An Introduction to the Hypoelliptic Laplacian

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Questions

The name *hypoelliptic Laplacian* is Jean-Michel Bismut's term for a family of operators constructed by him to make some very striking computations in Lie theory and spectral geometry.

I'll try to answer some of the obvious questions:

- *What is it?*
- *What does it do?*
- *How does it do what it does?*

The full theory requires a quite heavy dose of analysis (estimates). I won't discuss this at all, except to point out where it is needed.

Hypoelliptic Laplacian on the Circle

There are versions of the hypoelliptic Laplacian in a variety of contexts (Riemannian manifolds, Hermitian manifolds, symmetric spaces, compact groups, ...).

I'll mostly consider compact groups in this talk. In fact I'll mostly focus on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Here is Bismut's hypoelliptic Laplacian for the circle:

$$\mathsf{L}_b = \begin{bmatrix} \frac{1}{2b^2}(y^2 - \partial_y^2 - 1) + \frac{1}{b}y\partial_x & 0 \\ 0 & \frac{1}{2b^2}(y^2 - \partial_y^2 + 1) + \frac{1}{b}y\partial_x \end{bmatrix} \qquad (b > 0)$$

- L_b acts on (vector-valued functions on) $\mathbb{T}\times\mathbb{R}$, not $\mathbb{T}$, with x coordinatizing the circle and y coordinatizing the line.
- L_b is not elliptic and it is not self-adjoint.

Hypoellipticity

A linear partial differential operator P is hypoelliptic if

$$\forall U \quad Pf|_U \in C^\infty \quad \Rightarrow \quad f|_U \in C^\infty$$

Theorem (Hormander)

If $X_0, \ldots, X_d$ are vector fields and if $\mathrm{Lie}\langle X_0, \ldots, X_d \rangle$ spans each tangent space, then the operator

$$P = X_0 + X_1^2 + \cdots + X_d^2 + h$$

is hypoelliptic.

This applies to L_b and to the "heat operator" $\partial_t + L_b$.

Spectrum and Geodesics

A study of L_b on the circle of circumference c leads to the formula

$$\sum_{\lambda \in \mathrm{Spec}(\Delta)} e^{-t\lambda^2} = \frac{c}{\sqrt{4\pi t}} \sum_{\text{closed geodesics}} e^{-\ell(\gamma)^2/4t}$$

My goal is to explain how this comes about.

Of course, the formula just says that

$$\sum_{k \in \mathbb{Z}} e^{-4\pi^2 k^2 t/c^2} = \frac{c}{\sqrt{4\pi t}} \sum_{n \in \mathbb{Z}} e^{-n^2 c^2/4t},$$

which is a special case of the Poisson summation formula, among other things. So easy harmonic analysis applies.

In contrast, Bismut's method requires a considerable amount of effort!

But for other problems Bismut's method gives answers that *look quite different* from those given by harmonic analysis.

Methods from Spectral Geometry and Index Theory

On a closed manifold M, the Laplace operator (or similar) may be diagonalized in $L^2(M)$:

$$\Delta f_n = \lambda_n f_n \qquad\qquad \langle f_n, f_m\rangle|_{L^2(M)} = \delta_{mn} \quad \& \quad \lambda_n \to +\infty.$$

There is a one-parameter semigroup of operators $\exp(-t\Delta)$ for which

$$u_t = \exp(-t\Delta)u \quad \Rightarrow \quad \partial_t u_t + \Delta u_t = 0.$$

for $t \geq 0$. Each $\exp(-t\Delta)$ for $t > 0$ is an integral operator, with

$$\exp(-t\Delta)(p,q) = \sum e^{-t\lambda_n} f_n(p)\overline{f_n(q)},$$

and

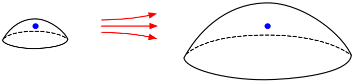
$$\mathrm{Tr}\big(\exp(-t\Delta)\big) = \int_M \exp(-t\Delta)(p,p)\,dp = \sum e^{-t\lambda_n}.$$

Rescaling Argument

Evidently

$$\lim_{t\to\infty} \exp(-t\Delta) = \text{orthogonal projection onto } \mathrm{kernel}(\Delta).$$

To understand the small t limit, rescale the metric $g \rightsquigarrow t^{-1}g$.



