined integral operators associated to smooth, compactly supported kernel functions $k(x, y)$ that are defined on the space $\{ (x, y) : x \sim_F y \}$



Thank you, Marco

This operator algebra captures crucial features of the foliation. For instance, the algebra (and even its K-theory) distinguishes distinct Kronecker foliations, as pictured.

John was familiar with Connes' work on foliations ...

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Finite propagation speed and Connes' foliation algebra

BY JOHN ROE
Mathematical Institute, University of Oxford

(*Received* 9 *February* 1987)

Introduction

In [**4**], A. Connes has defined the convolution algebra $C_c^\infty(\mathscr{G})$ associated to a foliation $\mathscr{F}$ of the compact manifold M. Here $\mathscr{G}$ is the *graph* or *holonomy groupoid* of the foliation $\mathscr{F}$ (Winkelnkemper [**15**]). By forming the completion of $C_c^\infty(\mathscr{G})$ in its regular representation, he obtains the C^*-algebra $C^*(M, \mathscr{F})$ associated to the foliation. The completeness of $C^*(M, \mathscr{F})$ makes it easier to handle in some analytical contexts, but in others it seems to be too big, and it is necessary to consider instead some carefully selected dense subalgebra (cf. [**6**]). The purpose of this note is to show that certain spectral functions of leafwise elliptic operators, which might *a priori* be expected to belong to $C^*(M, \mathscr{F})$, in fact belong to the more controllable dense subalgebra $C_c^\infty(\mathscr{G})$. We give a couple of applications, including a proof not passing through C^*-algebras of Connes' index theorem for measured foliations [**4**]. It should be emphasized that the proof of that result offered here is essentially Connes' one, but the presentation may perhaps be more congenial to those who are not C^*-algebra specialists.

He was also familiar with closely related work of Atiyah ...

Société Mathématique de France
Astérisque 32-33 (1976)

ELLIPTIC OPERATORS, DISCRETE GROUPS AND

VON NEUMANN ALGEBRAS *

by

M.F. ATIYAH

§ 1 - Introduction.

The global theory of elliptic equations on compact manifolds is very well established. In particular one has finite-dimensionality for the spaces of solutions and an explicit topological formula for the index [1]. For non-compact manifolds, on the other hand, the situation is much more difficult and there are few general results. The essential difficulties are :

(i) one has to decide which growth conditions to impose at infinity,

(ii) the spaces of solutions are usually infinite-dimensional.

He was also familiar with closely related work of Atiyah ...

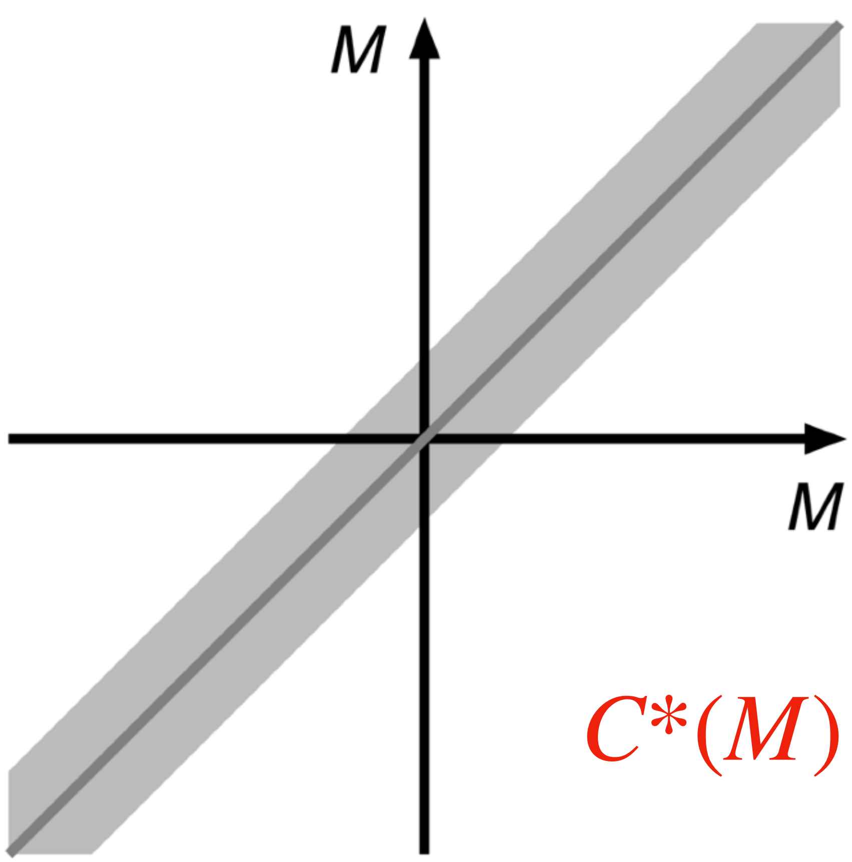
> The global theory of elliptic operators on compact manifolds is very well established ... For noncompact manifolds, on the other hand, the situation is much more difficult ...
>
> The essential difficulties are:
>
> (i) one has to decide which growth conditions to impose at infinity.
> (ii) the spaces of solutions are usually infinite-dimensional.
>
> In practice the most useful condition to impose under (i) is square-integrability.

