John Roe and Coarse Geometry

Part II

A Walk Around the Göttingen Stadtfriedhof

A Mathematical Epitaph for John?

$$\begin{tikzcd}
L_{n+1}(\pi) \arrow[r, dotted] \arrow[d] & \mathcal{S}(V) \arrow[r] \arrow[d] & \mathcal{N}(V) \arrow[r] \arrow[d] & L_n(\pi) \arrow[d] \\
K_{n+1}(C^*_\pi(V)) \arrow[r] & \mathcal{S}_{J.R.}(V) \arrow[r] & K_n(V) \arrow[r] & K_n(C^*_\pi(V))
\end{tikzcd}$$

I shall try to explain something (by no means everything) about this diagram, which expresses a relation between the topology of manifolds—surgery theory—and K-theory of C*-algebras.

There is a sort of addendum to the diagram—a third row—that I shall try to explain, too.

Surgery For Amateurs

In 1996 I was the Ulam Visiting Professor at the University of Colorado, Boulder. While I was there I gave a series of graduate lectures on high-dimensional manifold theory, which I whimsically titled Surgery for Amateurs. The title was supposed to express that I was coming to the subject from outside – basically, trying to answer to my own satisfaction the question "What is this Novikov Conjecture you keep talking about?"

J.R.

A Word of Caution

Sometimes in order to tell the truth one must lie … and so it is here.

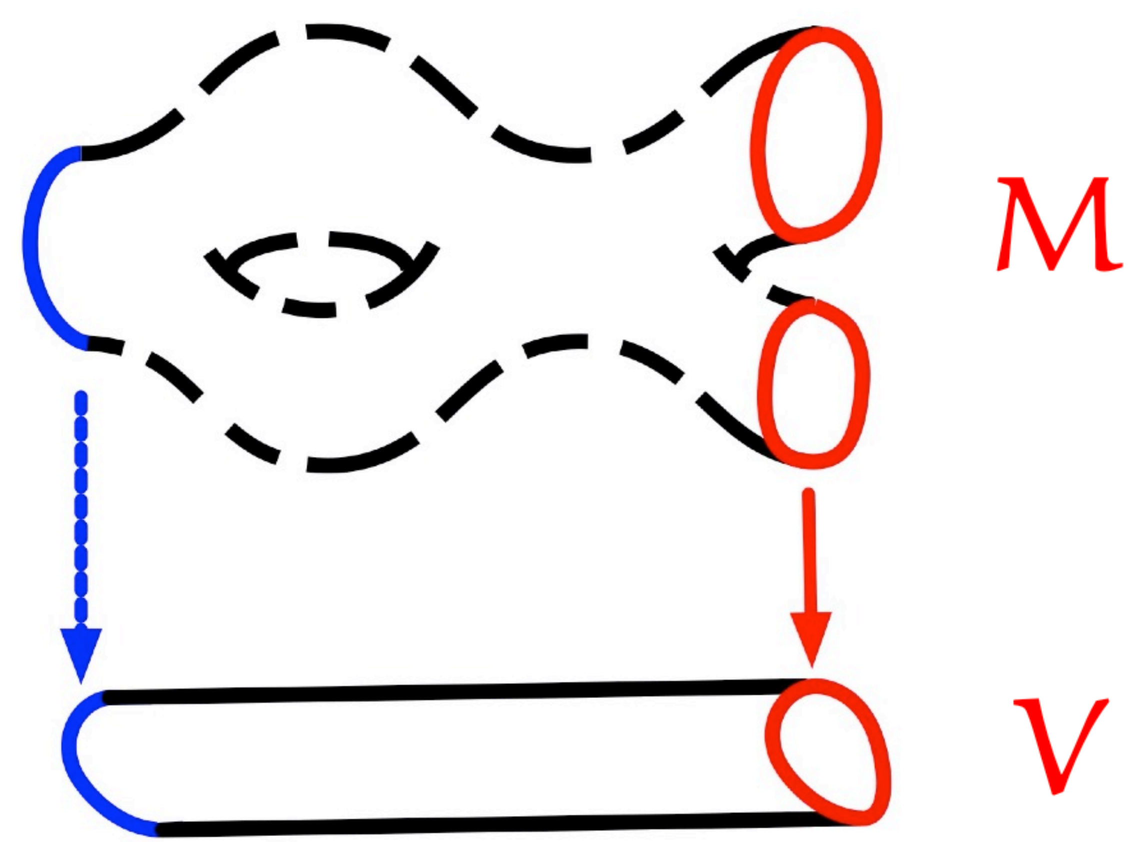
(In fact I have already misrepresented some details.)