At any $p \in M$, the rescaled manifolds M_t converge to T_pM, while

$$t\Delta_M \sim \Delta_{M_t}$$

For instance this leads to

$$\exp\big(-t\Delta_M\big)(p,p) \sim t^{-n/2} \exp\big(-\Delta_{T_pM}\big)(0,0)$$

and Weyl's asymptotic law.

The Supertrace and the Index Formula

In the $\mathbb{Z}_2$-graded context the supertrace is of course the functional

$$\mathsf{STr}\left(\begin{bmatrix} \mathsf{A}_{00} & \mathsf{A}_{01} \\ \mathsf{A}_{10} & \mathsf{A}_{11} \end{bmatrix}\right) = \mathsf{Tr}(\mathsf{A}_{00}) - \mathsf{Tr}(\mathsf{A}_{11}).$$

on (traceable) operators. It vanishes on supercommutators.

The Supertrace and the Index Formula

If $\mathsf{D} = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$ and $\Delta = \mathsf{D}^2$, then

$$\frac{d}{dt}\,\mathsf{STr}\big(\exp(-t\Delta)\big) = -\,\mathsf{STr}\Big(\big\{\mathsf{D}, \mathsf{D}\exp(-t\Delta)\big\}\Big) = 0$$

In the case of the Dirac operator, by rescaling both M and Clifford variables, Getzler showed that

$$\lim_{t\to 0} \mathsf{str}\Big(\exp\big(-t\Delta\big)(p,p)\Big) = \mathsf{str}_0\Big(\exp\big(-\Omega_{T_pM}\big)(0_p, 0_p)\Big)$$

for a suitable explicit limit operator Ω_{T_pM} and supertrace functional str_0 (there is no divergence in t).

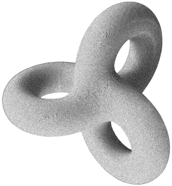
This gives the local index formula

$$\mathsf{Index}(D) = \int_M \mathsf{str}_0\Big(\exp\big(-\Omega_{T_pM}\big)(0_p, 0_p)\Big)\,dp$$

(LHS is the limit of $\mathsf{STr}(\exp(-t\Delta))$ as $t\to\infty$; RHS is the limit as $t\to 0$).

Selberg Trace Formula

Before examining how Bismut's theory works, here is an application that gives a somewhat more accurate impression of the reach of Bismut's method.



In dimension two, the analysis of L_b leads to the Selberg trace formula for a (closed) hyperbolic surface S:

$$\sum_{\lambda \in \mathrm{Spec}(\Delta)} e^{-t\lambda} = \frac{\mathrm{Area}(S)}{4\pi t} \cdot \frac{e^{-t/4}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \frac{x/2}{\sinh(x/2)} e^{-x^2/4t}\, dx$$

$$+ \frac{e^{-t/4}}{\sqrt{4\pi t}} \sum_{\text{closed geodesics}} \frac{\ell_0(\gamma)/2}{\sinh\big(\ell(\gamma)/2\big)} e^{-\ell(\gamma)^2/4t}$$

Orbital Integrals

If S is a hyperbolic surface, then $S \cong \Gamma \backslash SL(2,\mathbb{R})/SO(2)$, and

$$\exp(-t\Delta_S)(p,p) = \sum_{\gamma\in\Gamma} \exp(-t\Delta_H)(P,\gamma P)$$

where $H = SL(2,\mathbb{R})/SO(2)$. An elementary calculation shows

$$\mathrm{Tr}(\exp(-t\Delta_S)) = \sum_{\langle\gamma\rangle} \mathrm{vol}\big(Z_\Gamma(\gamma)\big\backslash Z_G(\gamma)\big) \cdot \mathrm{Tr}^{\langle\gamma\rangle}\big(\exp(-t\Delta_H)\big)$$

The sum is over representatives of conjugacy classes in Γ, and

$$\mathrm{Tr}^{\langle\gamma\rangle}\big(\exp(-t\Delta_H)\big) = \int_{Z_G(\gamma)\backslash G} \exp(-t\Delta_H)(gP,\gamma gP)\,dg.$$

What Bismut actually does is evaluate these (semisimple) orbital integrals.

