Topology and Spectrum

Kemeny Lectures, Dartmouth College

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Matrices from 200 BC to now

A matrix calculation from 200-100BC China …

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 26 \\ 34 \\ 39 \end{bmatrix}$$

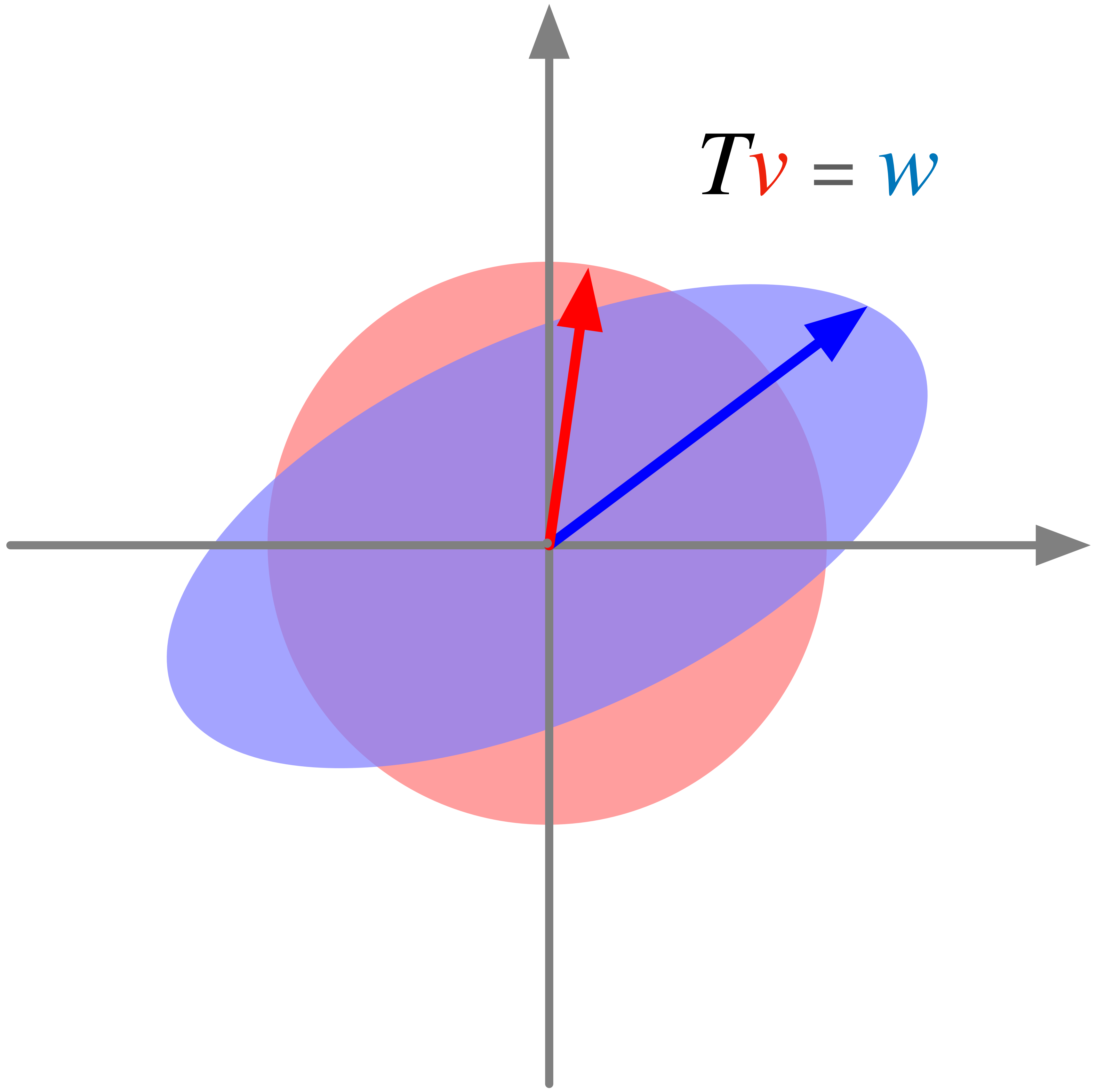
Manipulations with rectangular arrays of numbers (row operations) have been around for a long time.

But the term *matrix* and the discipline of matrix theory is surprisingly modern, dating from the 2000 years after the equation above was solved.

Matrix multiplication and linear transformations

I shall be dealing with square matrices. Any two of the same size can be added together, multiplied, etc. The order of the factors in a product is important, but otherwise the usual rules of algebra apply.

Matrices also have a geometric character: an $n \times n$ matrix is a linear transformation of n-dimensional space.



This is obviously related to the role of matrices in solving linear equations, but it opens up new possibilities, too.

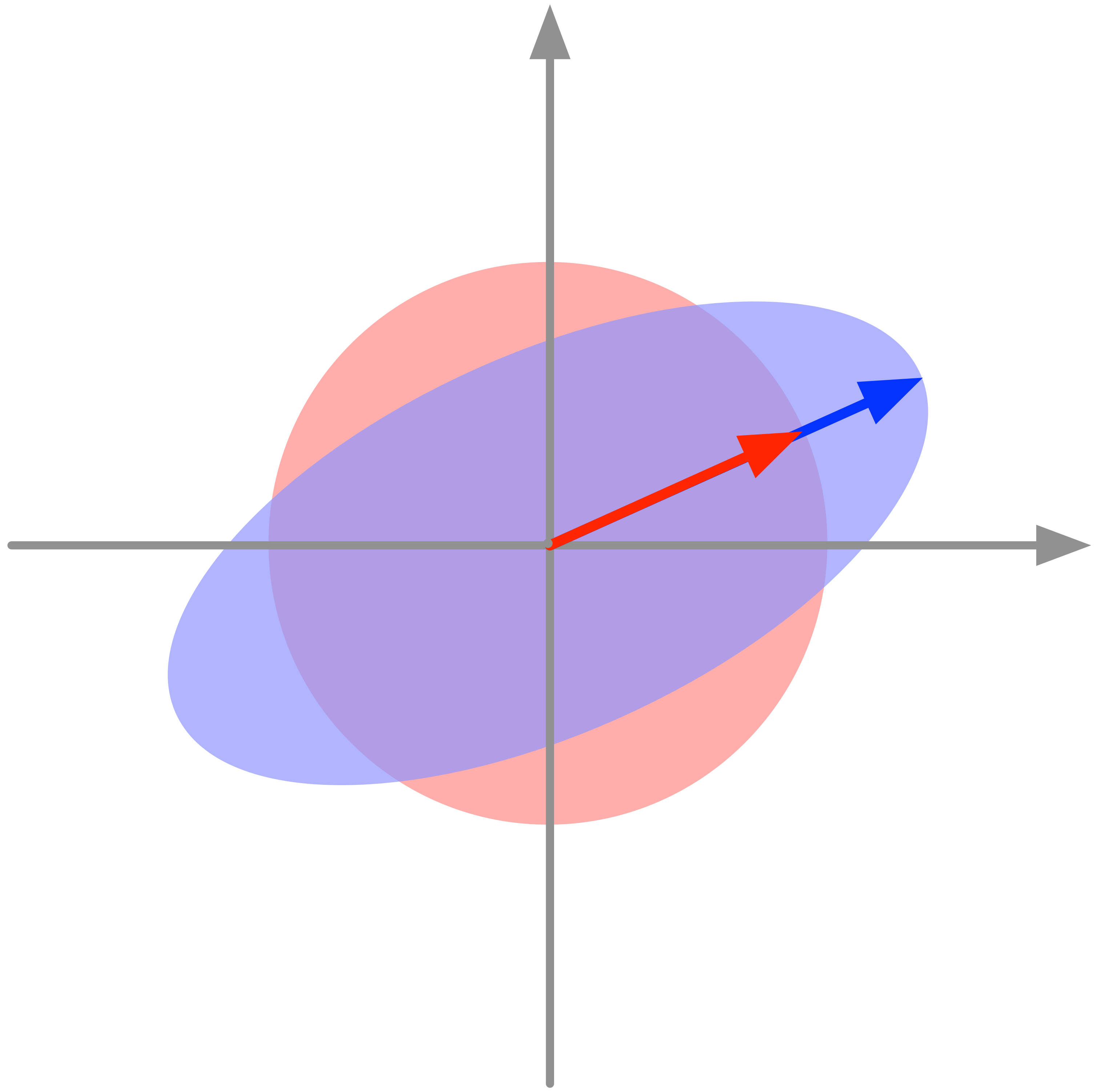
Eigenvalues

The eigenvalue equation for a matrix T is

$$Tv = \lambda \cdot v$$

eigenvalue

eigenvector (nonzero)



For example

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5}+1 \\ 2 \end{bmatrix} = \frac{\sqrt{5}+1}{2} \begin{bmatrix} \sqrt{5}+1 \\ 2 \end{bmatrix}$$

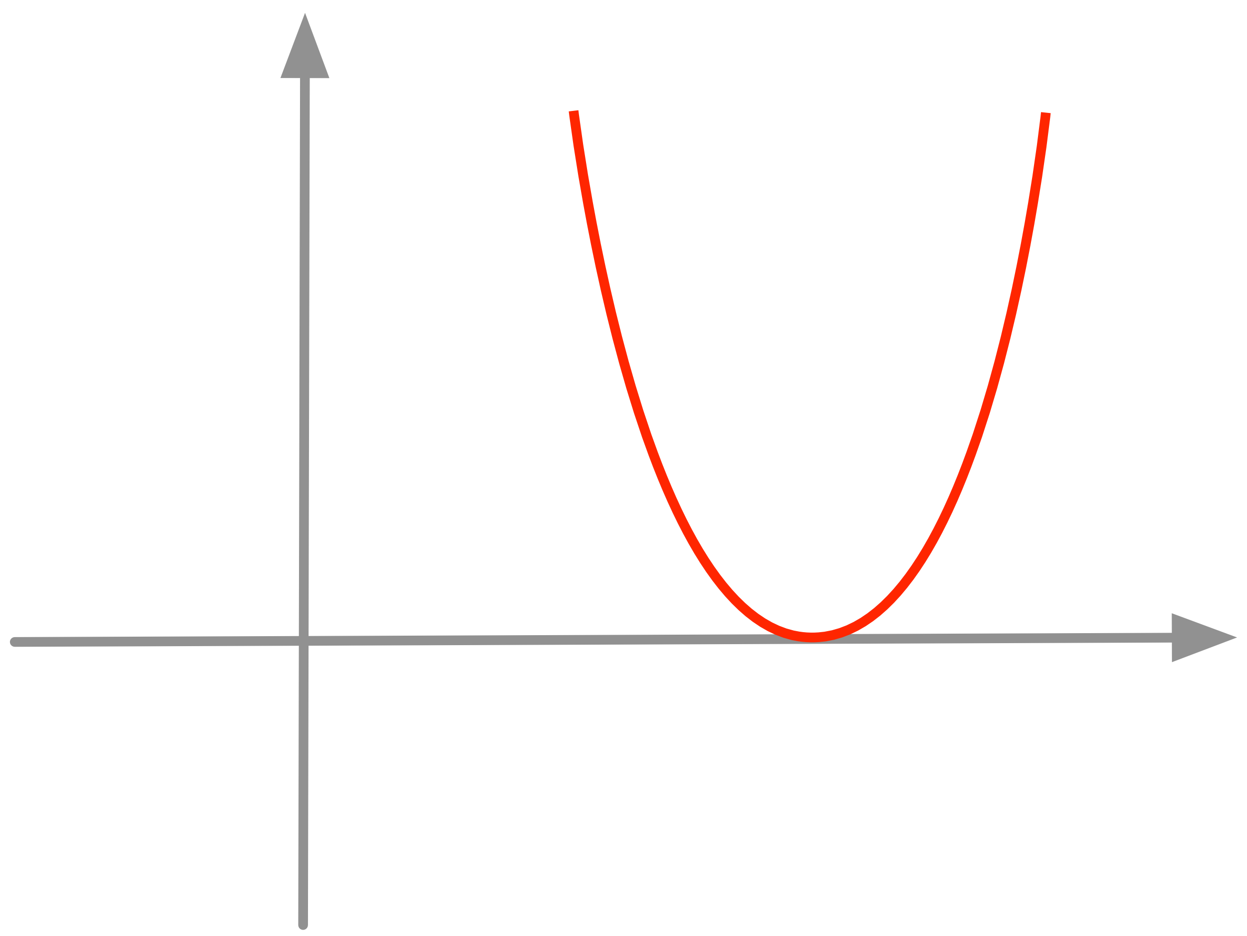
The presence of roots (from the quadratic formula?) hints that finding the eigenvalues of a matrix is a bit like finding the roots of a polynomial equation, which is exactly right …

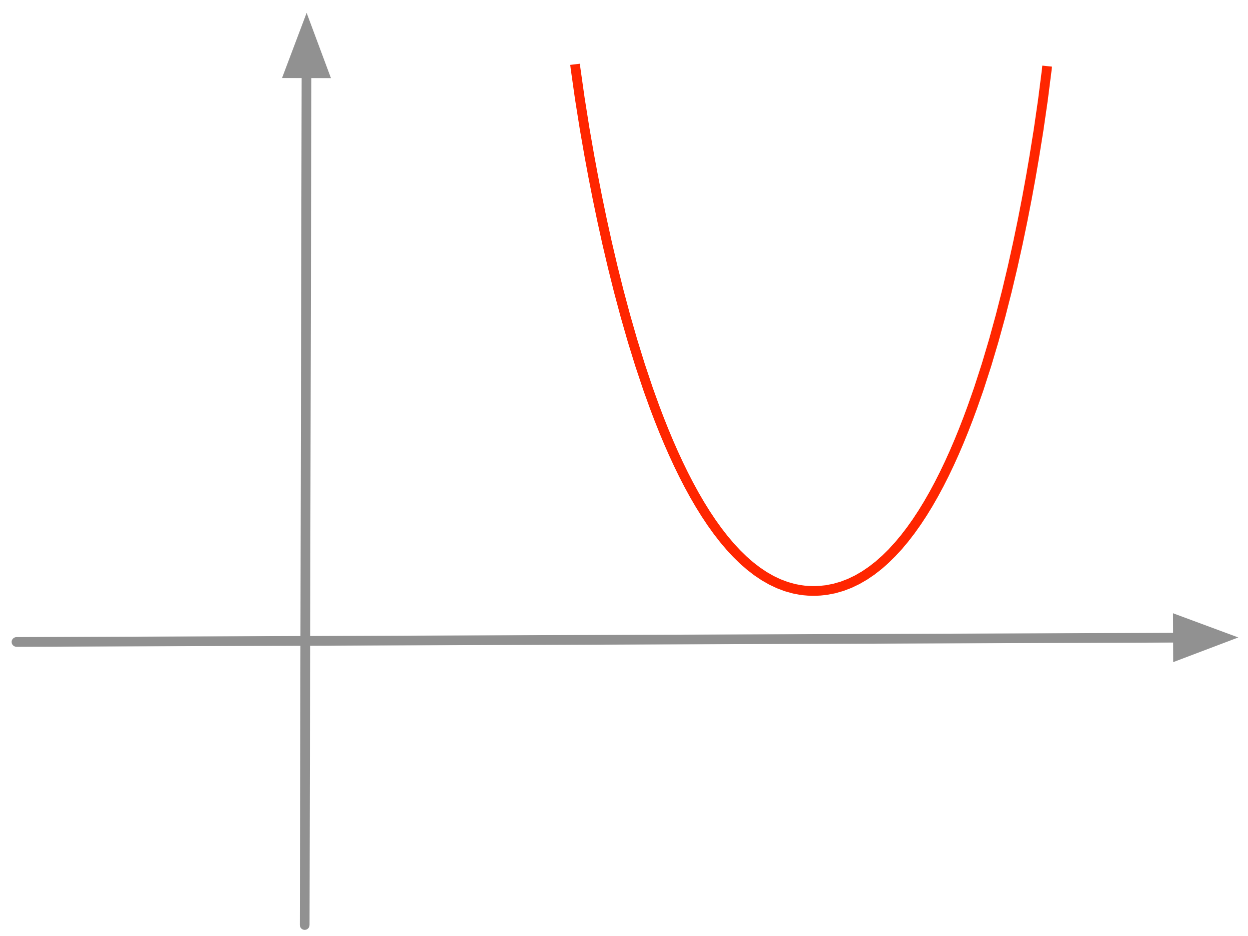
Spectrum

The spectrum of a (finite) matrix is the set of all its eigenvalues.

