The Oka Principle and a K-Theoretic Perspective on the Langlands Classification

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Introduction

I want to travel around a diagram that looks roughly like this:

$$\begin{tikzcd} \bullet \arrow[r, "\text{Paley-Wiener}"] \arrow[d, "\text{Connes-Kasparov}"'] & \bullet \arrow[d, "\text{Oka}"] \\ \bullet & \bullet \arrow[l, "\text{Langlands}"] \end{tikzcd}$$

Novodvorskii's Theorem

THEOREM Let A be a commutative Banach algebra. The Gelfand transform for A induces an isomorphism

$$K_*(A) \xrightarrow{\cong} K_*(C_0(\hat{A})).$$

This is from 1967 . . .

Polynomially Convex Sets and the Oka Principle

The polynomially convex hull of a compact set $K \subseteq \mathbb{C}^n$ is

$$\left\{ z \in \mathbb{C}^n : |p(z)| \leq \max_{w \in K} |p(w)| \quad \forall p \right\}.$$

THEOREM (Oka principle, after Grauert) If U is an open, polynomially convex subset of $\mathbb{C}^n$, then each complex topological vector bundle on U carries a unique holomorphic structure.

Polynomially Convex Sets and Banach Algebras

If $K \subseteq \mathbb{C}^n$ is compact and polynomially convex, and if

$$B(K) = \begin{array}{l}\text{uniform norm closure of the}\\ \text{polynomial functions on } K\end{array}$$

then the Gelfand transform for $B(K)$ is

$$B(K) \xrightarrow{\text{inclusion}} C(K)$$

When $A = B(K)$, Novodvorskii's theorem may be proved rather easily using Grauert's Oka principle.

Actually the two results are more or less equivalent (up to the usual distinction between isomorphism and stable isomorphism)

And actually, the case where $A = B(K)$ is the critical case for Novodvorskii's theorem.

Beyond Commutative Banach Algebras

Can one take the Oka principle beyond commutative Banach algebras?

This is an old question, usually asked in the context of group convolution algebras and the Baum-Connes conjecture.

I shall discuss one aspect of it, which involves travelling only a short distance beyond commutative algebras ... plus Langlands...

Tempered Representations and Admissible Representations

- G = real reductive group
  (for this talk, I'll take $G = SL(n, \mathbb{R})$
  or $G = SL(n, \mathbb{C})$, mostly)
- The irreducible tempered representations of G are defined and studied by Harish-Chandra. They constitute the spectrum of $C_r^*(G)$.
- The irreducible admissible representations are not necessarily unitary. They should be classified up to infinitesimal equivalence.

Langlands Classification

Langlands classified the irreducible admissible representations of G in terms of the irreducible tempered representations.

The method is remarkably simple, even geometric (polar decomposition)

EXAMPLE $G = \mathbb{C}^\times$ Write $G \cong \mathbb{T} \times \mathbb{R}_+^\times$. The parameters for irreducible admissible representations are pairs

$$(\text{irred. tempered rep. of } \mathbb{T},\ \text{complex number})$$

Standard Parabolic Subgroups

For $SL(n,\mathbb{R})$ or $SL(n,\mathbb{C})$ these are the block upper triangular subgroups, e.g.

$$P = \left\{ \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ 0 & 0 & \bullet \end{pmatrix} \right\}$$

Each has a Langlands decomposition $P = MAN$, e.g.

$$\underbrace{\left\{ \begin{pmatrix} \boxed{\mathrm{Det} = \pm 1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & \boxed{\pm 1} \end{pmatrix} \right\}}_{M} \times \underbrace{\left\{ \begin{pmatrix} \boxed{e^{sI}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & \boxed{e^{t}} \end{pmatrix} \right\}}_{A} \times \underbrace{\left\{ \begin{pmatrix} \boxed{I} & \begin{matrix} \bullet \\ \bullet \end{matrix} \\ 0 \quad 0 & \boxed{1} \end{pmatrix} \right\}}_{N}$$

Langlands Parameters

A Langlands parameter for an admissible irreducible representation is a triple (P, σ, ν) where

- $P = MAN$ is a standard parabolic subgroup
- σ is an irr. tempered rep. of M, up to equivalence
- $\nu : \mathfrak{a} \longrightarrow \mathbb{C}$ is $\mathbb{R}$-linear, with $\mathrm{Re}(\nu)$ dominant (in our $SL(3)$ example this means $\nu : \begin{pmatrix} s & & \\ & s & \\ & & t \end{pmatrix} \longmapsto as + bt$ with $\mathrm{Re}(a) > \mathrm{Re}(b)$).