Bismut's Orbital Integral Formula

Bismut's formulas for orbital integrals even apply to the orbital integral over the one-element conjugacy class of the identity element.

For $SL(2, \mathbb{R})$ one gets

$$\exp(-t\Delta_H)(P, P) = (4\pi t)^{-3/2} \int_{-\infty}^{\infty} \frac{x/2}{\sinh(x/2)} e^{-x^2/4t} \, dx$$

In contrast, from the Plancherel formula one gets

$$\exp(-t\Delta_H)(P, P) = \int_{\widehat{G}_{\text{spherical}}} e^{-t\| \operatorname{inf.ch}(\pi) \|^2} \, d\mu_{\text{Plancherel}}(\pi)$$

In general, it is an interesting challenge to compare and reconcile Bismut's formulas with the explicit Plancherel formula of Harish-Chandra (c.f. work of Shu Shen, Yanli Song, Xiang Tang).

Components of the Hypoelliptic Laplacian

From now on I shall focus on the circle (of circumference one). But actually there would be few changes for compact groups, and not so many more for the noncompact case.

1. I shall describe the parts from which L_b is built:
   - The Dirac operator
   - The square root of the quantum harmonic oscillator
2. Then I shall explain how these are assembled into L_b.
3. Finally I shall examine the features of L_b that lead to the Bismut's formulas.

Square Roots of the Laplacian

On the circle, Bismut essentially uses the following Dirac operator

$$\mathsf{D} = \begin{bmatrix} 0 & -i\,\partial_x \\ -i\,\partial_x & 0 \end{bmatrix}$$

(which is more or less the de Rham operator). It acts as a self-adjoint operator.

In the general case Bismut uses Kostant's cubic Dirac operator on a (real reductive) group, acting on $\Lambda^\bullet(\mathfrak{g})$.

One has

$$\mathsf{D}^2 = \mathsf{Casimir} + \frac{1}{24}\,\mathsf{tr}(\mathsf{Casimir}_{\mathfrak{g}}) \cdot I$$

So the square is Casimir, plus a scalar. This is important.

Disclosure: Bismut actually uses $i\,\mathsf{D}$, which will cause me to insert some square roots of minus one later.

Spectral Geometry on a Vector Space

The operator

$$H = -\partial_y^2 + y^2$$

on $L^2(\mathbb{R})$ is the well-known quantum harmonic oscillator. It has simple spectrum $\{1, 3, 5, \dots\}$ and the ground state is $\exp(-y^2/2)$.

There is an almost equally well-known "square root:"

$$\mathsf{Q} = \begin{bmatrix} 0 & -\partial_y + y \\ \partial_y + y & 0 \end{bmatrix} : L^2(\mathbb{R}, \Lambda) \longrightarrow L^2(\mathbb{R}, \Lambda)$$

(with Λ the exterior algebra of $\mathbb{R}$) for which

$$\mathsf{Q}^2 = \begin{bmatrix} -\partial_y^2 + y^2 - 1 & 0 \\ 0 & -\partial_y^2 + y^2 + 1 \end{bmatrix}$$

There is a counterpart on any euclidean vector space, with Q acting on functions valued in the exterior algebra, and

$$\mathsf{Q}^2 = H + \dim(V)I - N$$

Products, Geometric and Operator-Theoretic

The Laplace operator on a product manifold can be written as the sum of Laplace operators acting on each factor:

$$\Delta_{M_1 \times M_2} = \Delta_1 + \Delta_2 \colon L^2(M_1 \times M_2) \longrightarrow L^2(M_1 \times M_2)$$

What about the square roots—the Dirac operators D?

In index theory one defines

$$\mathrm{D}_1 \# \mathrm{D}_2 \colon L^2(M_1 \times M_2, \Lambda \otimes \Lambda) \longrightarrow L^2(M_1 \times M_2, \Lambda \otimes \Lambda)$$

which is almost the sum of D_1 and D_2, as above. The difference: some $\pm$ signs are added strategically.