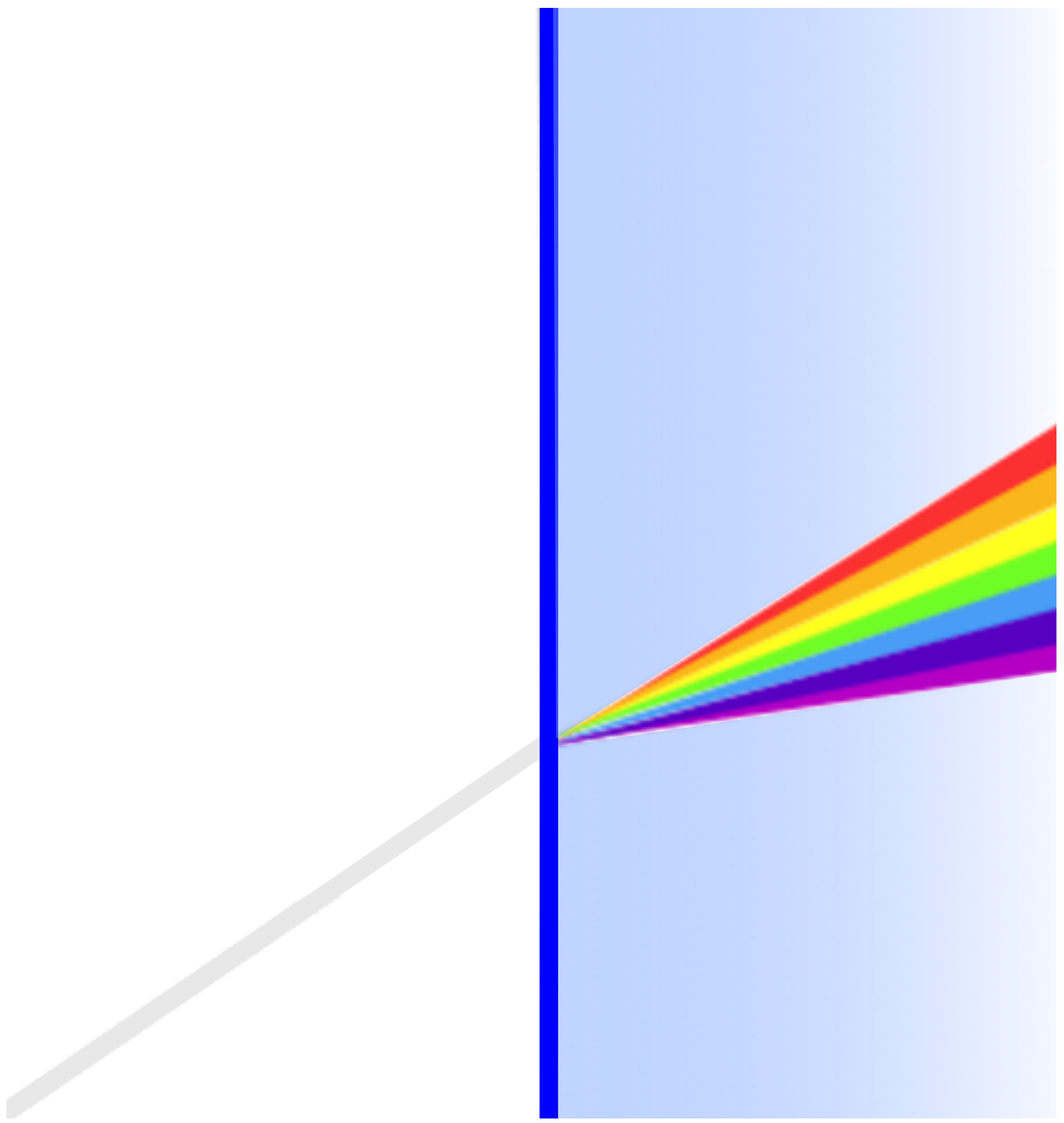
As with polynomial equations, an $n \times n$ matrix has n eigenvalues … if counted correctly.

- As with polynomial equations, there are multiplicities to cope with.
- And as with polynomial equations, there might be complex eigenvalues.





Spectrum of light



Newton's discovery during the plague years …

All this and more was figured out just in time for the arrival of quantum mechanics, which is organized around (the geometric perspective on) matrix theory … with matrices of infinite size.

The frequencies of light emitted by say hydrogen are the differences between eigenvalues in the spectrum of an associated matrix.

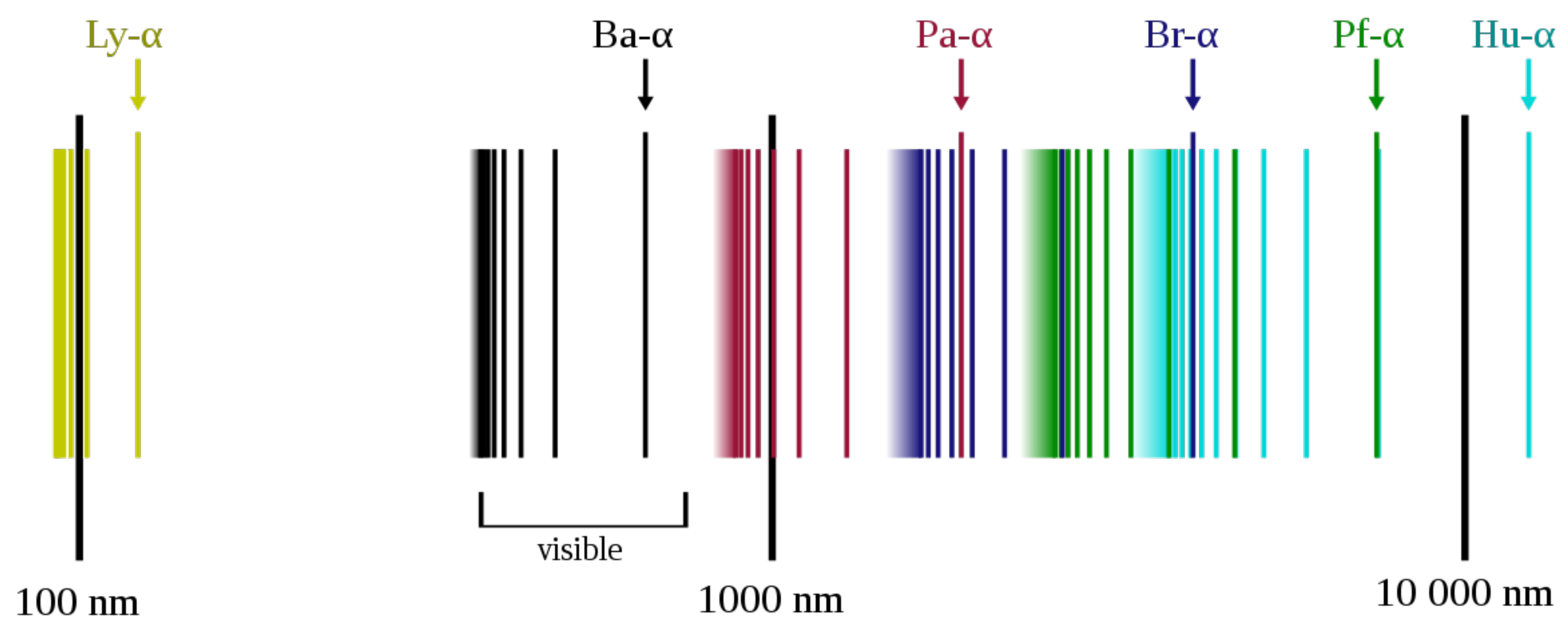
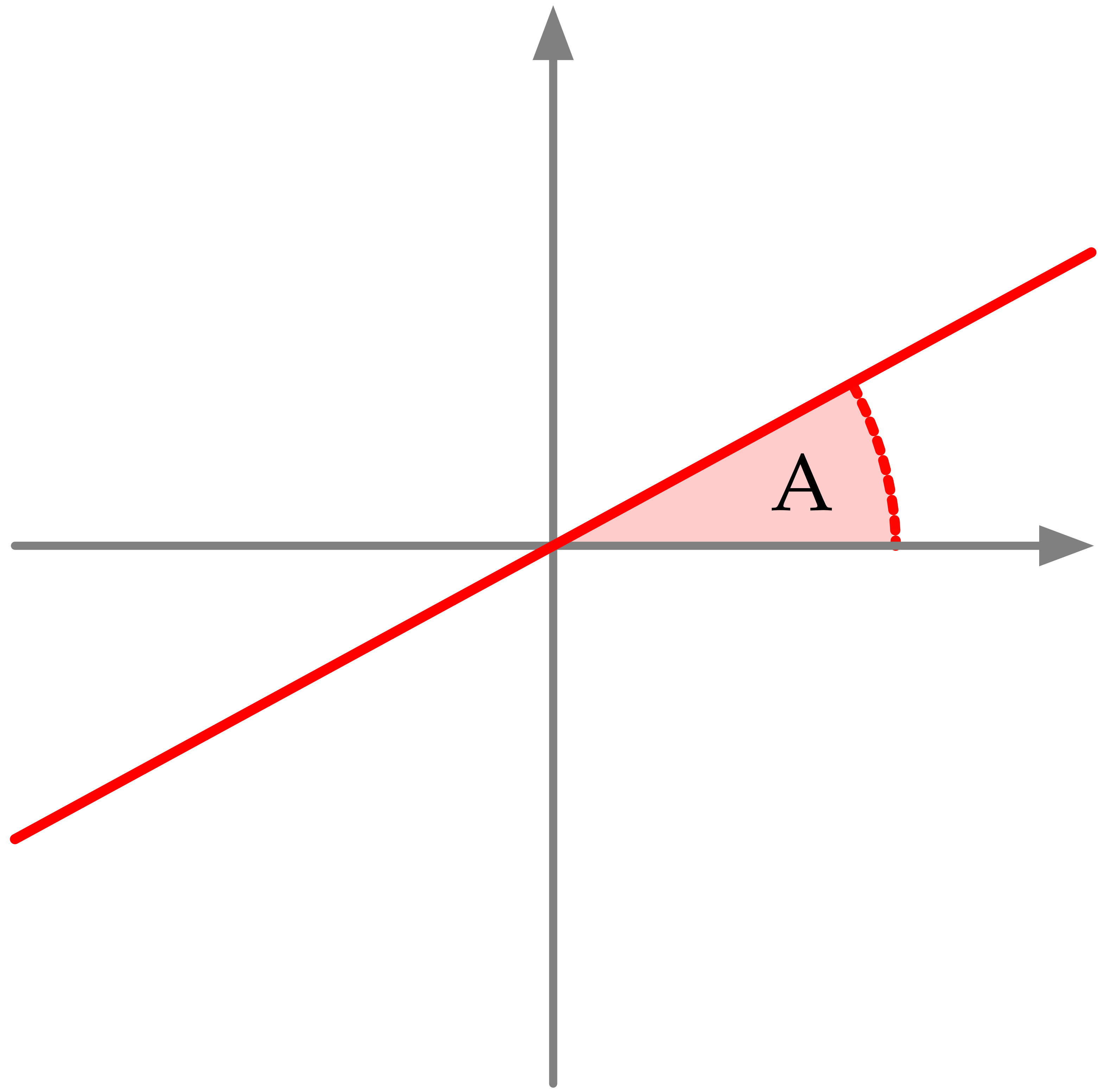


Image credit: Wikipedia

Topology and spectrum

Now I come to the main theme of the lecture …

Sample questions: what does the geometric space of *all* real two by two matrices with spectrum $\{\ 0,1\ \}$ look like?