(There will be more of the same to come … but they are just details.) (Mostly.)

Surgery Problems

$$L_{n+1}(\pi) \dashrightarrow \mathcal{S}(V) \longrightarrow \mathcal{N}(V) \longrightarrow L_n(\pi)$$

- V is a closed n-manifold with fundamental group π
- $\mathcal{S}(V)$ is comprised of *manifold structures* $M \xrightarrow{\sim} V$
- $\mathcal{N}(V)$ is comprised of *normal maps* $M \to V$
- $L_n(\pi)$ is the group of *surgery obstructions*



Analytic Surgery Problems

$$K_{n+1}(C^*_\pi(V)) \longrightarrow \mathcal{S}_{\mathrm{J.R.}}(V) \longrightarrow K_n(V) \longrightarrow K_n(C^*_\pi(V))$$

This comes from C*-algebra K-theory:

$$0 \longrightarrow C^*_\pi(V) \longrightarrow D^*_\pi(V) \longrightarrow D^*_\pi(V)/C^*_\pi(V) \longrightarrow 0$$

- $C^*_\pi(V)$ is the C*-algebra of π-equivariant operators in $C^*(\widetilde{V})$
- $D^*(\widetilde{V})$ is the C*-algebra of bounded propagation, pseudolocal operators on $L^2(\widetilde{V})$
- $D^*_\pi(V)$ is the C*-algebra of π-equivariant operators in $D^*(\widetilde{V})$