The product $\mathrm{D}_1 \# \mathrm{D}_2$ has the fundamental properties that

$$(\mathrm{D}_1 \# \mathrm{D}_2)^2 = \mathrm{D}_1^2 + \mathrm{D}_2^2 \quad \text{and} \quad \mathrm{Ind}(\mathrm{D}_1 \# \mathrm{D}_2) = \mathrm{Ind}(\mathrm{D}_1) \cdot \mathrm{Ind}(\mathrm{D}_2)$$

Asymptotics (a Baby Case)

Returning to the circle $\mathbb{T}$, introduce a parameter $T > 0$, and form

$$\mathrm{D} \# T\mathrm{Q} \colon L^2\big(\mathbb{T} \times \mathbb{R}, \Lambda \otimes \Lambda\big) \longrightarrow L^2\big(\mathbb{T} \times \mathbb{R}, \Lambda \otimes \Lambda\big)$$

Here is why one introduces T:

Theorem

If $L^2(\mathbb{T}, \Lambda)$ *is identified with the kernel of* Q *in* $L^2(\mathbb{T}\times\mathbb{R}, \Lambda\otimes\Lambda)$*, then*

$$\lim_{T\to+\infty} \exp\big(-t(\mathrm{D} \# T\mathrm{Q})^2\big) = \exp\big(-t\mathrm{D}^2\big)$$

for all $t > 0$ *(convergence of operators).*

Proof. Use $(\mathrm{D} \# T\mathrm{Q})^2 = \mathrm{D}^2 + T^2\mathrm{Q}^2$. The rest is easy.

Remark

The kernel of Q consists of functions, degree zero in the 2nd Λ-factor, that behave like $e^{-y^2/2}$ in the y-direction. It is preserved by D.

The theorem refines the identity $\mathrm{Ind}(\mathrm{D} \# T\mathrm{Q}) = \mathrm{Ind}(\mathrm{D})$.

An Accidental Discovery?

Look again at the product

$$\mathsf{D} \# T\mathsf{Q}\colon L^2\big(\mathbb{T}\times\mathbb{R}, \Lambda\otimes\Lambda\big) \longrightarrow L^2\big(\mathbb{T}\times\mathbb{R}, \Lambda\otimes\Lambda\big)$$

Remember that in real life Λ is an exterior algebra ...

...so it's tempting to multiply the tensor factors together, and build an operator

$$\mathsf{D} \mathbin{\bar{\#}} T\mathsf{Q}\colon L^2(\mathbb{T}\times\mathbb{R}, \Lambda) \longrightarrow L^2(\mathbb{T}\times\mathbb{R}, \Lambda),$$

say

$$\mathsf{D} \mathbin{\bar{\#}} T\mathsf{Q} = \mathsf{D} + T\mathsf{Q}.$$

Then

$$(\mathsf{D} \mathbin{\bar{\#}} T\mathsf{Q})^2 = \mathsf{D}^2 + T\{\mathsf{D},\mathsf{Q}\} + T^2\mathsf{Q}^2$$

and the cross-term, now non-zero, is

$$\{\mathsf{D},\mathsf{Q}\} = 2y\partial_x.$$

From a geometrical point of view, $y\partial_x$ is the generator of the geodesic flow on (the tangent bundle of) $\mathbb{T}$ ...which is interesting.

Convergence of Heat Operators?

But does it amount to anything? For instance what about the formula

$$\lim_{T\to+\infty} \exp\bigl(-t(\mathsf{D} \mathbin{\bar{\#}} T\mathsf{Q})^2\bigr) = \exp\bigl(-t\mathsf{D}^2\bigr)?$$

The initial indications are not promising, since D doesn't preserve the kernel of Q as it did before. In fact D *exchanges* the kernel and its orthogonal complement (just look at the $\mathbb{Z}_2$-grading).

But D^2 *does* preserve the kernel of Q. Let's examine the matrix decomposition

$$(\mathsf{D} + T\mathsf{Q})^2 = \begin{bmatrix} \mathsf{D}^2 & T\mathsf{DQ} \\ T\mathsf{QD} & \mathsf{D}^2 + T\{\mathsf{D},\mathsf{Q}\} + T^2\mathsf{Q}^2 \end{bmatrix}$$

with respect to

$$L^2(\mathbb{T}\times\mathbb{R},\Lambda) = \operatorname{Ker}(\mathsf{Q}) \oplus \operatorname{Ker}(\mathsf{Q})^\perp \cong L^2(\mathbb{T}) \oplus L^2(\mathbb{T})^\perp$$

not with respect to any $\mathbb{Z}_2$-grading.