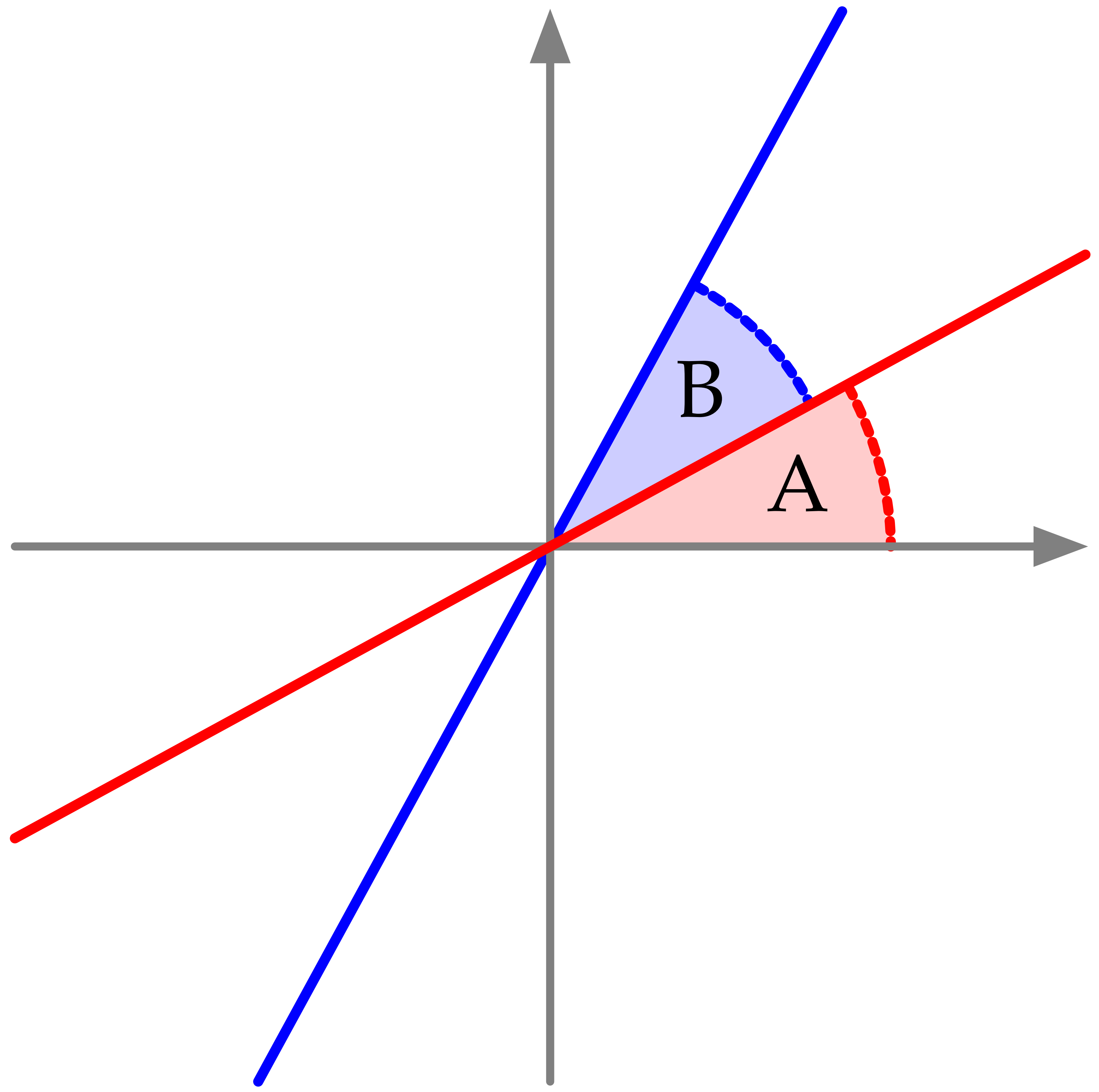
Since the real number 1 is an eigenvalue of T, there is a nonzero vector v such that $Tv = v$.

Color the vector v and all its multiples red, so as to obtain a red line in the plane.

The line makes some angle A with the horizontal axis in the plane.

The angle is between 0 and 180 degrees, *inclusive*, and *the two extreme angles correspond to the same line*.

Topology and spectrum



Similarly since the real number 0 is an eigenvalue of T, there is a nonzero vector w such that $Tw = 0$.

Color the w and its multiples blue, so as to obtain a blue line in the plane.

The red and blue lines determine T, and vice versa.

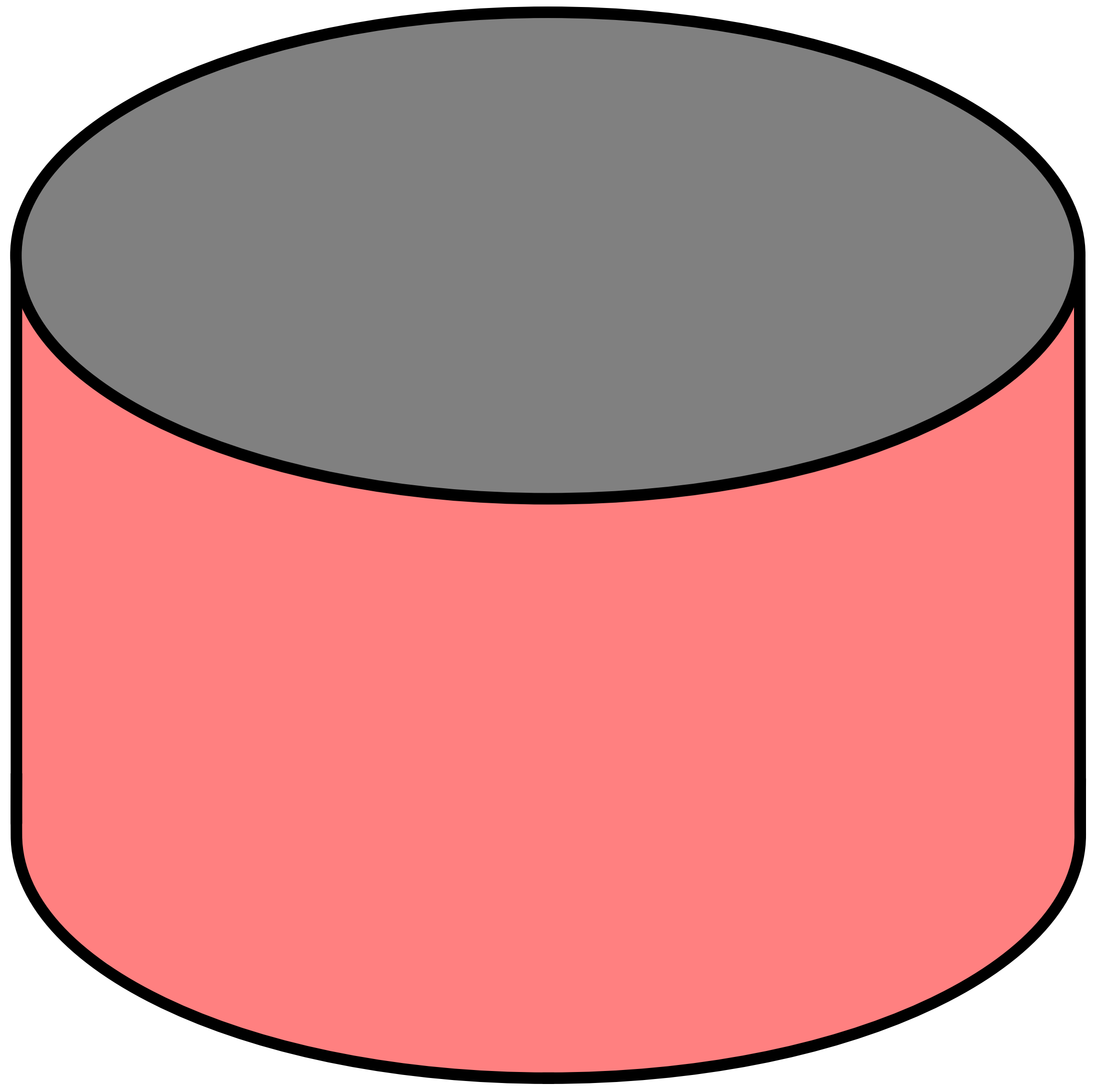
The blue line makes some angle B with the red line.

This angle is *bigger* than 0 degrees (the two lines are distinct) and *less than* 180 degrees.

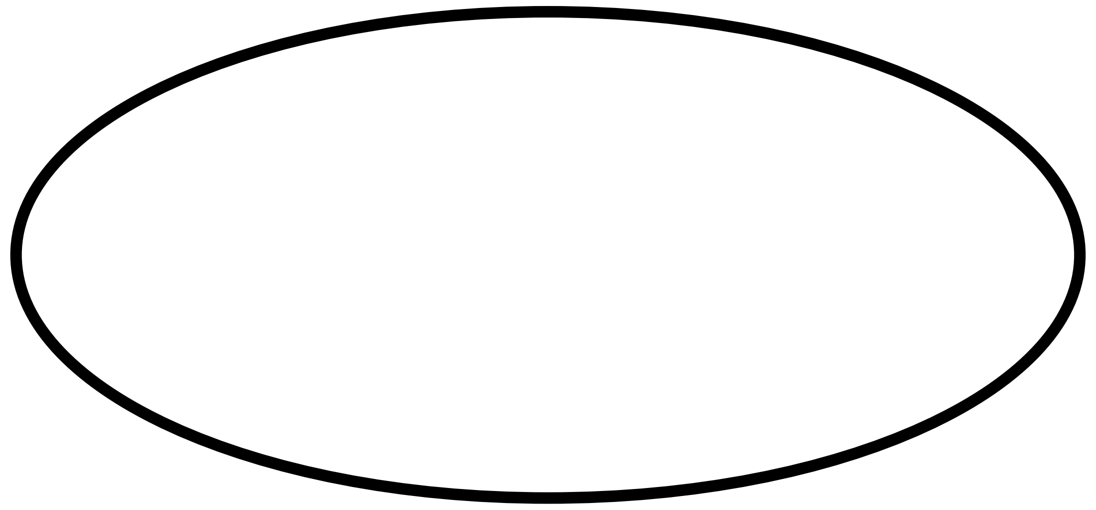
Topology and spectrum

So, what does the geometric space of all real two by two matrices with spectrum $\{ 0,1 \}$ look like?

It looks like a cylinder, with the angle B giving the “height” above the bottom of the cylinder, and that angle A giving (half the) angle around the base.



But often one is interested only in the shape up to deformation, a.k.a homotopy. Since the cylinder squashes down to a circle, the answer is then that up to homotopy the shape is a circle.

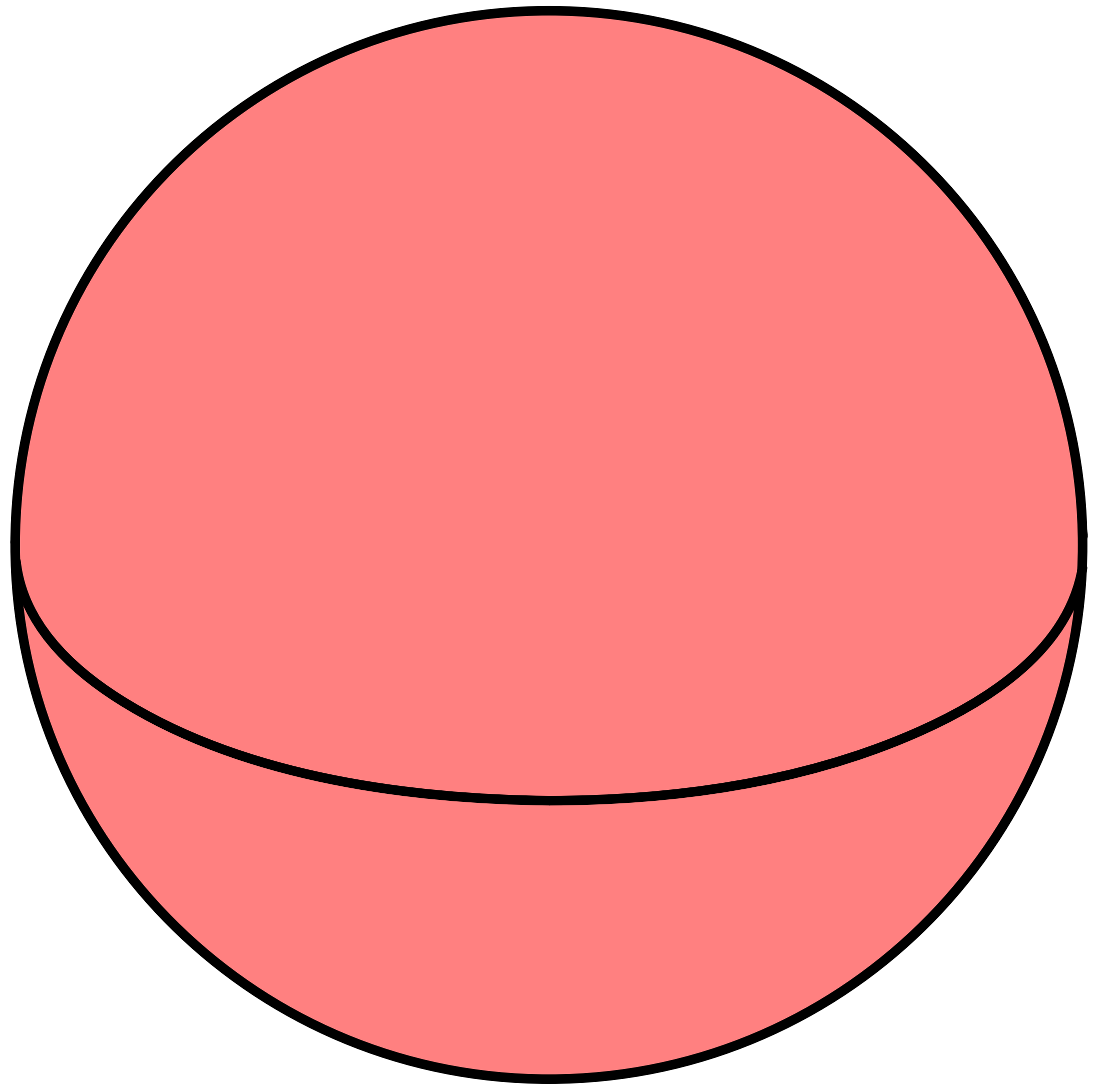


Topology and spectrum

What about complex two by two matrices with spectrum $\{\, 0,1 \,\}$?

It's relevant to ask this because in many contexts the complex answer is simpler than the answer using real numbers.

The answer here is that up to homotopy the shape is a two-dimensional sphere.



For the record, the exact answer is that the shape is the space of all tangent vectors to the sphere (a.k.a the tangent bundle). For comparison, note that the space of all tangent vectors to a circle is a cylinder.

And note that the real space is indeed a subspace of its complex cousin.

Symmetric and anti-symmetric matrices

Here are examples of symmetric and anti-symmetric (real) matrices:

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & b & c \\ -b & 0 & e \\ -c & -e & 0 \end{bmatrix}$$

The symmetric matrices behave especially simply with regard to eigenvalues and eigenvectors. I shall discuss them, or at least mention them, in the next lecture.

Anti-symmetric matrices will be of special interest in this lecture.

An interesting fact: all the eigenvalues of an anti-symmetric matrix are purely imaginary complex numbers. And they always come in pairs $\pm i\lambda$.