Parabolic Induction

Because $P = MAN = MA \ltimes N$, we can attach to (P, σ, μ) an induced representation:

$$(P, \sigma, \mu) \longmapsto \mathrm{Ind}_P^G \, \sigma \otimes \exp(\mu)$$

(I'm ignoring a small normalizing factor).

THEOREM For each Langlands parameter, the induced representation above has a unique irreducible quotient, and every irreducible admissible rep. arises this way precisely once.

A Case Study: $SL(2,\mathbb{R})$

For $SL(2,\mathbb{R})$ there are two standard parabolics

- G itself, which accounts for the tempered dual in the Langlands theorem.

- The minimal parabolic subgroup

$$P = \underbrace{\left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\right\}}_{M \cong \mathbb{Z}_2} \times \underbrace{\left\{\begin{pmatrix} e^{s} & 0 \\ 0 & e^{-s} \end{pmatrix}\right\}}_{A \cong \mathbb{R}} \times \underbrace{\left\{\begin{pmatrix} 1 & \bullet \\ 0 & 1 \end{pmatrix}\right\}}_{N}$$

of all upper triangular matrices, which accounts for all the non-tempered reps

A Case Study: $SL(2,\mathbb{R})$

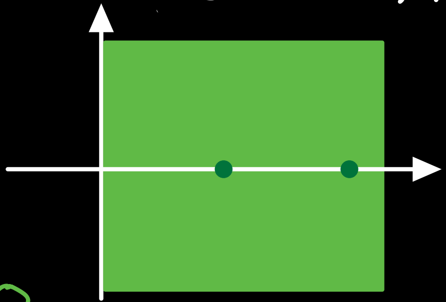
So the non-tempered irreducible admissible reps are classified by

$$(P, \sigma, \mu)$$

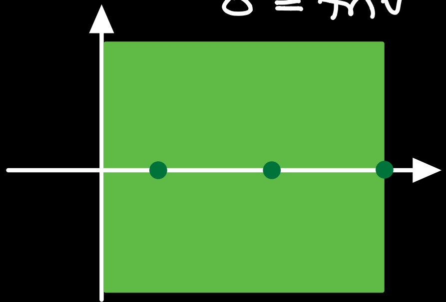
$\sigma \in \widehat{\mathbb{Z}_2}$

$\mu \in \mathbb{C}, \mathrm{Re}(\mu) > 0$

$\sigma =$ nontriv



$\sigma =$ triv



Generically the induced rep is irreducible. At the indicated points the Langlands quotient is non-trivial (& finite-dimensional).

Fourier Transform Picture of Group Convolution Algebras

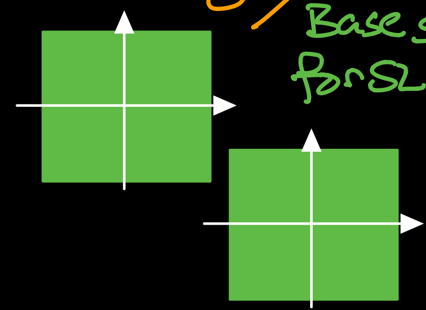
We have traveled quite far from Banach algebras and K-theory! Let us begin the journey back...

I'll use the minimal parabolic subgroup of upper triangular matrices, and the full space of parameters (σ, μ) for parabolically induced representations (For now I do not require $\mathrm{Re}(\mu)$ to be dominant)

$$\boxed{\sigma \in \hat{M} \text{ (discrete)} \qquad \mu \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})}$$

$$\boxed{\sigma \in \hat{M} \ \text{(discrete)} \qquad \mu \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})}$$

Form the vector bundle over this parameter space whose fiber over (σ, μ) is the space $\mathrm{Ind}_P^G H_\sigma$ of the parabolically induced representation $\mathrm{Ind}_P^G \sigma \otimes \exp(\mu)$ — this is independent of μ.

