Automorphisms of the Reduced C*-Algebra of a Reductive Lie Group

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Copenhagen, October 19, 2021

An Overview

C*-algebras and tempered unitary representation theory

The reduced group C^*-algebra of a locally compact group was introduced almost 75 years ago (by Irving Segal, in 1947).

For most of its history it has had only limited use in representation theory (e.g. existence of sufficiently many irreducible representations, existence of an abstract Plancherel decomposition, both due to Segal).

But newer tools, especially K-theory, have changed this, to a certain extent, and have led to further significant interactions between C^*-algebra theory and representation theory.

I'm going to talk about a phenomenon—and a theorem—in representation theory that probably would not have been discovered except for such interactions, along with a second theorem that uses the language of C^*-algebras in a crucial way.

These theorems concern what is now called the Mackey bijection. Its existence was speculated upon by Mackey in the 1970's.

Mackey's ideas were kept alive by Alain Connes, who noticed an interesting resonance between them and K-theory. The bijection was finally put on a solid mathematical footing by Alexandre Afgoustidis in the last several years.

Before I get to the Mackey bijection, to set the scene I shall describe how some established topics in tempered representation theory look from the C^*-algebra point of view:

- Discrete series representations
- Parabolic induction
- The fundamental principle that all tempered irreducible unitary representations are accessible through discrete series representations and parabolic induction.

The Reduced C*-Algebra of a Reductive Group

Real reductive groups

The groups G under consideration in this talk will be closed, connected (or very nearly connected) groups of invertible real matrices that are stable under the transpose operation.

Examples include the unimodular groups $SL(n, \mathbb{R})$ (just focus on these, if you like) as well as $SO(p, q)$, $Sp(2n, \mathbb{R})$, etc.

Suppose $G \subseteq GL(n, \mathbb{R})$. Fundamental roles are played by the maximal compact subgroup

$$K = G \cap O(n)$$

and the decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \quad \text{and} \quad G = K \cdot \exp[\mathfrak{s}],$$

where $\mathfrak{s}$ is the subspace of symmetric matrices in $\mathfrak{g} = \mathrm{Lie}(G)$.

The tempered dual

The tempered dual of G is the reduced dual of $C_r^*(G)$.

It is also the support of the Plancherel measure in Segal's abstract Plancherel decomposition

$$L^2(G) \cong \int_{\widehat{G}}^{\oplus} H_\pi \otimes H_\pi^* \, d\mu(\pi)$$

Harish-Chandra made this decomposition explicit, and in doing so determined much of the tempered dual, and set the course for much of the rest of the representation theory of reductive groups, too.

The final steps in the determination of the tempered dual were taken by Knapp and Zuckerman. The details, even of the statement of the result, are quite formidable.

Structure of a reductive group at infinity

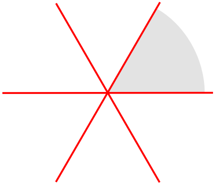
Viewed in retrospect, much of the broad form of Harish-Chandra's theory can be seen as being a consequence of the geometric form of the group G near infinity. I shall try to explain this from a C^*-perspective.

The starting point is the KAK-decomposition of G:

If $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{s}$, for instance the diagonal matrices in $\mathfrak{sl}(n, \mathbb{R})$, then

$$G = K \cdot \exp[\mathfrak{a}] \cdot K$$

So $\mathfrak{a}$, or rather $\mathfrak{a}/N_K(\mathfrak{a})$, determines the structure of G infinity.



Parabolic subgroups

Now fix an element $X \in \mathfrak{a}$ and define

$$N = \{\, g \in G : \lim_{t \to +\infty} \exp(tX) g \exp(-tX) = e \,\}$$

Example If X is diagonal with non-decreasing entries, then this is a group of block upper triangular unipotent matrices in G.

In addition define

$$L = \{\, g \in G : \exp(tX) g \exp(-tX) = g \ \forall t \,\}$$

Example For the same X, this is a group of block diagonal matrices in G.

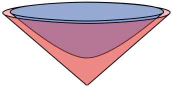
Obviously L normalizes N. The product $P = LN$ is a closed subgroup, called a parabolic subgroup of G.

Remark When X is central in $\mathfrak{g}$, the group N is trivial and $L = G$. This case requires special treatment below, so for simplicity I will assume from now on assume that G has compact center.