So for example, 0 must be an eigenvalue of the anti-symmetric matrix above.

The Pfaffian and the topology of anti-symmetric matrices

I'll generally be examining invertible anti-symmetric matrices, to avoid the eigenvalue 0, which is special for anti-symmetric matrices.

The space of all invertible anti-symmetric matrices separates into two parts according to the sign of the Pfaffian, which is

$$\mathrm{Pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + cd$$

Its square is the determinant (so the determinant is always positive).

Topology of anti-symmetric matrices

Now let's take a closer look at invertible 4×4 anti-symmetric matrices …

$$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$

There are 4 eigenvalues, in two imaginary pairs, $\pm\lambda$ and $\pm\mu$. I'm assuming that λ and μ are nonzero. I shall also consider matrices where the two eigenvalue pairs are distinct (so, no multiplicities). The other case, where this isn't assumed, is interesting, too. But the present case is the most interesting.

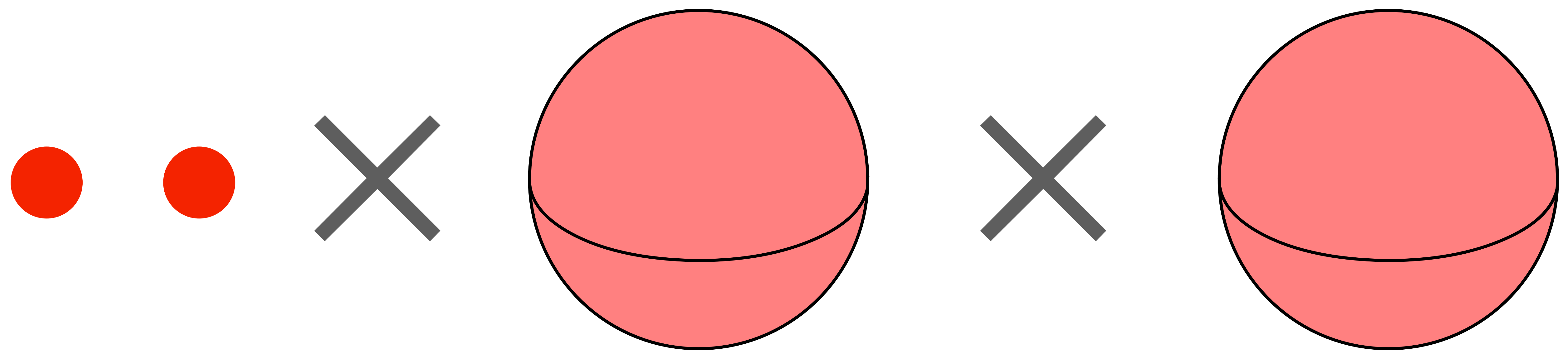
What is the shape of this space?

Topology of anti symmetric matrices

The values λ^2 and μ^2 give two coordinates on this space. But they're like the angle B in the 2×2 matrix example: after deformation they don't help describe shape of the space. I shall ignore them.

What is left as regards the shape is perhaps a bit surprising, and not so easy to calculate. What is left is the product of two 2-dimensional spheres (for each of the two components separated by the Pfaffian).

That's a 4-dimensional space, but there is plenty of room for it inside the space of anti-symmetric matrices, which is 6-dimensional.

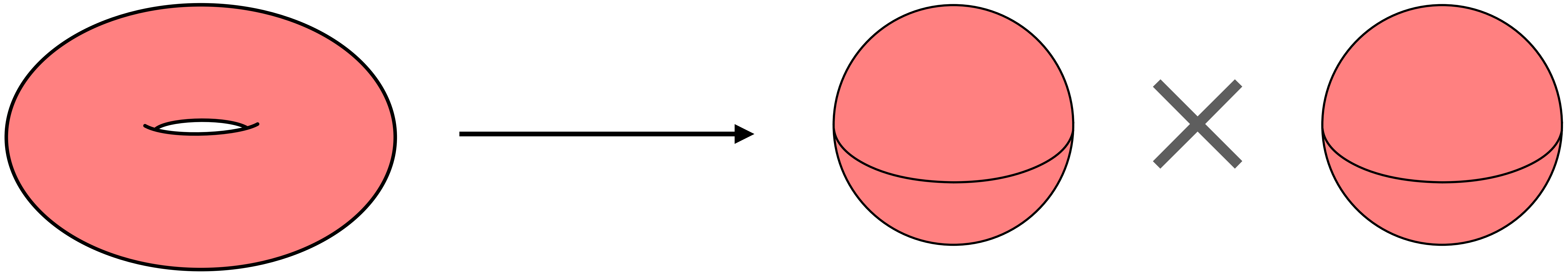


Higher topological Pfaffians

The title of this slide is a bit fanciful … it's supposed to suggest $\mathrm{Sign}(\mathrm{Pf}(A))$.

Suppose one is presented with a family of 4×4 antisymmetric matrices, with $\lambda^2 \neq \mu^2$, parametrized by points on a 2-torus. This isn't fanciful; it will actually happen later in the lecture!

How can one begin to understand family? One can analyze the map below (using degree theory) to find out at least something about it.



Some calculations

Where do those spheres come from? I'm glad you asked …

1. Let's assume that $\lambda^2 > \mu^2$. Take the $\pm i\lambda$ eigenvectors, and take their wedge product.

2. Take the $\pm i\lambda$ eigenvectors, and take their wedge product.

3. The wedge product lies in (i times) the second exterior power of 4-space.

4. The 6-dimensional second exterior power of 4-space decomposes into a 3-dimensional self-dual part, and a 3-dimensional anti-self-dual part.

5. Project the wedge product into each part, and then onto the unit sphere in each part.

A brief comment about stable versus unstable

For stable topological invariants, the size of the matrices is not so important. For unstable topological invariants, size matters.

The higher degree invariants I've defined are unstable, but there sum is stable, and it is the sum that will have the most obvious application in what follows.

The higher degree invariants I've defined are unstable, but their sum is stable, and it is the sum that will have the most obvious application in what follows.

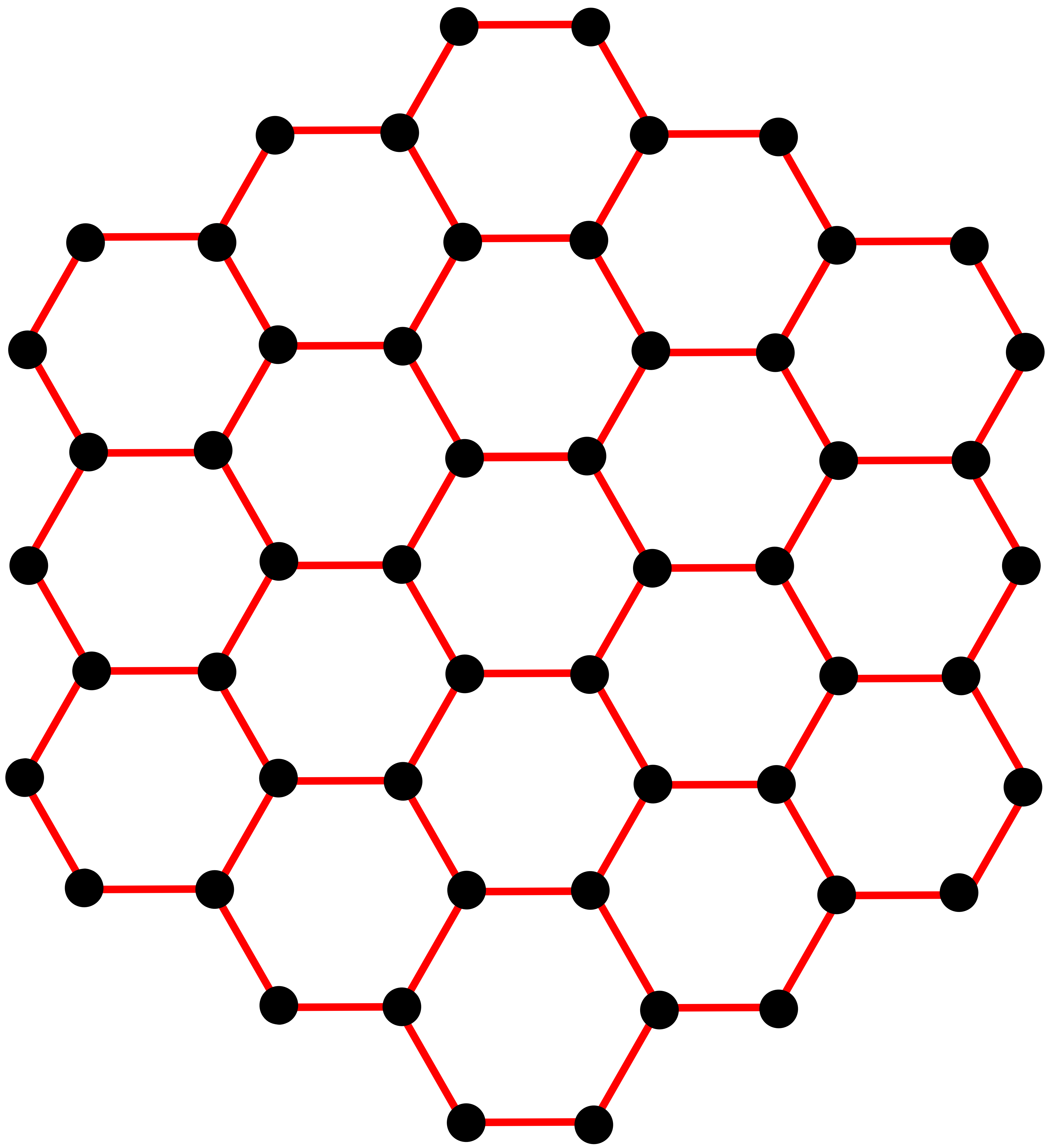
The most famous statement about stable invariants is the Bott periodicity theorem. It will feature in what follows (but I hope to manage without precisely formulating the result).

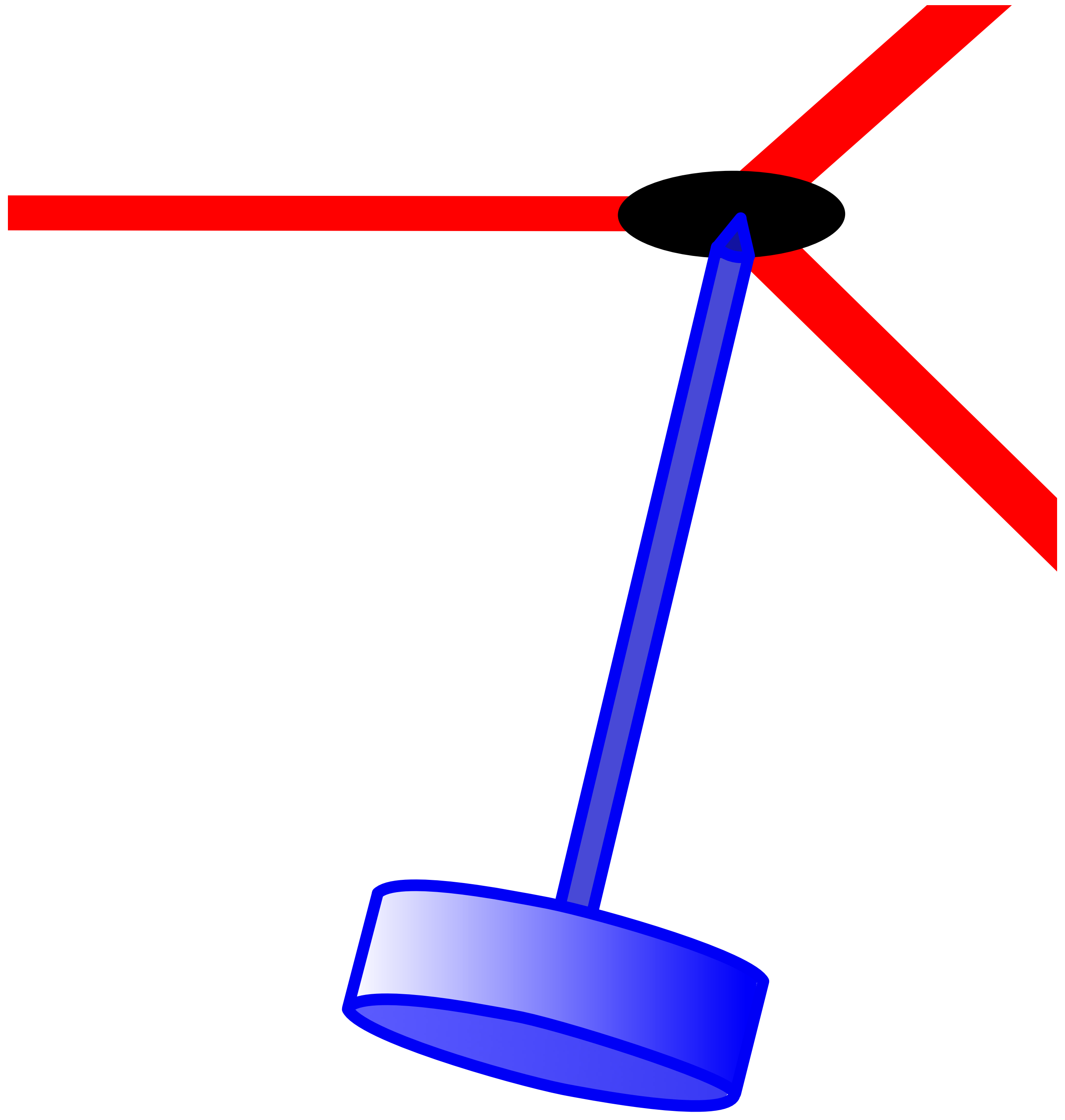
From the spectrum and topology to mechanics …

But first, any questions?

Image credit: Wikipedia

A 54-gyroscope network ...