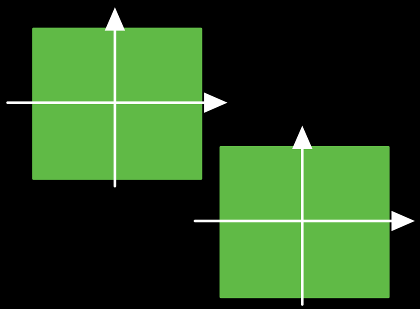


Base space for $SL(2,\mathbb{R})$

I want to build some algebras of endomorphisms of this bundle, starting with a purely algebraic construction...

$$\boxed{\sigma \in \hat{M} \text{ (discrete)} \qquad \mu \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})}$$

On each finite sum S of K-isotypic subbundles ($K \subseteq G$ max. compact*) form the algebra $\mathcal{A}(\mathfrak{g}, S)$ (Hecke algebra) generated by the action of $\mathfrak{g}$, compressed to S. This is an algebra of matrix-valued polynomial functions.



* For simplicity, I'll assume K connected.

Here's what $\mathcal{A}(\mathfrak{g}, S)$ looks like for $SL(2,\mathbb{R})$.

I'll take $S = \{-2, -1, 0, 1, 2\}$ (weights of $SO(2)$).

(on the $\sigma \neq$ trivial plane)

$$\left\{ \begin{pmatrix} p_{11}(z^2) & z\, p_{12}(z^2) \\ z\, p_{21}(z^2) & p_{22}(z^2) \end{pmatrix} \right\}$$

(on the $\sigma =$ trivial plane)

$$\oplus \left\{ \begin{pmatrix} q_{11}(z^2) & (z+1)\, q_{12}(z^2) & (z^2-1)\, q_{13}(z^2) \\ (z-1)\, q_{21}(z^2) & q_{22}(z^2) & (z-1)\, q_{23}(z^2) \\ (z^2-1)\, q_{31}(z^2) & (z+1)\, q_{32}(z^2) & q_{33}(z^2) \end{pmatrix} \right\}$$

Fréchet Algebras of Continuous Matrix-Valued Functions

The algebras $\mathcal{A}(\mathfrak{g}, S)$ are extremely interesting. (the full category of admissible representations of G can be reconstructed from them).

DEFINITION Denote by $\mathcal{C}(G, S)$ the Fréchet algebra of continuous, matrix-valued functions that:

- On finite sets in $\hat{M} \times \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})$ agree with elements of $\mathcal{A}(\mathfrak{g}, S)$.
- Converge rapidly to 0 as $\mathrm{Im}(\mu) \to \infty$, uniformly in compact sets of $\mathrm{Re}\,\mu$).

At last, the reduced group C^*-algebra!

Denote by $C_r^*(G,S)$ the compression of $C_r^*(G)$ to the S-K-isotypical part of $L^2(G)$.

"THEOREM" There is a unique homomorphism

$$\alpha : \mathcal{C}(G,S) \longrightarrow C_r^*(G,S)$$

such that if a tempered irreducible rep. occurs as a subquotient of $\mathrm{Ind}_P^G\, \sigma \otimes \exp(\mu)$, then the operators induced from $f \in \mathcal{C}(G,S)$ and $\alpha(f)$ agree.

"THEOREM" The homomorphism

$$\alpha : \mathcal{C}(G, S) \longrightarrow C_r^*(G, S)$$

induces an isomorphism

$$\alpha_* : K_*(\mathcal{C}(G, S)/\ker(\alpha)) \xrightarrow{\cong} K_*(C_r^*(G, S))$$

(The quotation marks indicate we have not yet proved this in full generality . . .)

The Connes-Kasparov Problem

Two questions present themselves:

- What is the K-theory of $\ker(\alpha)$?
- What is the K-theory of $\mathcal{C}(G, S)$?

I want to indicate that the Langlands theorem has a lot to do with the first question, and that the Oka principle has a lot to do with the second.