Theorem. *The K-theory of $D^*_\pi(V)/C^*_\pi(V)$ is the K-homology of V.*

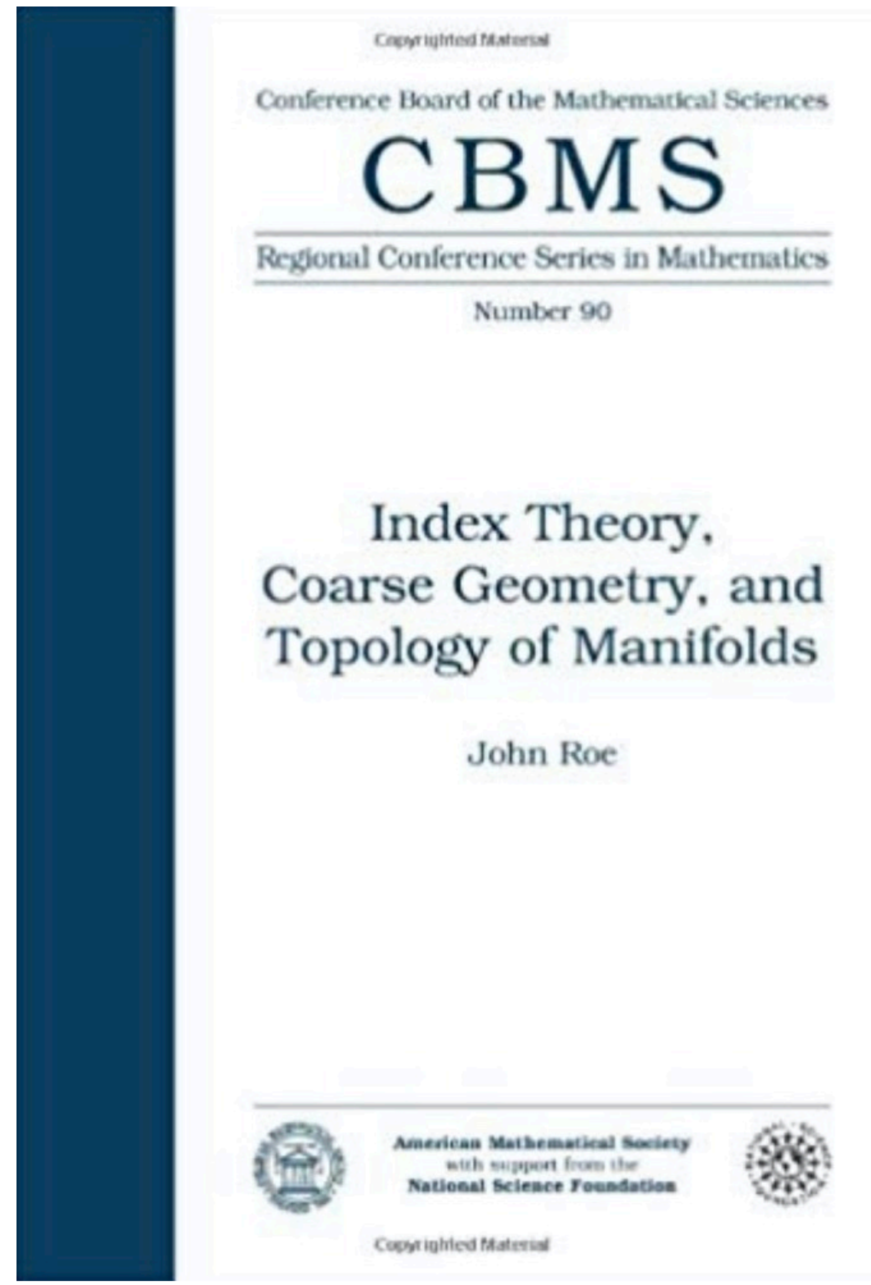
Mapping Surgery to Analysis

$$
\begin{tikzcd}
L_{n+1}(\pi) \arrow[r, dotted] \arrow[d] & \mathcal{S}(V) \arrow[r] \arrow[d] & \mathcal{N}(V) \arrow[r] \arrow[d] & L_n(\pi) \arrow[d] \\
K_{n+1}(C^*_\pi(V)) \arrow[r] & \mathcal{S}_{J.R.}(V) \arrow[r] & K_n(V) \arrow[r] & K_n(C^*_\pi(V))
\end{tikzcd}
$$

- A normal map $M \to V$ is sent to $\mathrm{Signature}(M) - \mathrm{Signature}(V) \in K_n(C^*_\pi(V))$
- So *the signature is a homotopy invariant*
- And if the *analytic structure set* $\mathcal{S}_{J.R.}(\pi)$ is zero, then the image of $M \to V$ in $K_n(V)$ is a homotopy invariant. This is the *Novikov conjecture*

Mapping Surgery to Analysis

Read all about it in
John's CBMS notes …



Boulder 1996

K-Theory (2004) 33:277–299
DOI 10.1007/s10977-005-1561-8

And here:

Mapping Surgery to Analysis I: Analytic Signatures*

NIGEL HIGSON and JOHN ROE
Department of Mathematics, Penn State University, University Park, Pennsylvania 16802, USA. e-mail: higson@math.psu.edu, roe@math.psu.edu

(Received: February 2004)

Abstract. We develop the theory of analytically controlled Poincaré complexes over C^*-algebras. We associate a *signature* in C^*-algebra K-theory to such a complex, and we show that it is invariant under bordism and homotopy.

K-Theory (2004) 33:301–324
DOI 10.1007/s10977-005-1559-2

And here:

Mapping Surgery to Analysis II: Geometric Signatures*

NIGEL HIGSON and JOHN ROE
Department of Mathematics, Penn State University, University Park, Pennsylvania 16802, USA. e-mail: higson@math.psu.edu; roe@math.psu.edu

(Received: February 2004)

Abstract. We give geometric constructions leading to analytically controlled Poincaré complexes in the sense of the previous paper. In the case of a complete Riemannian manifold we identify the signature of the associated complex with the coarse index of the signature operator.

K-Theory (2004) 33:325–346
DOI 10.1007/s10977-005-1554-7

And here:

Mapping Surgery to Analysis III: Exact Sequences

NIGEL HIGSON and JOHN ROE
Department of Mathematics, Penn State University, University Park, Pennsylvania 16802.
e-mail: roe@math.psu.edu; higson@math.psu.edu

(Received: February 2004)

Abstract. Using the constructions of the preceding two papers, we construct a natural transformation (after inverting 2) from the Browder–Novikov–Sullivan–Wall surgery exact sequence of a compact manifold to a certain exact sequence of C^*-algebra K-theory groups.

Co-authors (by number of collaborations)

Deforest, Russell Hanke, Bernhard **Higson, Nigel**
Jamshidi, Sara Kotschick, Dieter Pedersen, Erik Kjær
Qiao, Yu[3] Rabinovich, Vladimir S. Roch, Steffen Schick, Thomas Siegel, Paul Weinberger, Shmuel Willett, Rufus
Yu, Guo Liang[1]

The last paper in the surgery to analysis series …

Pure and Applied Mathematics Quarterly
Volume 6, Number 2
(*Special Issue: In honor of Michael Atiyah and Isadore Singer*)
555—601, 2010

K-Homology, Assembly and Rigidity Theorems for Relative Eta Invariants

Nigel Higson and John Roe

Abstract: We connect the assembly map in C*-algebra K-theory to rigidity properties for relative eta invariants that have been investigated by Mathai, Keswani, Weinberger and others. We give a new and conceptual proof of Keswani's theorem that whenever the C*-algebra assembly map is an isomorphism, the relative eta invariants associated to the signature operator are homotopy invariants, whereas the relative eta invariants associated to the Dirac operator on a manifold with positive scalar curvature vanish.