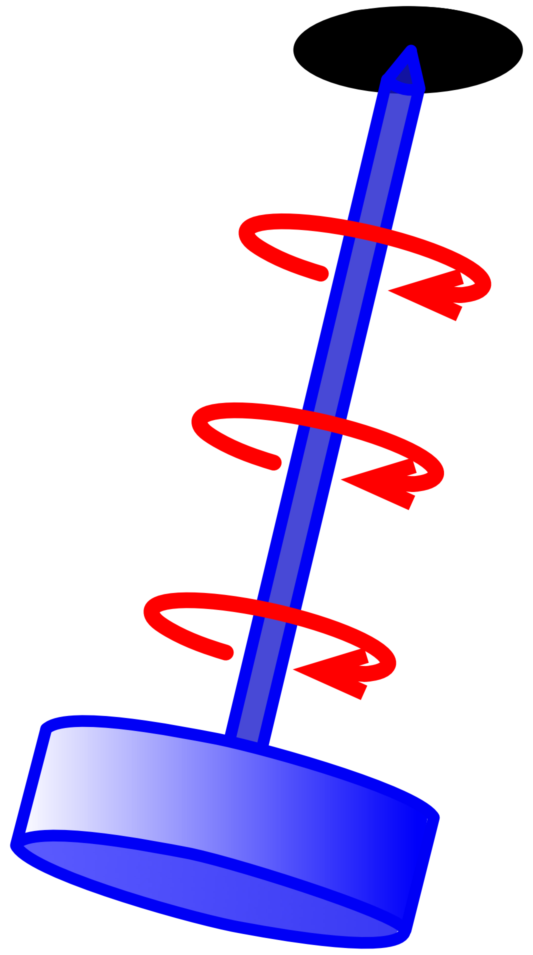


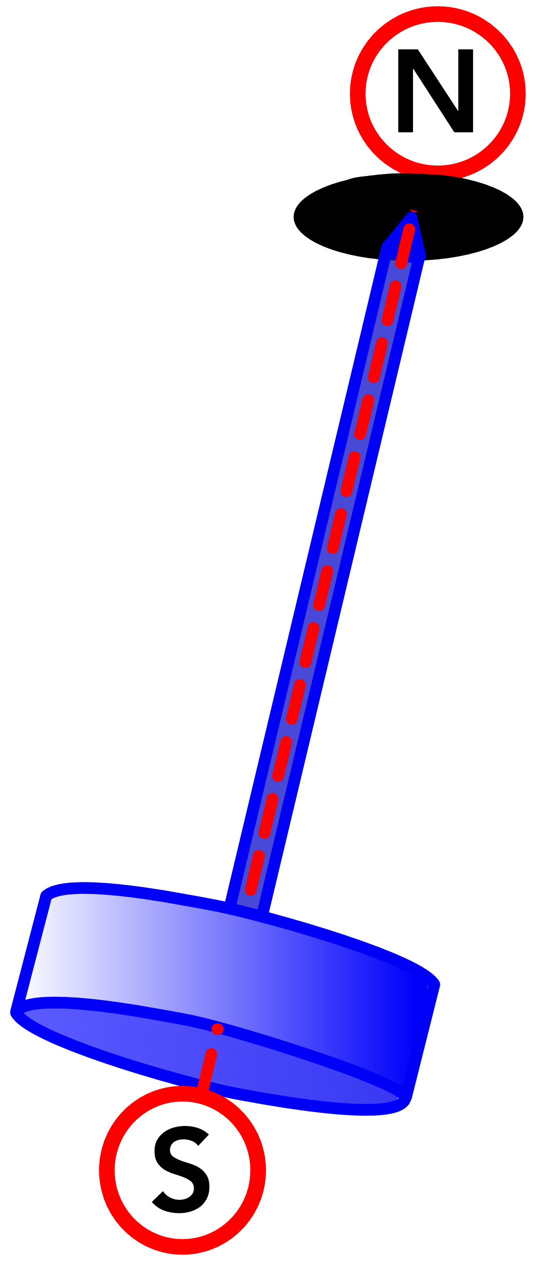


For the rest of the talk, I shall examine a network like the one here … under each node of which there is suspended a rapidly spinning gyroscope. The gyroscopes are magnetized, and so interact with one another via magnetism. What happens?

Mechanical details of the system

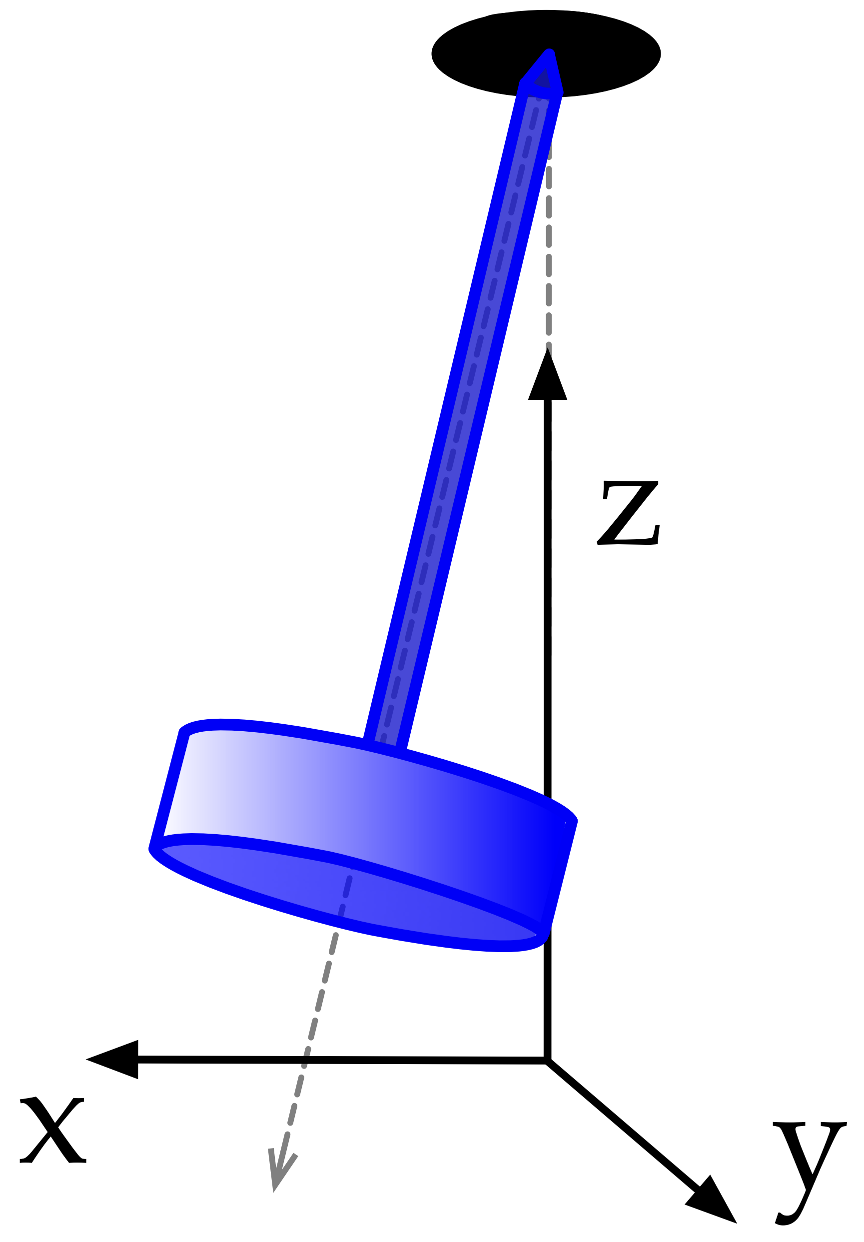
Once again, the gyroscopes are rapidly spinning, all in the same direction and at the same rate.





And they are all magnetized in the same way, to the same degree.

Mathematical details



The state of each gyroscope is described by a point in the (x, y) plane. A state of the gyroscope network is therefore described by a list of 54 points in the (x, y) plane.

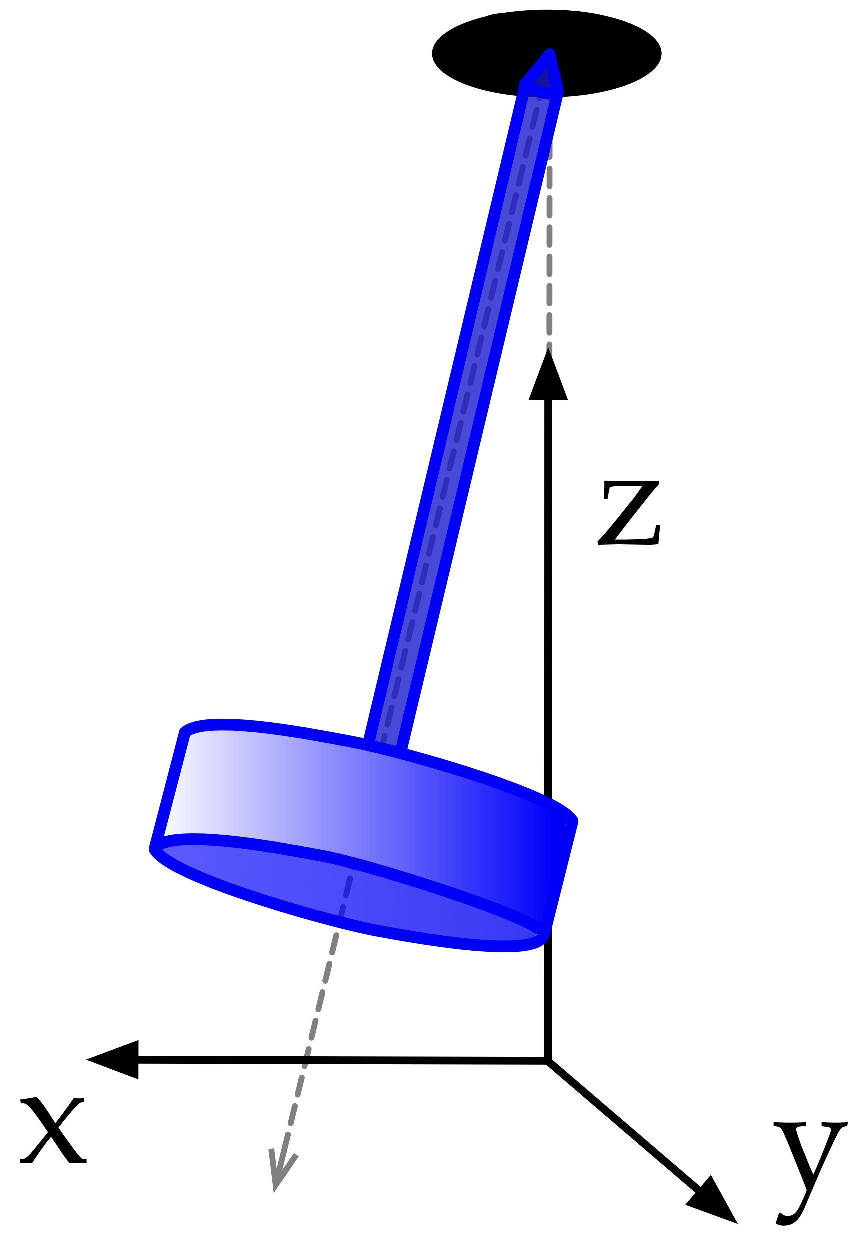
That is, a state ψ of the gyroscope network is described by vector of length 108.

The (first order!) differential equation describing the evolution has the form

$$J\frac{d\psi}{dt} = (\alpha I + \beta d^* P d + \gamma d^* P^{\perp} d)\psi$$

J and everything in the parentheses are matrices, and ψ is a vector-valued function of t.

Mathematical details



$$J\frac{d\psi}{dt} = (\alpha I + \beta d^* P d + \gamma d^* P^\perp d)\psi$$

Here P is an orthogonal projection matrix, and $\alpha, \beta, \gamma > 0$.
So the matrix in parentheses is positive-definite. Also:

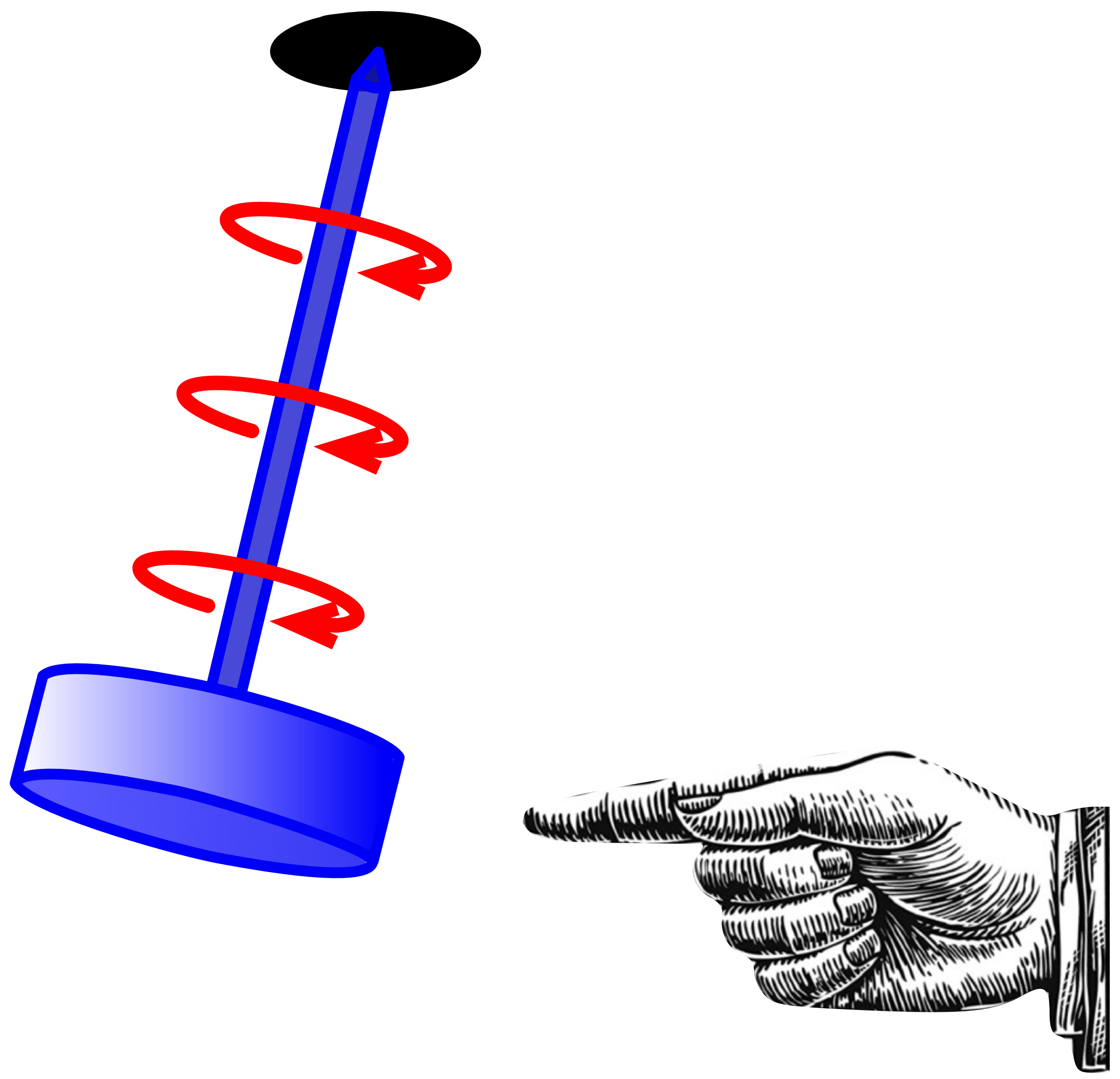
$$J = \begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{bmatrix}$$

This gives

$$\frac{d\psi}{dt} = -JD\psi$$

This matrix is similar to $D^{1/2}JD^{1/2}$, which is anti-symmetric

Anti-symmetric matrices and the gyroscope network



The system is governed by a first-order equation

$$\frac{d\psi}{dt} = A\psi$$

where A is (basically) an anti-symmetric matrix. By design, this is very similar in mathematical form to Schrodinger's equation from quantum mechanics.