Atiyah used von Neumann's Type II traces to formulate and prove an index theorem for covering spaces ...

The Roe Algebra

Let M be a complete Riemannian manifold (of bounded geometry). John introduces the C*-algebra generated by all bounded integral operators on $L^2(M)$ given by smooth integral kernels $k(x, y)$ with

$$\sup\{\, d(x,y) : k(x,y) \neq 0 \,\} < \infty$$



$$(k \star f)(x) = \int_M k(x,y) f(y)\, dy$$

$$\text{Propagation} = \sup\{\, d(x,y) : k(x,y) \neq 0 \,\}$$

$C^*(M) =$ C*-algebra completion of the algebra of finite propagation smoothing operators on M.

The Roe Algebra and Foliation Algebras

If M is a leaf in a compact foliated manifold, then the restriction of one of Connes' integral operators to M belongs to the Roe algebra. If the leaf is dense, this leads to an *embedding* of Connes' foliation algebra into the Roe algebra.

If M is the universal cover of a closed manifold, then a π -equivariant integral operator with π-compact support lies in the Roe algebra. This more or less leads to an embedding of the algebra of operators considered by Atiyah into the Roe algebra (except that Atiyah considers a von Neumann algebra completion).

The Roe Algebra and Coarse Geometry

The *Roe algebra* of M depends on M in a rather simple way, and in fact only on the *coarse structure* of M.

Two Riemannian manifolds are *coarsely equivalent* if their large-scale features are the same. For instance, an infinite cylinder (of some fixed finite radius) and a line are coarsely equivalent.

This makes the Roe algebra simple to understand in some cases, and its K-theory simple to understand in many more cases. But not all cases.

The Roe Algebra and Index Theory

Theorem. *If D is a Dirac-type operator on a complete Riemannian manifold M, then the resolvent operators $(D - \lambda)^{-1}$ belong to the Roe algebra of M*

For example the resolvent operators for $D = -i\, d/dx$ lie in the Roe algebra of the line $\mathbb{R}$.

Corollary. *If D is a Dirac-type operator on a complete Riemannian manifold M, then D has a well-defined index in the K-group of the algebra of smoothing operators on M with finite propagation.*

Partitioning non-compact manifolds and the dual Toeplitz problem

John Roe
Mathematical Institute
24-29 St. Giles'
Oxford.

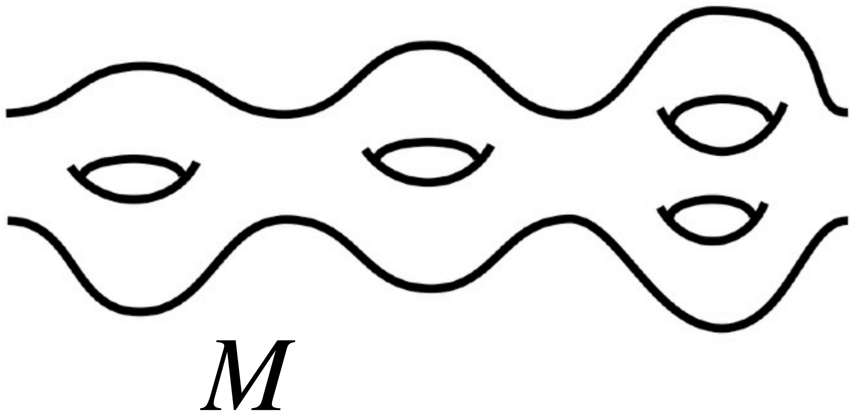
Introduction

In [13], I introduced the idea of considering the index of an elliptic operator on a non-compact manifold as an element of the K-theory of a certain operator algebra, called there the algebra of "uniform operators". The results of [13] showed that in certain circumstances a trace could be constructed on this operator algebra, giving rise to a real-valued dimension function on its K-theory; the resulting real-valued index was then calculated by the heat equation method. This approach had already been employed in a different geometrical context by Connes [5].