Twisting A Differential Operator

- Let $D = -i\, d/dx$ on $L^2(\mathbb{R}/\mathbb{Z})$
- For $\theta \in \mathbb{R}$, let $L^2_\theta(\mathbb{R}/\mathbb{Z})$ be the Hilbert space of θ-twisted-periodic functions on $\mathbb{R}$:

$$f(x+1) = e^{i\theta} f(x)$$

- Define $D_\theta = -i\, d/dx$ on $L^2_\theta(\mathbb{R}/\mathbb{Z})$
- $\mathrm{Spectrum}(D_\theta) = \{\, 2\pi n + \theta : n \in \mathbb{Z} \,\}$

Twisted Dirac Operators

The operators D_θ have generalizations far beyond the circle …

- D is the Dirac operator on a closed (spin) manifold V
- θ is now a *unitary representation*

$$\theta\colon \pi \longrightarrow U(N)$$

- It is again interesting to study the *spectral asymmetry* of the operators D_θ

Eta Invariants

The *eta-function* of a self-adjoint operator D is

$$\eta_D(s) = \sum_n \mathrm{sign}(\lambda_n)|\lambda_n|^{-s}$$

and the *eta-invariant* of D is

$$\eta(D) = \eta_D(0)$$

So it is a regularization of the number of positive eigenvalues, minus the number of negative eigenvalues.

Relative Eta Invariants

The *relative eta-invariants* of D are

$$\mathrm{Ind}_{\theta_1,\theta_2}(D) = \eta(D_{\theta_1}) - \eta(D_{\theta_2})$$

These have some remarkable properties:

- They are differential invariants (Atiyah, Patodi, Singer)
- They are homotopy invariants modulo the integers (Weinberger)
- If the surgery structure set is trivial, they are homotopy invariants on the nose (Weinberger)
- If the analytic structure set is trivial, they are again homotopy invariants on the nose (Keswani)

Surgery to Analysis, Again

All this is explained by a third row of the surgery to analysis diagram:

$$\begin{tikzcd}
L_{n+1}(\pi) \arrow[r] \arrow[d] & \mathcal{S}(V) \arrow[r] \arrow[d] & \mathcal{N}(V) \arrow[r] \arrow[d] & L_n(\pi) \arrow[d] \\
K_{n+1}(C^*_\pi(V)) \arrow[r] \arrow[d] & \mathcal{S}_{\text{J.R.}}(V) \arrow[r] \arrow[d] & K_n(V) \arrow[r] \arrow[d] & K_n(C^*_\pi(V)) \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

First, the image of a structure or normal map is the relative eta invariant

Surgery to Analysis, Again

All this is explained by a third row of the surgery to analysis diagram:

$$
\begin{tikzcd}
L_{n+1}(\pi) \arrow[r] \arrow[d] & \mathcal{S}(V) \arrow[r] \arrow[d] & \mathcal{N}(V) \arrow[r] \arrow[d] & L_n(\pi) \arrow[d] \\
K_{n+1}(C^*_\pi(V)) \arrow[r] \arrow[d] & \mathcal{S}_{J.R.}(V) \arrow[r] \arrow[d] & K_n(V) \arrow[r] \arrow[d] & K_n(C^*_\pi(V)) \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

This plus a Novikov argument implies homotopy invariance mod $\mathbf{Z}$

Surgery to Analysis, Again

All this is explained by a third row of the surgery to analysis diagram:

$$\begin{tikzcd}
L_{n+1}(\pi) \arrow[r] \arrow[d] & \mathcal{S}(V) \arrow[r] \arrow[d] & \mathcal{N}(V) \arrow[r] \arrow[d] & L_n(\pi) \arrow[d] \\
K_{n+1}(C^*_\pi(V)) \arrow[r] \arrow[d] & \mathcal{S}_{\mathrm{J.R.}}(V) \arrow[r] \arrow[d] & K_n(V) \arrow[r] \arrow[d] & K_n(C^*_\pi(V)) \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

And if the structure set vanishes, we get exact homotopy invariance

Thank You!

John Roe, 1959-2018