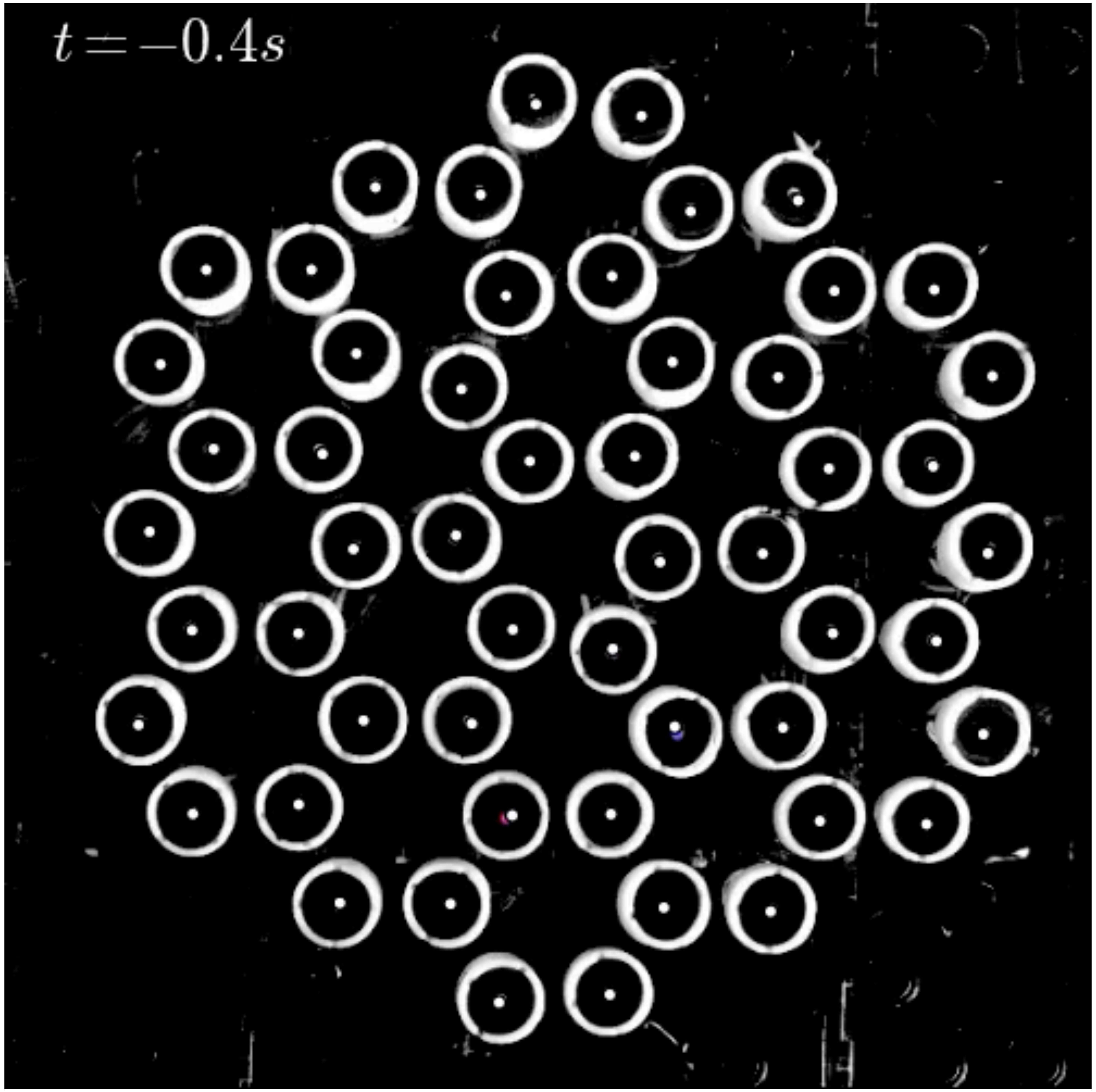
This takes advantage of an odd feature of gyroscopes, that a force applied produces a torque in a direction *orthogonal* to the force and the axis.

An experiment

Lisa M. Nash et al
Topological mechanics of gyroscopic meta materials
PNAS **112** (2015)

The view is directly from below the gyroscope network.

The network (with the gyroscopes already rapidly spinning) is going to be excited at a precise frequency at a single gyroscope on the upper right of the array.

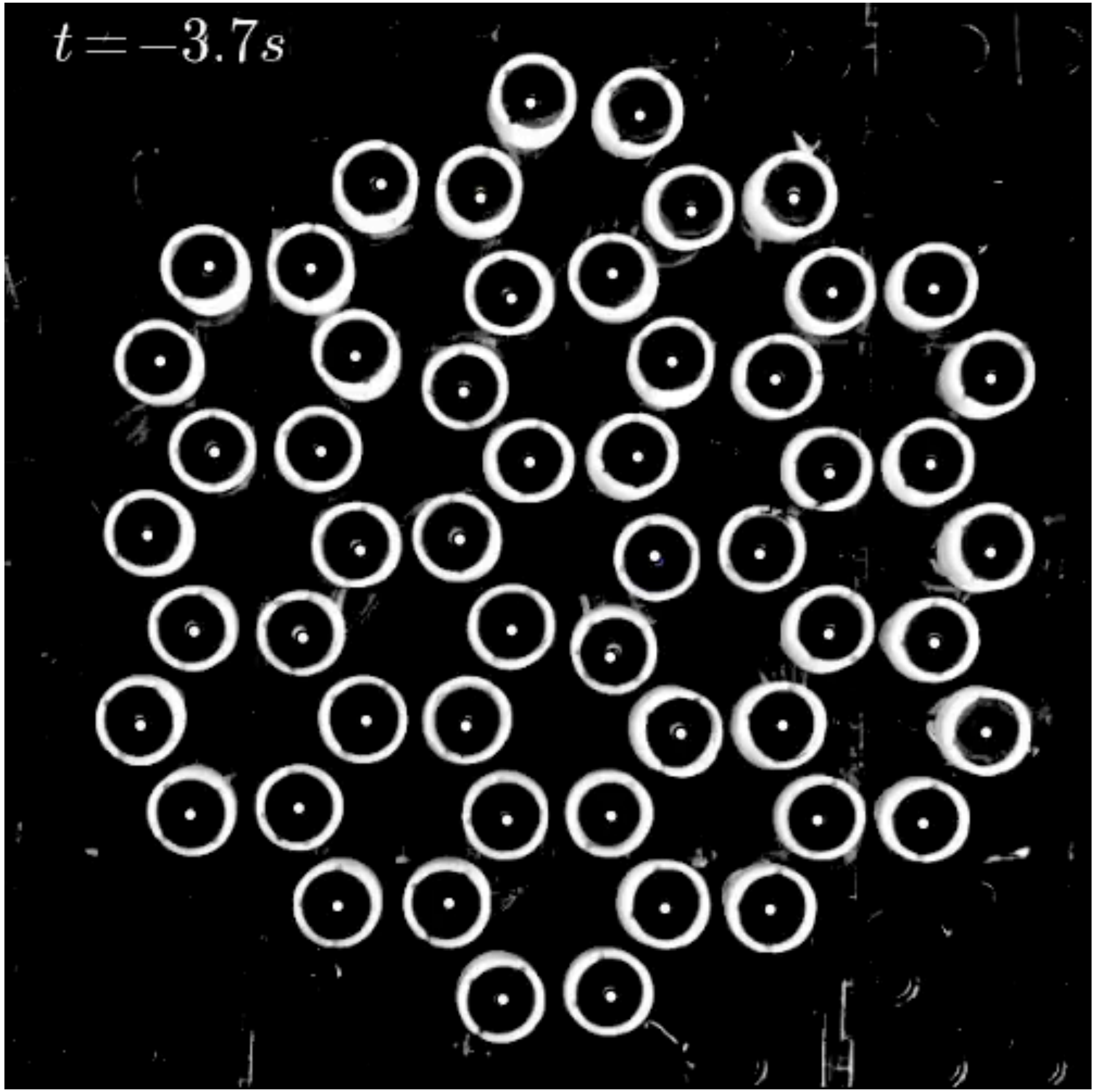


Another experiment (edge modes)

Lisa M. Nash et al
Topological mechanics of gyroscopic meta materials
PNAS **112** (2015)

The experiment will be repeated.

This time a gap mode frequency will be used (I shall try to explain in the following slides what these frequencies are).

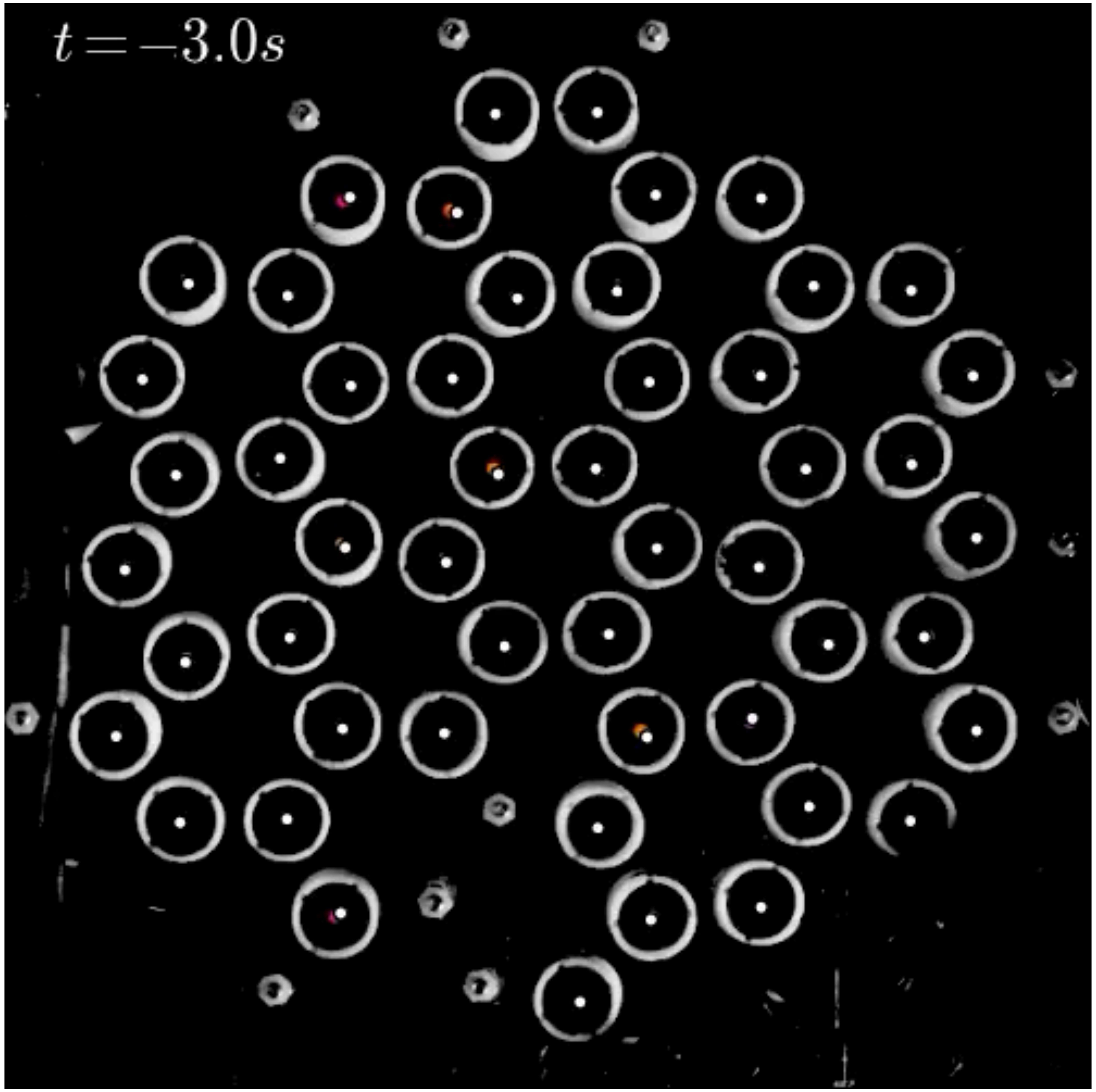


Topological nature of the boundary modes

Lisa M. Nash et al
Topological mechanics of gyroscopic meta materials
PNAS **112** (2015)

Three gyroscopes have been removed from the network, at the bottom.

The same gap mode frequency will be used to excite the network.