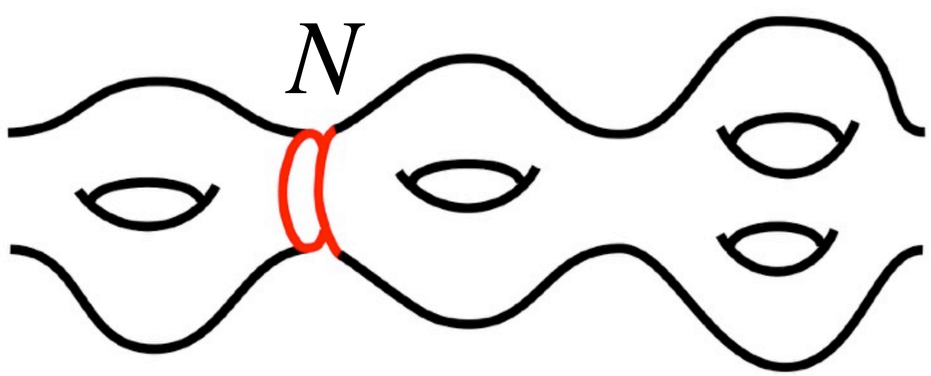
In this paper I address a special case of the following general question: are there cyclic cocycles of dimension > 0 on the algebra of uniform operators?

John’s Partioned Manifold Index Theorem

The theorem concerns complete (spin) manifolds like this one …



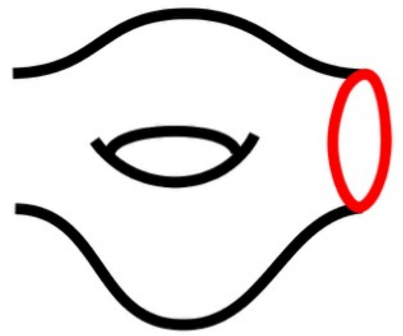
… that can be chopped in two by a closed hypersurface:



$$M = M_+ \cup M_-$$

(but actually they should be odd-dimensional)

Lemma. *If U is a unitary operator on $L^2(M)$ that is a perturbation of the identity operator by an operator in the Roe algebra, then the compression of U to $L^2(M_+)$ is a Fredholm operator.*



M_+

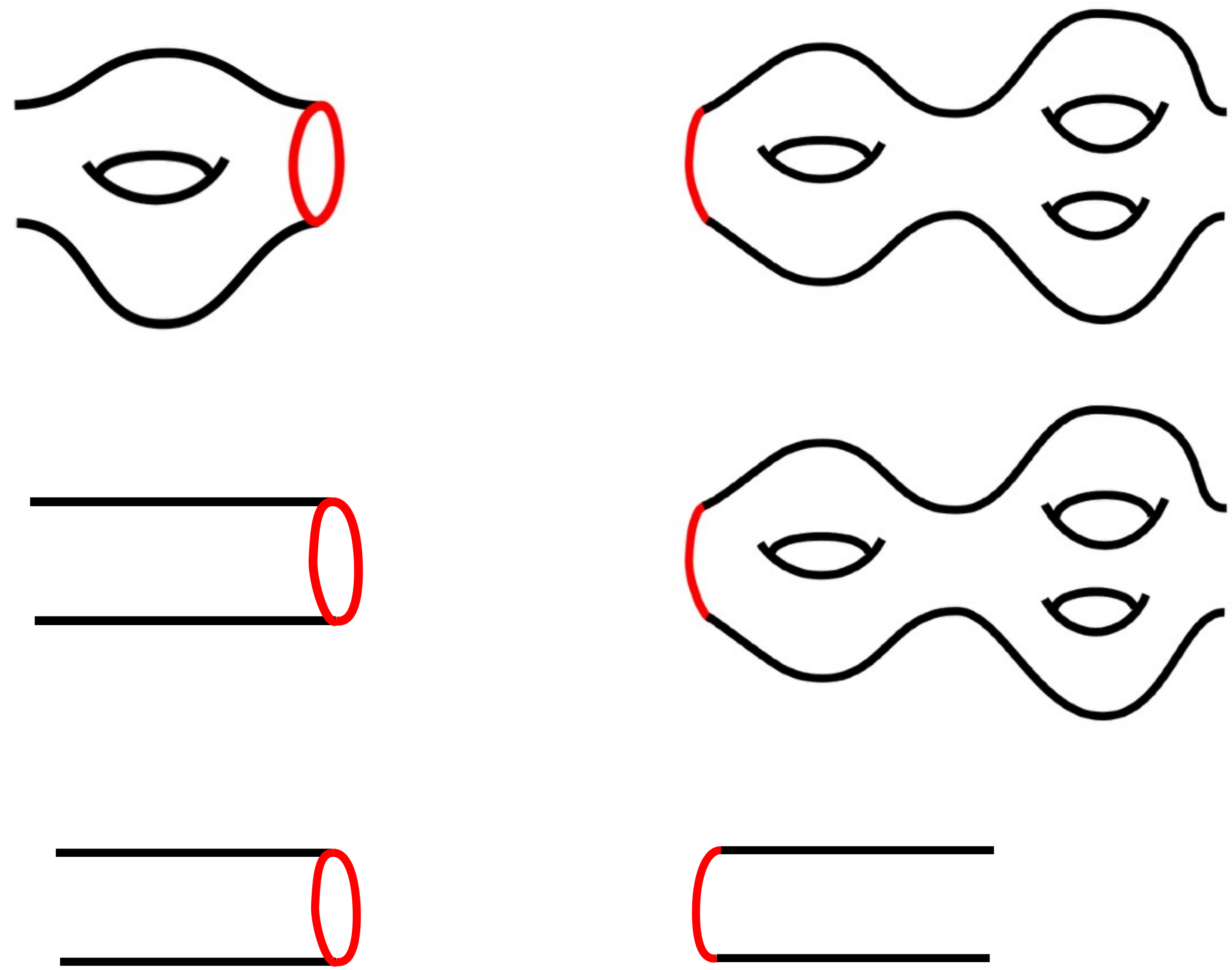
This applies to the Cayley transform of the Dirac operator on M; for instance in applies to the Cayley transform of the operator $D = -i\, d/dx$ on the line $\mathbb{R}$.