Two by Two Block Matrix Calculations

Let's examine $\exp\bigl(-t(\mathsf{D}+T\mathsf{Q})^2\bigr)$ via

$$\exp(-t(\mathsf{D}+T\mathsf{Q})^2) = \frac{1}{2\pi i}\int e^{-t\mu}(\mu-(\mathsf{D}+T\mathsf{Q})^2)^{-1}\,d\mu,$$

using

$$\mu-(\mathsf{D}+T\mathsf{Q})^2 = \begin{bmatrix}\mu & 0\\ 0 & \mu\end{bmatrix} - \begin{bmatrix}\mathsf{D}^2 & T\mathsf{D}\mathsf{Q}\\ T\mathsf{Q}\mathsf{D} & \mathsf{D}^2+T\{\mathsf{D},\mathsf{Q}\}+T^2\mathsf{Q}^2\end{bmatrix}$$

If the bottom right entry d of any block matrix is invertible, then

$$\begin{bmatrix}\mathsf{a} & \mathsf{b}\\ \mathsf{c} & \mathsf{d}\end{bmatrix} = \begin{bmatrix}1 & \mathsf{b}\mathsf{d}^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathsf{e} & 0\\ 0 & \mathsf{d}\end{bmatrix}\begin{bmatrix}1 & 0\\ \mathsf{d}^{-1}\mathsf{c} & 1\end{bmatrix}$$

where

$$\mathsf{e} = \mathsf{a}-\mathsf{b}\mathsf{d}^{-1}\mathsf{c}$$

The $(1,1)$-entry of the inverse matrix is therefore e^{-1}. But in our case

$$\mathsf{e} = \mu-\mathsf{D}^2+\mathsf{D}\cdot\frac{\mathsf{T}^2\mathsf{Q}^2}{\mathsf{D}^2+\mathsf{T}\{\mathsf{D},\mathsf{Q}\}+\mathsf{T}^2\mathsf{Q}^2-\mu}\cdot\mathsf{D}$$

and as a result $\mathsf{e}\to\mu$ as $T\to\infty$. The operator D disappears!

Two by Two Block Matrix Calculations

What this means is that

$$\exp\bigl(-t(\mathsf{D}+T\mathsf{Q})^2\bigr) \approx \begin{bmatrix} I & \star \\ \star & \star \end{bmatrix}$$

as $T \to \infty$ (and actually the $\star$-terms are 0).

However, a simple adjustment presents itself. Start not with

$$\mu - (\mathsf{D} + T\mathsf{Q})^2 = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} - \begin{bmatrix} \mathsf{D}^2 & T\mathsf{DQ} \\ T\mathsf{QD} & \mathsf{D}^2 + T\{\mathsf{D},\mathsf{Q}\} + T^2\mathsf{Q}^2 \end{bmatrix}$$

but with

$$\mu - (\mathsf{D} + T\mathsf{Q})^2 + \mathsf{D}^2 = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} - \begin{bmatrix} 0 & T\mathsf{DQ} \\ T\mathsf{QD} & T\{\mathsf{D},\mathsf{Q}\} + T^2\mathsf{Q}^2 \end{bmatrix}$$

Then we get

$$\mathsf{e} = \mu + \mathsf{D} \cdot \frac{\mathsf{T}^2\mathsf{Q}^2}{\mathsf{D}^2 + \mathsf{T}\{\mathsf{D},\mathsf{Q}\} + \mathsf{T}^2\mathsf{Q}^2 - \mu} \cdot \mathsf{D}$$

Hence

$$\mathsf{e} \to \mu + \mathsf{D}^2|_{\ker(\mathsf{Q})} \quad \text{as} \quad \mathsf{T} \to \infty.$$