Normal modes

Let me return to the differential equation

$$\frac{d\psi}{dt} = -JD\psi$$

If v, a complex vector of length 108, is an eigenvector with eigenvalue $i\lambda$ for JD (which is a 108×108 matrix) then the functions

$$\psi(t) = \mathrm{Re}\left[e^{i\lambda t}v\right] \qquad \text{and} \qquad \psi(t) = \mathrm{Im}\left[e^{i\lambda t}v\right]$$

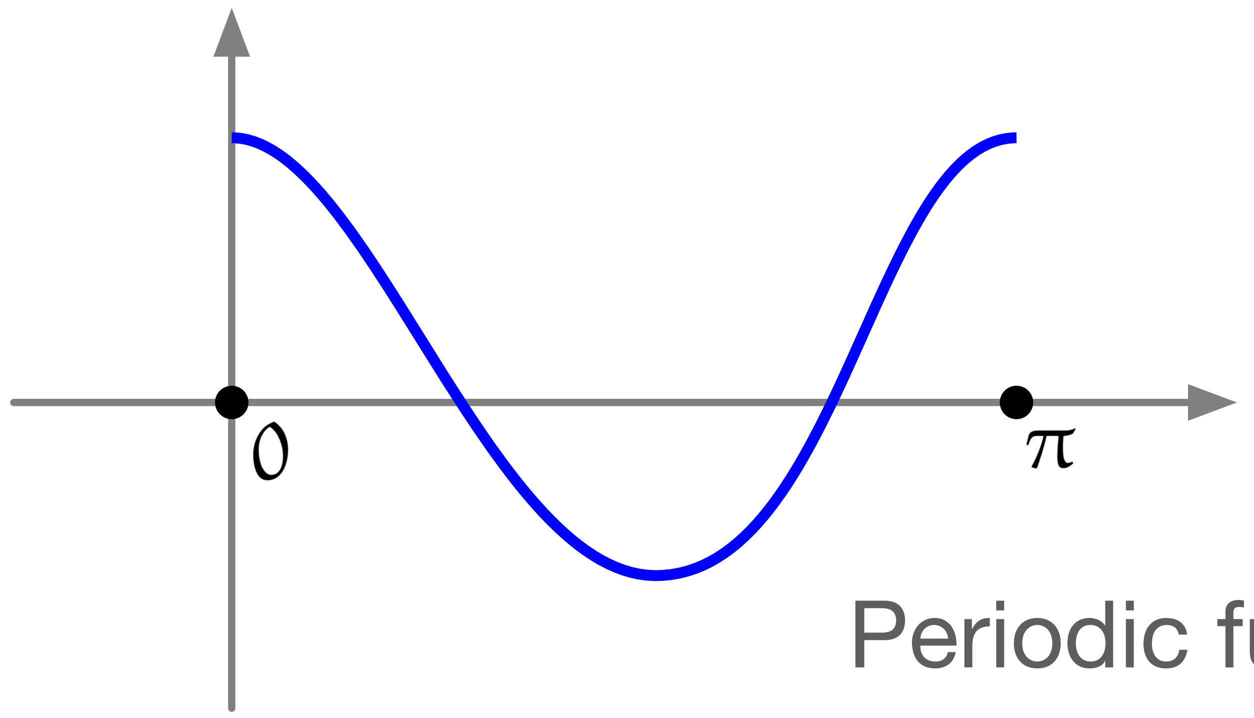
are solutions. These are the *normal modes*, and all solutions are built from them.

So everything about the system is in principle understandable in terms of the eigenvalues and eigenvectors of JD. Are some eigenvalues more special than others?

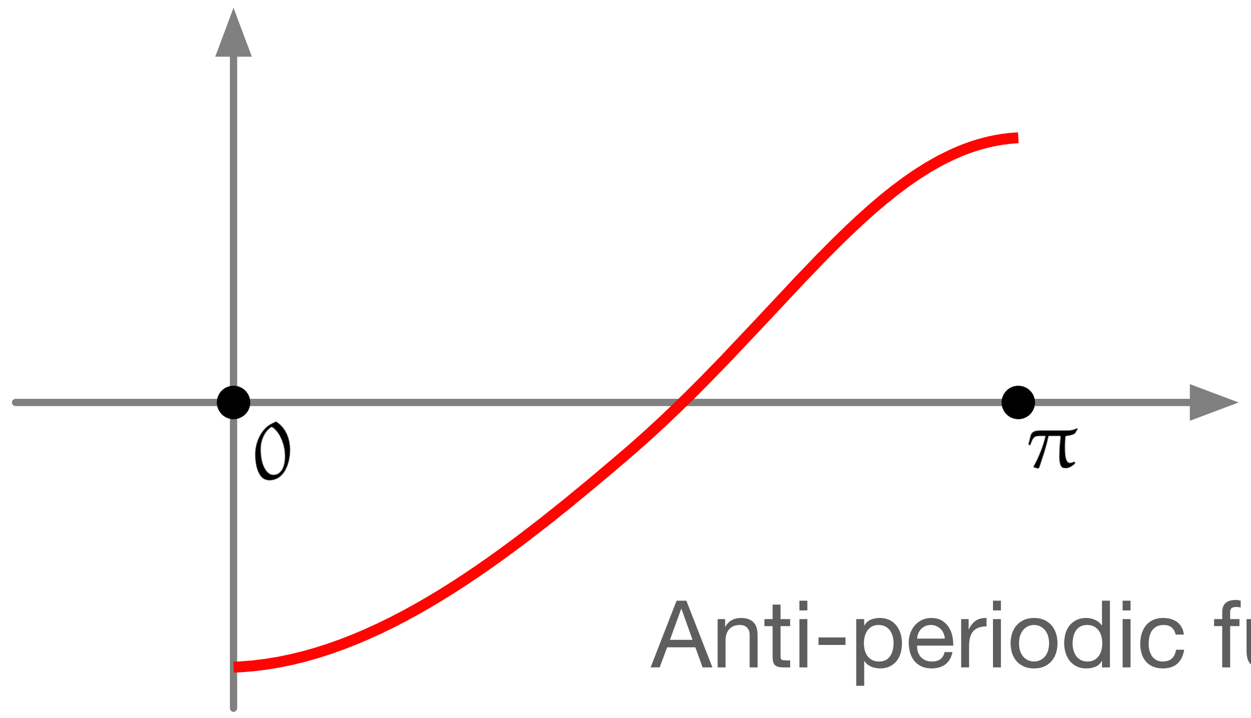
Topologically protected spectrum

Sometimes, some eigenvalues *are* more special than others. For example if an anti-symmetric real matrix has odd size, then 0 is definitely an eigenvalue, and no amount of adjusting the entries of the matrix will make it go away.

One might say that the value 0 is *topologically protected*. It is invariant under (in this case extreme) deformations of the matrix.



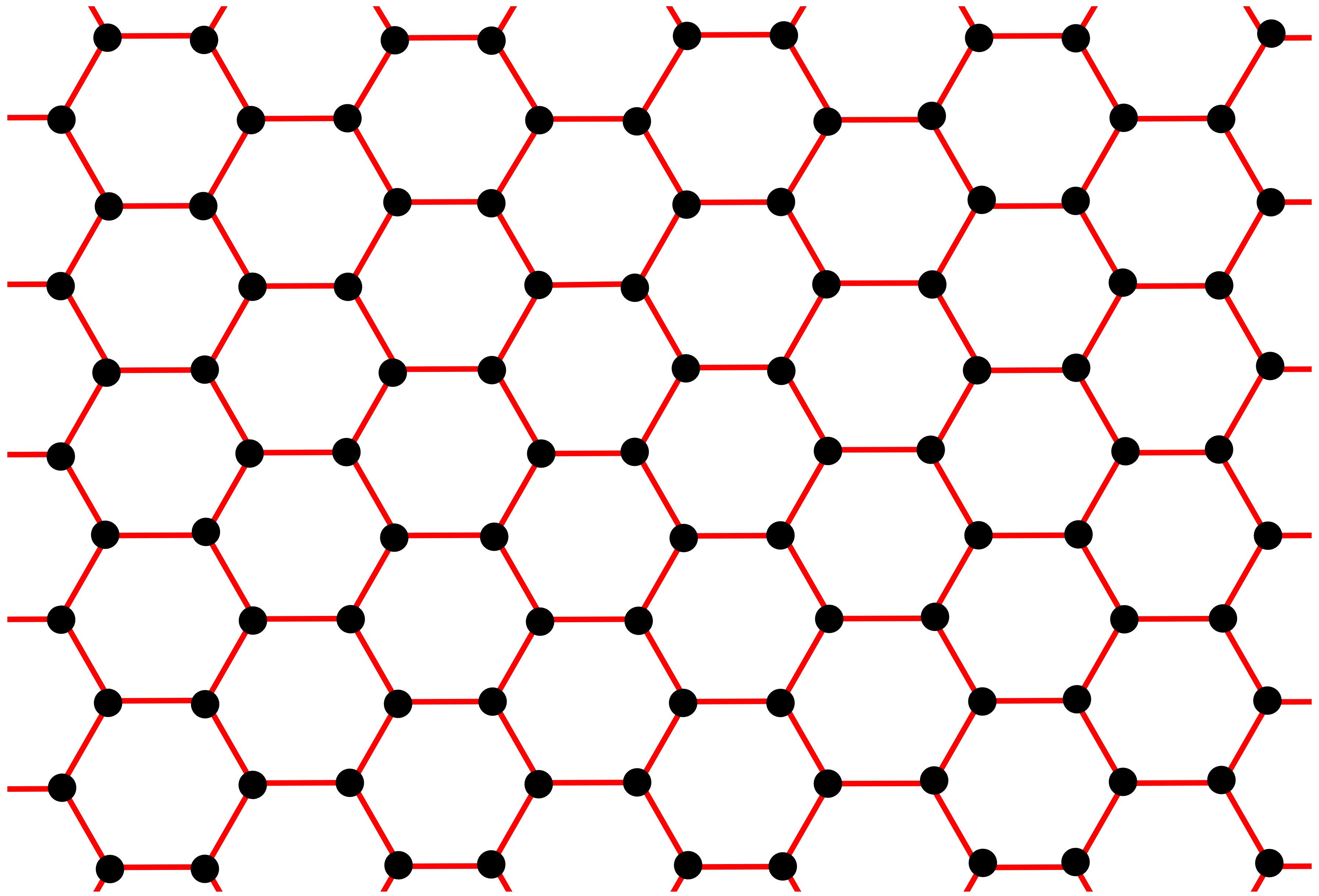
Periodic function



Anti-periodic function

Special features about the eigenvalue 0 were exploited to the full by Atiyah and Singer in the index theorem.

Topology, the spectrum and the gyroscope network



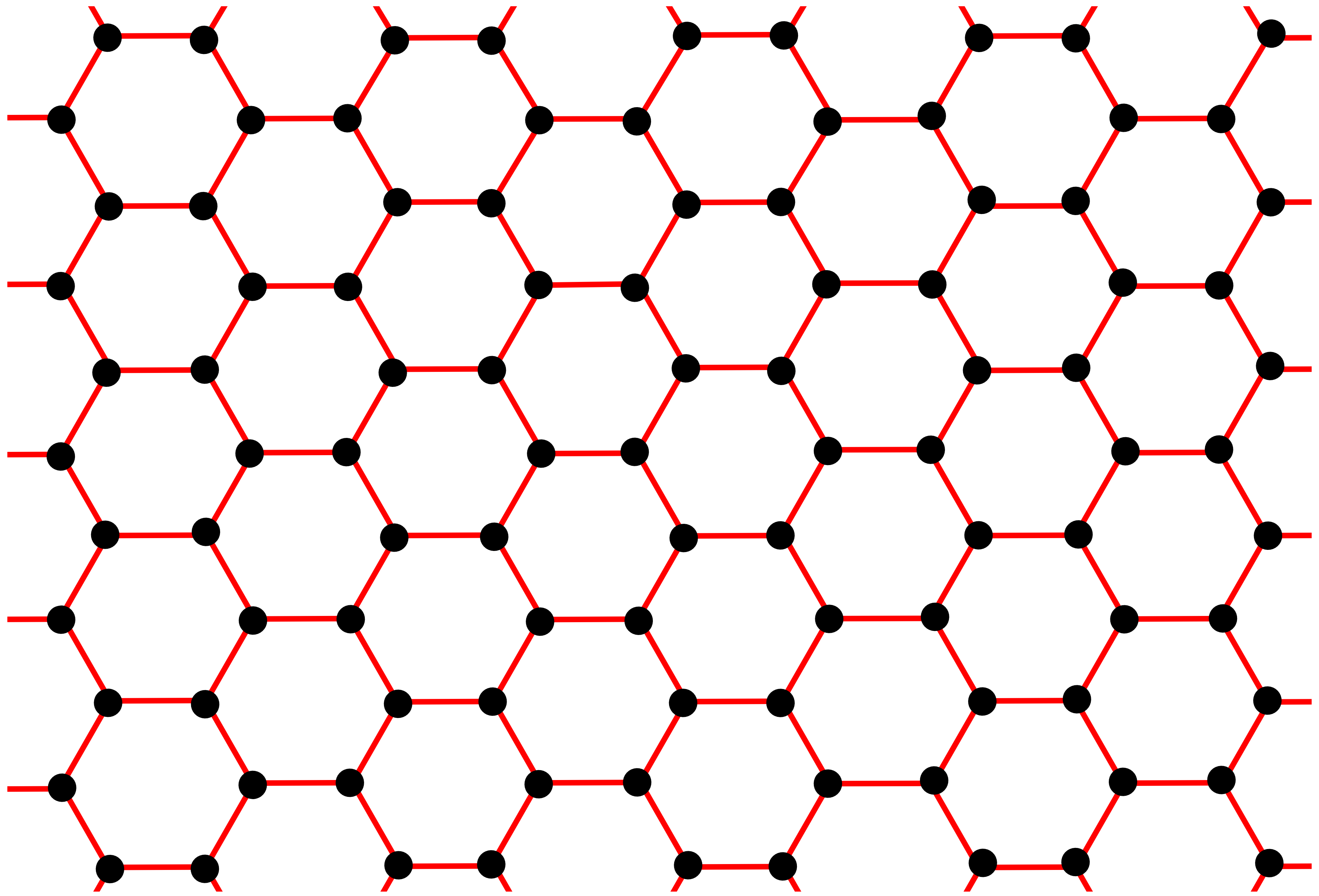
To bring topology into the gyroscope problem, let's examine an *infinite* network.

(With more care, one could replace *infinite* networks by *very large* networks, or perhaps even somewhat large networks.)

What is the spectrum of JD in the infinite network case?

An interesting (but not terribly surprising) fact: the frequencies (spectral values) corresponding to boundary modes are missing from the spectrum in the infinite case.