Langlands and the Structure of $\ker(\alpha)$

Each $f \in \mathcal{C}(G, S)$ is determined by its restriction to the closure of the chamber

$$\{ (\sigma, \mu) : \mathrm{Re}(\mu) \text{ dominant} \}$$

Moreover $\mathcal{C}(G, S)$ is an inverse limit of Fréchet algebras, each an algebra of matrix-valued functions on the tubes

$$\| \mathrm{Re}(\mu) \| \leq N$$

Let's examine our algebra for one N in the case of $SL(2,\mathbb{R})$, and study in particular the ideal $\ker(\alpha_N)$*. Take $S = \{-2, 0, 2\}$.

$$\ker(\alpha_N) = \left\{ f : (N \geq \mathrm{Re}(z) \geq 0) \to M_3(\mathbb{C}) : f\big|_{\mathrm{Re}(z)=0} = 0,\ f(1) = \begin{pmatrix} 0 & \bullet & 0 \\ 0 & \bullet & 0 \\ 0 & \bullet & 0 \end{pmatrix} \right\}$$

EXERCISE This has zero K-theory!

This holds for all $SL(2,\mathbb{R})$ examples!

* $\ker(\alpha_N)$ is well defined for fixed S & $N \gg 0$

One should be careful because $SL(2,\mathbb{R})$ is in many respects very special...

FIRST ISSUE It is rarely true that the "rest" of $\mathrm{Ind}_P^G \sigma \otimes \exp(\mu)$, apart from the Langlands quotient, is tempered. At first glance, a plausible $\ker(\alpha_N)$ might be

$$\left\{ f : (N \geq \mathrm{Re}(z) \geq 0) \to M_2(\mathbb{C}) : \; f\big|_{\mathrm{Re}(z)=0} = 0,\; f(1) = \begin{pmatrix} b & \bullet \\ 0 & a \end{pmatrix},\; f(2) = \begin{pmatrix} a & \bullet \\ 0 & b \end{pmatrix} \right\}$$

EXERCISE

$\boxed{K_0 \cong \mathbb{Z},\ K_1 \cong \mathbb{Z}}$

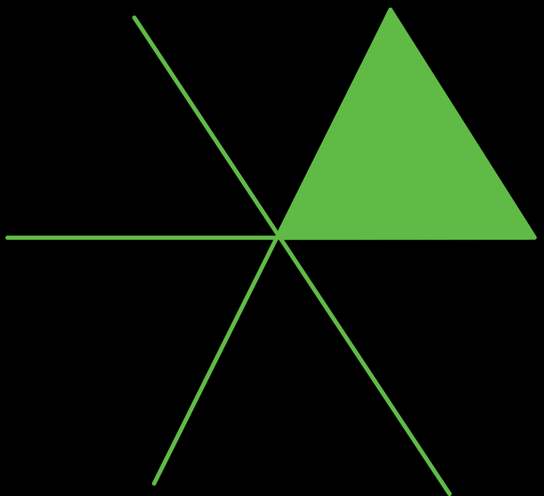
But actually this cannot occur. There is an ordering "by exponents" built into the Langlands classification, and the closest one can get for $\mathrm{Re}(\alpha_N)$ to the above is:

$$\left\{ f : (N \geq \mathrm{Re}(z) \geq 0) \to M_2(\mathbb{C}) : \; f\big|_{\mathrm{Re}(z)=0} = 0,\ f(1) = \begin{pmatrix} 0 & \bullet \\ 0 & a \end{pmatrix},\ f(2) = \begin{pmatrix} a & \bullet \\ 0 & b \end{pmatrix} \right\}$$

EXERCISE This has zero K-theory.

SECOND ISSUE Another problem is that the boundary of $\{\mathrm{Re}(\mu)\text{ dominant}\}$ is more than $\{\mathrm{Re}(\mu)=0\}$.

To handle this, use the rest of the Langlands classification, and an induction argument.



Dominant $\mathrm{Re}(\mu)$ for $SL(3,\mathbb{R})$

"THEOREM" In general

$$K_*(\ker(\alpha)) = 0.$$

Oka and the K-theory of $C(G,S)$

Let's define a new(-ish) Fréchet algebra:

DEFINITION Denote by $\mathcal{H}(G,S)$ the Fréchet algebra of holomorphic, matrix-valued functions that:

- On finite sets in $\hat{M} \times \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})$ agree with elements of $\mathcal{A}(\mathfrak{g}, S)$.
- Converge rapidly to 0 as $\mathrm{Im}(\mu) \longrightarrow \infty$, uniformly in compact sets of $\mathrm{Re}(\mu)$.