Geometry and harmonic analysis of parabolic subgroups

Parabolic subgroups have important geometric and harmonic-analytic properties:

Geometry: The spaces G/K and $G/K_L N$ are asymptotic to one another in the direction of X. Here $K_L = K \cap L$.



Harmonic Analysis: The homogeneous space G/N is a left G-space and a proper right L-space. As a result we can form the Hilbert $C_r^*(L)$ module $C_r^*(G/N)$, à la Kasparov, and it carries a left action of $C_r^*(G)$. This action is through Kasparov's compact operators.

Lemma (From the Harmonic Analysis Property)

Every tempered irreducible representation of G that is zero on the kernel of

$$C_r^*(G) \longrightarrow \mathfrak{K}\big(C_r^*(G/N)\big)$$

is a subrepresentation of some $C_r^*(G/N) \otimes_{C_r^*(L)} H$

The latter are called parabolically induced representations.

Discrete series

The cuspidal ideal in $C_r^*(G)$ consists of those $f \in C_r^*(G)$ that act by compact operators on the Hilbert spaces

$$L^2(G)^\sigma = \{ \psi \in L^2(G) : \psi \star p_\sigma = \psi \}$$

for all $\sigma \in \widehat{K}$ (here p_σ is the isotypical projection for σ).

An irreducible tempered unitary representation is a discrete series representation if it is nonzero on the cuspidal ideal. It is then square-integrable and an isolated point in the tempered dual.

It is remarkable that any f (a convolution operator on a noncompact homogeneous space) could act as a nonzero compact operator. Indeed for many groups (e.g. complex groups) the cuspidal ideal is zero.

The discrete series representations for $SL(2,\mathbb{R})$ were famously parametrized by Bargmann by integers $n \in \mathbb{Z}$, $n \neq 0$.

Harish-Chandra/Langlands principle

From the geometric property of parabolic subgroups we obtain:

Theorem The intersection over all parabolic subgroups of the kernels of

$$C^*_r(G) \longrightarrow \mathfrak{K}\big(C^*_r(G/N)\big)$$

is the cuspidal ideal of $C^*_r(G)$.

Proof. Because $G/K_L N$ resembles G/K at infinity, if $f \in C^*_r(G)$ acts as zero on $C^*_r(G/N)$, and hence on each $L^2(G/N)^{K_L}$, it must essentially act as zero on $L^2(G)^K$, and indeed on each $L^2(G)^\sigma$.

From this we obtain:

Theorem If an irreducible tempered representation embeds in no parabolically induced representation, then it is a discrete series.

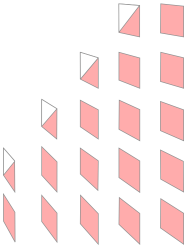
Theorem Every irreducible tempered representation embeds in a representation parabolically induced from a discrete series, mod center, representation.

Families of tempered representations

I want to extract two points from the foregoing discussion.

The first is that the geometric structure of G at infinity plays a determining role in shaping the overall form of the tempered dual. This is in sharp contrast to what will follow.

The second is that as a result, tempered representations come in families. See for instance the picture of the tempered dual of $GL(2, \mathbb{C})$ to the right.



Each L-subgroup has a direct product Langlands decomposition $L = M_P A_P$, where A_P is abelian and positive-definite (e.g. it contains $\{\exp(tX)\}$) and where M has compact center.

Each plane is determined by a single discrete series of M_P, and is spanned by the characters of A_P.

Structure of the reduced C*-algebra

By analyzing families of parabolically induced representations, and in particular by studying equivalences among representations in the family, Wassermann constructed a Morita equivalence

$$C_r^*(G) \underset{\text{Morita}}{\approx} \bigoplus_{[P,\sigma]} C_0\big(\mathfrak{a}_P^*/W'_{P,\sigma}\big) \rtimes R_{P,\sigma}$$

I won't explain the details, except to say that

- $W'_{P,\sigma}$ and $R_{P,\sigma}$ are finite groups of automorphisms of A_P, and
- these groups are difficult to determine, yet they are crucial.

Example When $G = SL(2,\mathbb{R})$ there are $\mathbb{C}$-summands in the above corresponding to the discrete series, and two other summands:

$$C_0(\mathbb{R}/\mathbb{Z}_2) \quad \text{and} \quad C_0(\mathbb{R}) \rtimes \mathbb{Z}_2$$

Their spectra are

 and 

and include, in the latter case two limits of discrete series representations associated with the non-zero R-group.