Computation of the spectrum for the infinite network



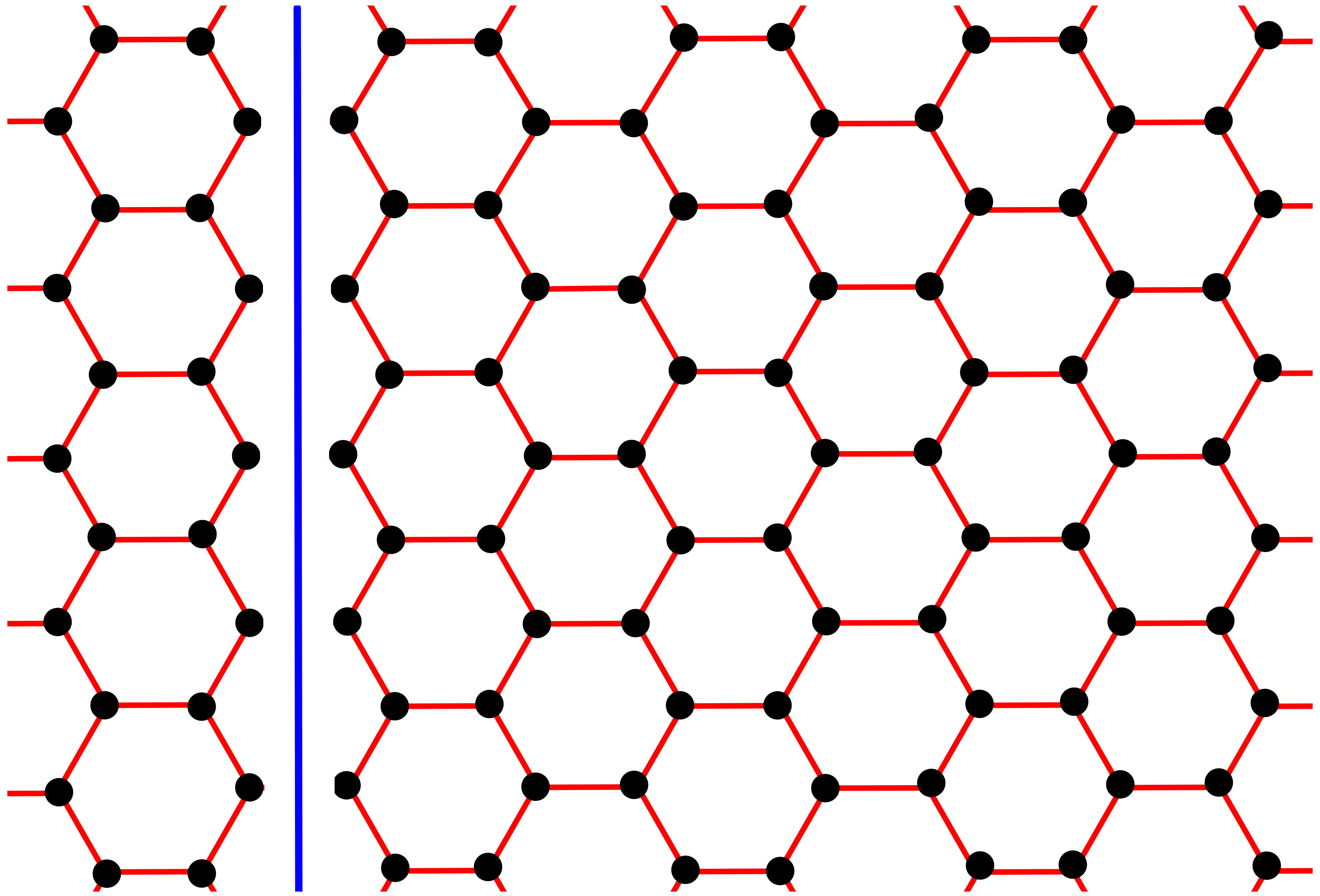
The computation of the spectrum of an $\infty \times \infty$ matrix is in general hugely difficult.

But here, the network has double translational symmetries, and double Fourier series can be used.

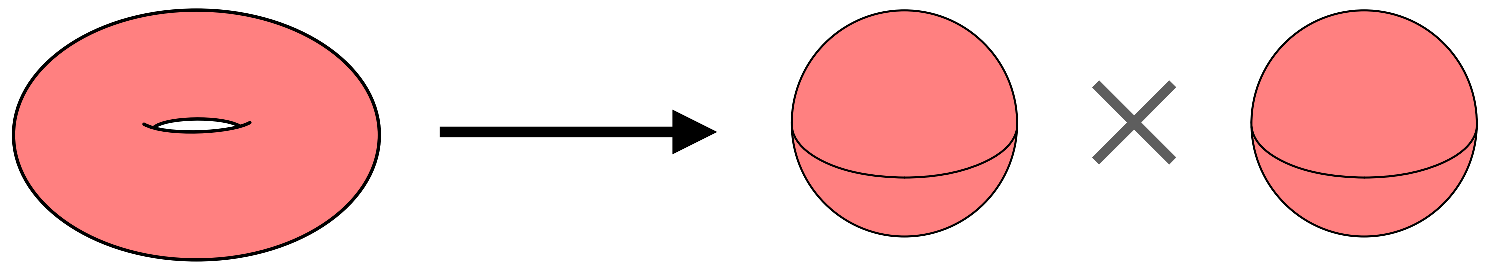
Conclusion: the spectrum is the same as the combined spectrum of a family of anti-symmetric 4×4 matrices … parametrized by a 2-torus.

And for all of these 4×4 matrices, if the mechanical parameters are tuned appropriately, then $\lambda^2 > \mu^2$. In fact $\lambda^2 \geq \ell > m \geq \mu^2$. So there is a gap in the spectrum of JD on the infinite network,

Consequences of the topological computation



Earlier I used, fancifully, the term *higher topological Pfaffians* (for topological degrees).

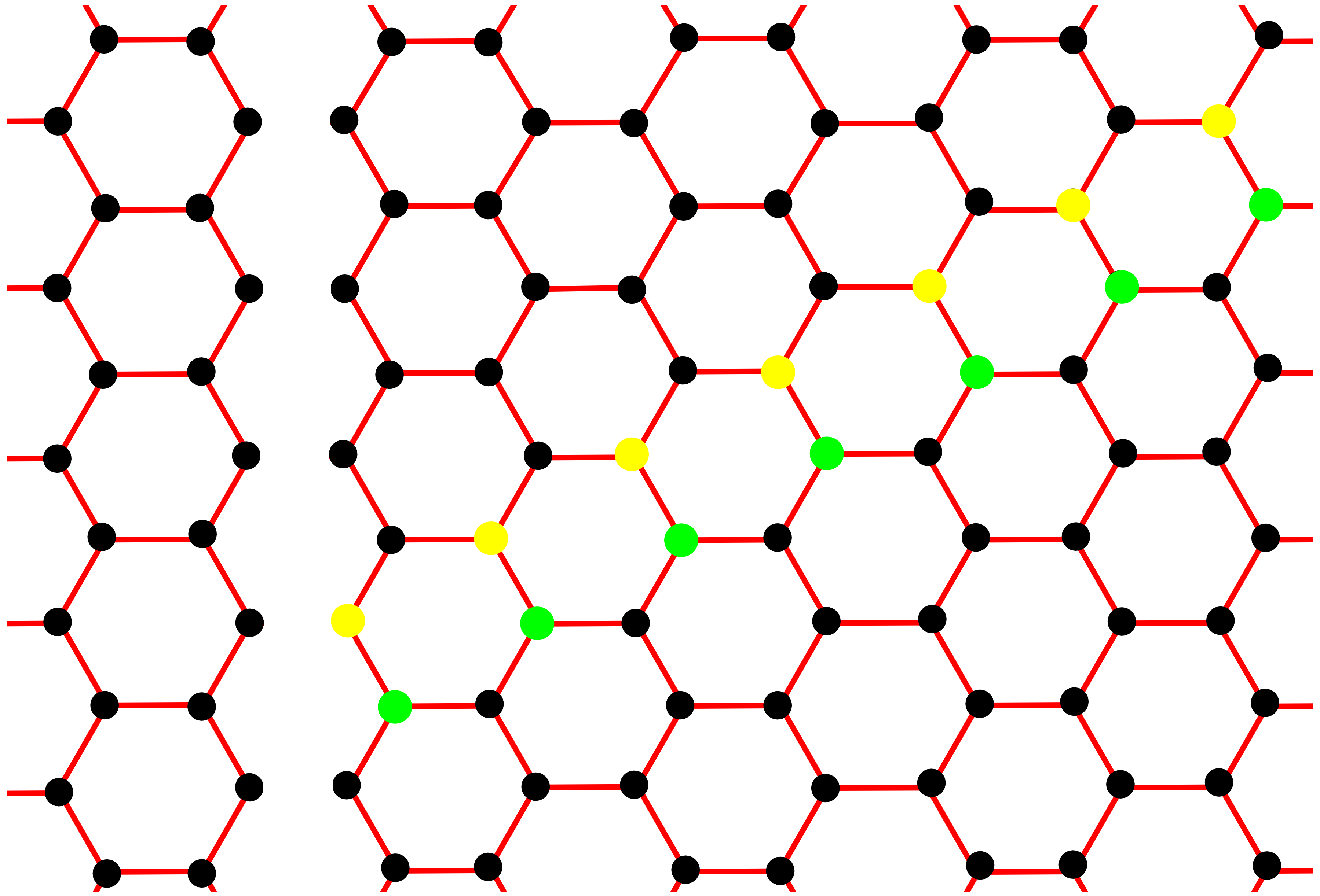


Here they are non-zero (the important number is the sum). This has a striking consequence …

Consider the infinite network to the right of a vertical cut. The spectrum of JD for this half-infinite network always includes the spectrum for the full infinite network.

Can it be larger? Yes, it must be larger, thanks to the topological degree computation!

Spectrum on the half-infinite network



The computation of the spectrum is much harder on the half-infinite network, since there are fewer symmetries.

But there is still one translational symmetry. The spectrum is the same as the combined spectrum of a family of anti-symmetric 4×4 block Toeplitz matrices, parametrized by circle.

Now, (thanks to Bott periodicity) we know a great deal about families of Toeplitz matrices, and we know in particular that there are *no nonzero higher topological invariants associated to them.*

Oddly, from the point of view of topology, Toeplitz matrices behave like 1×1 matrices!

What is a Toeplitz matrix?

They are $\infty \times \infty$ matrices that are constant along every diagonal, like this:

$$\begin{bmatrix} a_0 & a_1 & a_2 & & & \\ a_{-1} & a_0 & a_1 & a_2 & & \\ & a_{-1} & a_0 & a_1 & a_2 & \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & & & \end{bmatrix}$$

The extensive theory of Toeplitz matrices is organized around the analogy between the matrices and associated Fourier series $\sum a_n e^{2\pi i n\theta}$.

No-gap theorem

The gap between ℓ and m in the spectrum of the infinite network is entirely filled in the spectrum of the half-infinite network [which is a model for our finite network].

Proof.

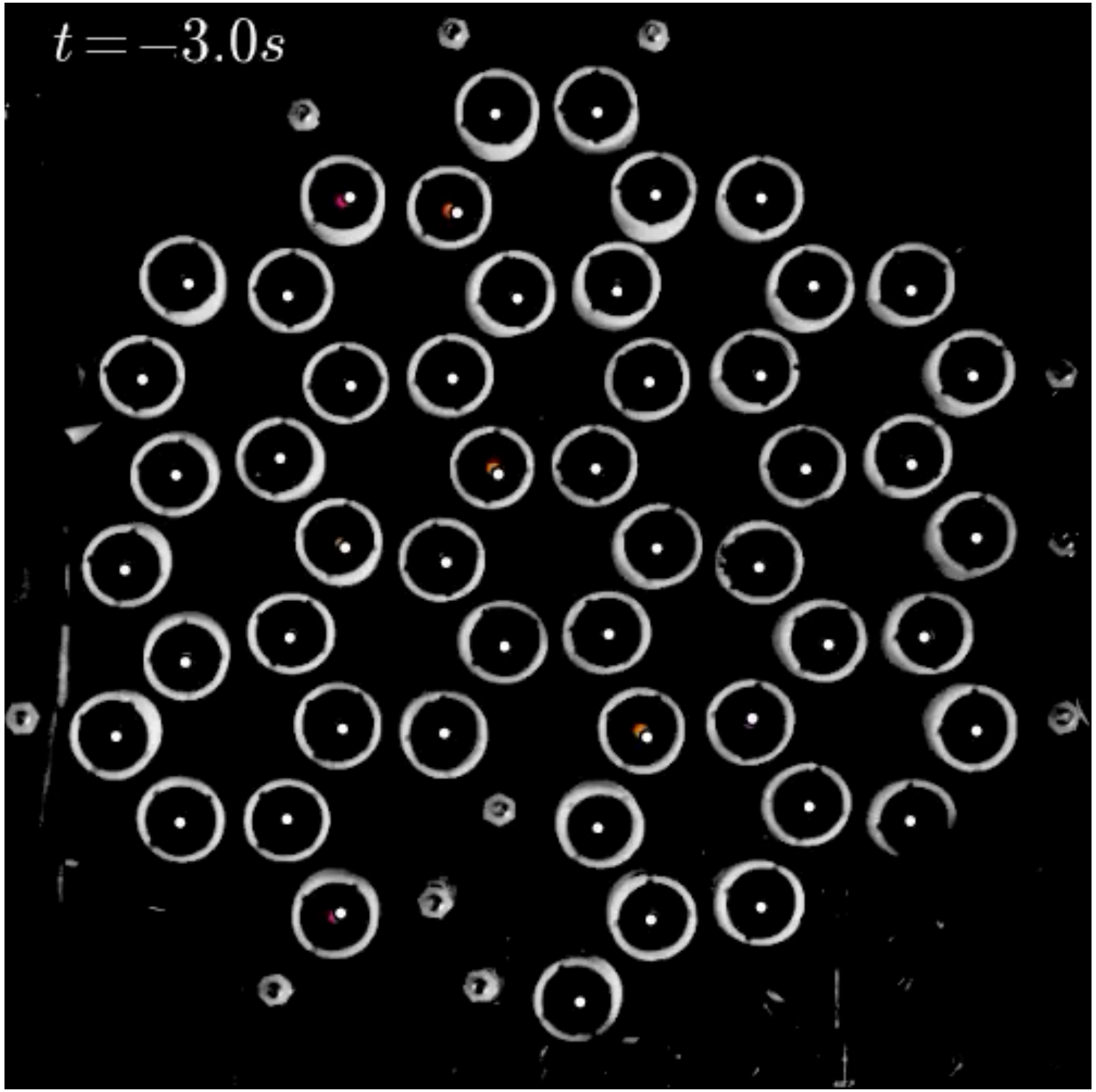
1. IF some point between ℓ and m was missing from the spectrum for the half-infinite network, THEN we could form the topological invariants of the torus family for the full network using circle families of Toeplitz operators for the half-infinite network.

2. BUT in that case the topological invariants for the torus family would be zero.

3. Which they're not.

4. THEREFORE the entire interval between ℓ and m belongs to the spectrum for the half-infinite network

Thank you!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