Naturally, we now formulate a version of the Oka principle (as a conjecture, for the time being):

OKA PRINCIPLE FOR REDUCTIVE GROUPS

c.f. Bernstein, Braverman, Gaitsgory

The inclusion

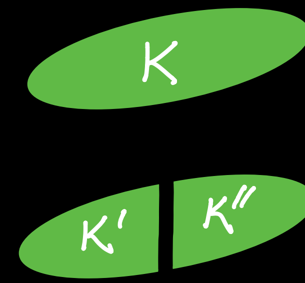
$$\mathcal{H}(G,S) \longrightarrow \mathcal{C}(G,S)$$

induces an isomorphism in K-theory.

Jacob has proved this for $SL(2,\mathbb{R})$ using a new approach to Novodvorskii's theorem.

A Word or Two More about Novodvorskii

The main step in the proof is the construction of a Mayer-Vietoris sequence in K-theory associated to a division of K into two parts by a hyperplane . . .



$$
\begin{tikzcd}
K_0(B(K)) \arrow[r] & K_0(B(K')) \oplus K_0(B(K'')) \arrow[r] & K_0(B(K' \cap K'')) \arrow[d] \\
K_0(B(K' \cap K'')) \arrow[u] & K_0(B(K')) \oplus K_0(B(K'')) \arrow[l] & K_0(B(K)) \arrow[l]
\end{tikzcd}
$$

THEOREM (Bost) Each pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d, "\varphi'"] \\ A'' \arrow[r, "\varphi''"'] & A''' \end{tikzcd}$$

with $\mathrm{Range}(\varphi') + \mathrm{Range}(\varphi'') = A'''$, and with $\mathrm{Range}(\varphi')$ dense, induces a Mayer-Vietoris sequence in K-theory.

Now iterate and prove Novodvorskii for $B(K)$ by induction on $\dim(K)$.

The Schwartz Algebra of G

The reductive group G is a Nash manifold (nonsingular real semi-algebraic set) and so has a natural Schwartz space $\mathcal{S}(G)$, which is a convolution algebra.

- This is not Harish-Chandra's Schwartz space!
- This is not holomorphically closed in $C^*_r(G)$ (not even close)!

(Conjectural) Paley-Wiener Theorem

The representation of $\mathcal{S}(G,S)$ on the principal series induces an inclusion

c.f. Van den Ban & Souaifi

$$\mathcal{S}(G,S) \longrightarrow \mathcal{H}(G,S)$$

Probably this is an isomorphism.

Jacob has proved this for $SL(2,\mathbb{R})$.

Connes-Kasparov Isomorphism

THEOREM The Connes-Kasparov index homomorphism may be identified with the K-theory map induced from the inclusion

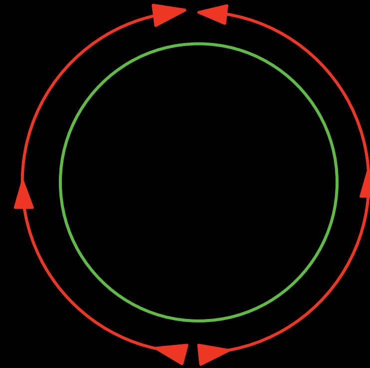
$$\mathcal{S}(G) \longrightarrow C_r^*(G).$$

This point of view has been emphasized by Lafforgue.

Summary (Including Lots of Questions, Still)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
K_*(\mathcal{S}(G,S)) \arrow[r, "\text{Paley-Wiener}", "\overset{?}{\cong}"'] \arrow[d, "\text{Connes-Kasparov}"', "\overset{?}{\cong}"] & K_*(\mathcal{H}(G,S)) \arrow[d, "\text{Oka}", "\overset{?}{\cong}"'] \\
K_*(C_r^*(G,S)) & K_*(\mathcal{C}(G,S)) \arrow[l, "\text{Langlands}", "\overset{?}{\cong}"']
\end{tikzcd}
$$

A Final Remark



Ultimately what may be most important from Langlands are its conceptual underpinnings — the long intertwiner

$$J : \mathrm{Ind}_P^G\, \sigma \otimes \exp(\mu) \longrightarrow \mathrm{Ind}_{\bar{P}}^G\, \sigma \otimes \exp(\mu)$$

and the limit formula

$$\lim_{t \to +\infty} \exp(\rho(tX) - \mu(tX)) \langle h, \exp(tX) f \rangle$$
$$= \langle h(e), (Jf)(e) \rangle_\sigma$$

Are there any lessons here for Baum-Connes?

THANK YOU !