The Connes-Kasparov isomorphism

Wassermann showed that the summands in his Morita equivalence each contribute either nothing or a single free generator to K-theory. From here it is still a considerable challenge to usefully list the generators, however:

Theorem (Connes-Kasparov Isomorphism)

Kasparov's Fredholm index gives an isomorphism from the Grothendieck group of G-equivariant Dirac-type operators on G/K [*this is not so far from the representation ring $R(K)$ and in particular this group is quite computable*] *to the K-theory of $C_r^*(G)$.*

But given the intricacy of the computations involved, compared to the beautiful simplicity of the statement, one can hope that this isn't the end of the story ...

Representations of the Cartan Motion Group

Cartan motion group

Remember the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. The Cartan motion group associated with G is the semidirect product

$$G_0 = K \ltimes \mathfrak{s}$$

Think of G_0 as a simplified version of G. It acts properly and by isometries on $\mathfrak{s}$ in a way that recalls the proper and isometric action of G on G/K.

So for instance when $G = SL(2, \mathbb{R})$, the Cartan motion group G_0 is (essentially) the group of isometries of the Euclidean plane, while G is (essentially) the group of isometries of the hyperbolic plane.

Geometrically, G_0 is the normal bundle for the inclusion of K into G (which is a group since K and G are groups).

The normal bundle resembles G somewhat closely since

$$G_0 = K \ltimes \mathfrak{s} \quad \text{while} \quad G = K \cdot \exp[\mathfrak{s}]$$

Unitary dual of the Cartan motion group

Using the Fourier transform,

$$C_r^*(G_0) \cong C_0\big(\mathfrak{s}^*, \mathfrak{K}(L^2(K))\big)^K$$

From this one can read off the irreducible unitary representations of G_0. They are given by indecomposable equivariant vector bundles over K-orbits in $\mathfrak{s}^*$.

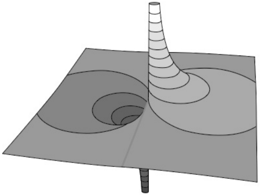
Elements of the subgroup $\mathfrak{s} \subseteq G_0$ act on sections of such a vector bundle through pointwise multiplication by the functions

$$\pi_X : \mathfrak{s}^* \longrightarrow \mathbb{T}, \qquad \varphi \longmapsto \exp(i\varphi(X))$$

Elements of K act in the obvious way, using equivariance.

Mackey's DNC space

Because G_0 is a normal bundle, we can apply a standard construction from geometry to obtain a smooth family $\mathbb{G}$ of groups



$$G_t = \begin{cases} G & t \neq 0 \\ G_0 & t = 0 \end{cases}$$

This is the deformation to the normal cone, a close relative of the tubular neighborhood construction (see below).

Example The picture shows the DNC associated to the embedding of the trivial group into the circle group.

Example In the case $G = SL(2, \mathbb{R})$, one can think of G_t as essentially the group of isometries of the plane of curvature $-t^2$.

A continuous field of C*-algebras

I shall examine the DNC from a geometric point of view in a moment. But first:

Lemma

The smooth, compactly supported functions on $\mathbb{G}$ [*or, better, fiberwise densities*] *generate the sections of a continuous field of* C^**-algebras with fibers* $C^*_r(G_t)$.

Alain Connes made the following interesting observation:

Lemma

The following two assertions are equivalent:

- *The Connes-Kasparov index homomorphism is an isomorphism.*
- *The* K*-theory groups of the* C^**-algebras* $C^*_r(G_t)$ *are the stalks of a constant sheaf over* $\mathbb{R}$.

Mackey's proposal

. . . the physical meaning of the unitary representations of G_0, and the fact that G/K is a possible model for physical space, suggests there might be a much closer relationship between the unitary representation theories of G and G_0 than [*first*] *considerations would lead one to expect.*

We wish to set up a natural one-to-one correspondence between almost all equivalence classes of irreducible unitary representations of G and "most" equivalence classes of irreducible unitary representations of G_0.

We have not yet ventured to formulate a precise conjecture along the lines suggested by the speculations in the preceding paragraphs . . . however we feel sure that some such result exists . . .

George Mackey, 1975

Some issues with the proposal

Mackey seems to have been concerned about some obvious mismatches between $\widehat{G}$ and $\widehat{G}_0$, for instance

- Discrete series representations of G (when they exist) have no obvious counterparts in $\widehat{G}_0$.