Theorem. *The Fredholm index of the compression to $L^2(M_+)$ of the Cayley transform of the Dirac operator on M is equal to the Fredholm index of the Dirac operator on N.*

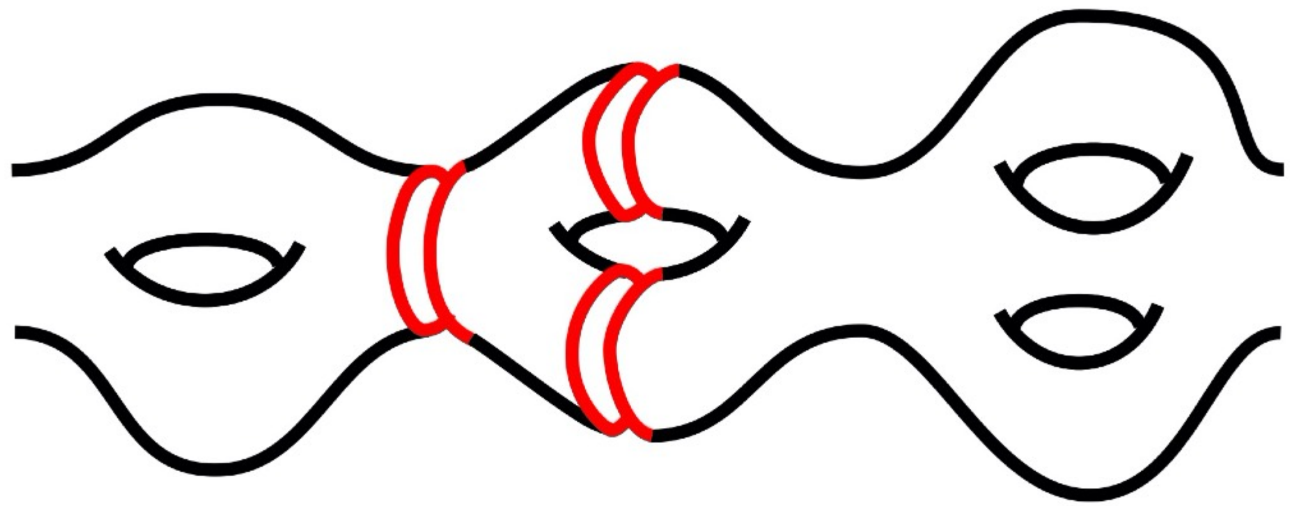
$$\mathrm{Index}(P_+\, \mathrm{Cayley}(D_M)\, P_+) = \mathrm{Index}(D_N)$$

Proof by Pictures

Index = n Index = $-n$



Corollary. *The Dirac index is a cobordism invariant.*



$$\operatorname{Index}(P_+^{(1)}\,\operatorname{Cayley}(D_M)\,P_+^{(1)}) = \operatorname{Index}(P_+^{(2)}\,\operatorname{Cayley}(D_M)\,P_+^{(2)})$$

$$\therefore \quad \operatorname{Index}(D_{N^{(1)}}) = \operatorname{Index}(D_{N^{(2)}})$$

Time for Cake

The Original Recipe

"So you think the proof of the index theorem is a piece of cake?"

Michael Atiyah

Cake design by John Roe

Thank You!