- $\widehat{G}_0$ includes many finite-dimensional representations, while $\widehat{G}$ typically does not.

Moreover Mackey scarcely considered at all the subleties involving R-groups (e.g. the two limits of discrete series for $SL(2, \mathbb{R})$), and in addition

- Mackey's proposal went directly against the prevailing approach to representation theory, involving the asymptotic geometry of G.

But despite all its apparent shortcomings, not to mention its vagueness, Mackey's idea was kept alive by Connes' observation ...

The Mackey Bijection

Mackey bijection via the continuous field

Connes' observation was about the topology of the tempered duals of G and G_0, while Mackey's proposal was measure-theoretic . . .

There is an interesting tension between the two, which is most easily resolved by guessing something much stronger—namely that the duals are the same . . .

Theorem (Afgoustidis)

The continuous field $\{C^*_r(G_t)\}$ is assembled from constant fields by extensions and Morita equivalences.[1]

This means that

- the duals of G_t are partitioned into identical locally closed sets (corresponding to the constant fields), and
- the parts are assembled within each tempered dual in ways that may be (and in fact are) different, but the differences don't affect K-theory.

[1] There is also a direct limit over an increasing sequence of ideals at the end.

Geometry of Mackey's Deformation

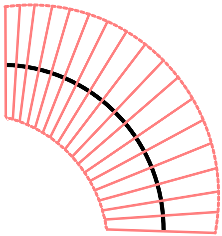
Euler-like vector fields and tubular neighborhoods

Let M be a smooth submanifold of V.

An Euler-like vector field for the embedding of M into V is a vector field X on V such that

$$X(f) = f + r$$

for all smooth f that vanish on M, where the remainder vanishes to order 2 or more on M.



Example The usual Euler vector field on a vector space (or vector bundle) is an Euler-like vector field for the embedding of zero (or the embedding of the zero section).

Theorem (Bursztyn, Lima, Meinrenken)

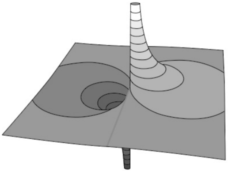
Every Euler-like vector field is the Euler vector field for a unique tubular neighborhood embedding of the normal bundle $N_V M$ into the manifold V.

Tubular neighborhoods from the C*-algebra point of view

Here is how the BLM theorem works from the point of view of function algebras.

- Fix an Euler-like vector field X and denote by α_t the flow on V that it generates (here $t > 0$ so that $\alpha_1 = \mathrm{id}$).

- Given a test function f on the normal bundle $N_V M$ (the zero fiber of the DNC), extend it to a test function $\boldsymbol{f} = \{f_t\}$ on the DNC.



- The formula

$$\alpha(f) = \lim_{t\to 0} \alpha_{1/t}(f_t)$$

defines an embedding

$$\alpha\colon C_0(N_V M) \longrightarrow C_0(V)$$

and a tubular neighborhood embedding of $N_V M$ into V.

The above from the group C*-algebra point of view

Theorem (NH and Angel Roman)

Let G be a complex reductive group. There is a flow

$$\alpha_t \colon C^*_r(G) \longrightarrow C^*_r(G) \qquad (t > 0)$$

for which the formula $\alpha(f) = \lim_{t\to 0} \alpha_{1/t}(f_t)$ defines an embedding of C^*-algebras

$$\alpha \colon C^*_r(G_0) \longrightarrow C^*_r(G)$$

Remarks

- The continuous field can be reconstructed from α, which is an isomorphism in K-theory.
- Good news: the Mackey bijection is easy to express in terms of α (see the next slide).
- Bad news: Currently the construction of $\{\alpha_t\}$ involves an analysis of R-groups (which is why at present the theorem is limited to complex groups).

The Mackey bijection, again

Theorem (NH and Angel Roman)

Let G be a complex reductive group. There is a unique bijection

$$\beta\colon \big\langle \textit{tempered dual of } G \big\rangle \longrightarrow \big\langle \textit{unitary dual of } G_0 \big\rangle$$

such that for every tempered irreducible representaton π of G, $\beta(\pi)$ is a subrepresentation of

$$\pi \circ \alpha\colon C^*_r(G_0) \longrightarrow \mathfrak{B}(H_\pi)$$

Thank You!

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