Block Matrix Calculations Summarized

To cope with the minus sign, we make a small adjustment, and write

$$\mathsf{L}(T) = (\sqrt{-1}\mathsf{D} + T\mathsf{Q})^2 + \mathsf{D}^2.$$

Then $\mathsf{L}(T)$ converges in the resolvent sense to $\mathsf{D}^2|_{\mathrm{kernel}(\mathsf{Q})}$:

Theorem

For fixed $t > 0$

$$\lim_{T\to\infty} \exp\bigl(-t\,\mathsf{L}(T)\bigr) = \exp\bigl(-t\,\mathsf{D}^2|_{\mathrm{kernel}(\mathsf{Q})}\bigr)$$

and

$$\lim_{T\to\infty} \mathrm{STr}\,\bigl(\exp\bigl(-t\,\mathsf{L}(T)\bigr)\bigr) = \mathrm{Tr}\,\Bigl(\exp\bigl(-t\,\mathsf{D}^2|_{\mathrm{kernel}(\mathsf{Q})}\bigr)\Bigr)$$

Note the ordinary trace (not supertrace) on the RHS!

Definition of the Hypoelliptic Laplacian

Bismut uses b^{-1} instead of T, and divides by 2. So he defines the hypoelliptic Laplacian (on the circle, or on any compact group) to be

$$\mathsf{L}_b = \tfrac{1}{2}(\sqrt{-1}\mathsf{D} + b^{-1}\mathsf{Q})^2 + \tfrac{1}{2}\mathsf{D}^2$$

This acts on $L^2(G\times\mathfrak{g}, \Lambda^\bullet(\mathfrak{g}))$.

In the case of the circle $\mathbb{T}$ we get

$$\mathsf{L}_b = \begin{bmatrix} \frac{1}{2b^2}(y^2 - \partial_y^2 - 1) + \frac{1}{b}y\partial_x & 0 \\ 0 & \frac{1}{2b^2}(y^2 - \partial_y^2 + 1) + \frac{1}{b}y\partial_x \end{bmatrix}$$

which is the operator we saw before.

Fundamental Properties

Theorem

For each $b>0$ the hypoelliptic Laplacian operator L_b is hypoelliptic, and the generator of a one-parameter semigroup $\exp(-t\,\mathsf{L}_b)$ of traceable operators.

Theorem

$$\lim_{b\to 0} \mathrm{STr}\big(\exp(-t\,\mathsf{L}_b)\big) = \mathrm{Tr}(\exp(-t\Delta/2))$$

Here Δ is the restriction of D^2 to 0-forms. Moreover:

Theorem

$$\frac{d}{db}\,\mathrm{STr}\big(\exp(-t\,\mathsf{L}_b)\big) = 0$$

For this it is essential that D^2 commute with Q (which is why in general one should use Kostant's version of the Dirac operator).

The Method of the Hypoelliptic Laplacian

As should now be clear, Bismut's approach to trace formulas using L_b is as follows:

1. Evaluate the limit of the supertrace of the heat kernel as b tends to zero:

$$\lim_{b\to 0} \mathrm{STr}\big(\exp(-t\mathsf{L}_b)\big) = \mathrm{Tr}\big(\exp(-t\Delta/2)\big)$$

2. Show that the b-derivative of the supertrace vanishes:

$$\frac{d}{db}\mathrm{STr}\big(\exp(-t\mathsf{L}_b)\big) = 0$$

3. Evaluate the limit

$$\lim_{b\to\infty} \mathrm{STr}\big(\exp(-t\mathsf{L}_b)\big)$$

I've already discussed the first two steps. The third requires still more new ideas, this time geometric, not spectral.

Geometry of the Hypoelliptic Laplacian

I shall work now with the *scalar* operator

$$L_b = \frac{1}{2b^2}(-\partial_y^2 + y^2) + \frac{1}{b} y\partial_x$$

for simplicity (if you strain your eyes, you'll see a different font in use).

Actually to begin with, I shall work with the even simpler operator

$$K = -\frac{1}{2}\partial_y^2 + y\partial_x$$

on the (x, y)-plane (this operator was initially studied by Kolmogorov).

I want to explain the influence of the term $y\partial_x$ on the behavior of solutions to the K-heat equation

$$\partial_t u_t + K u_t = 0.$$

The Drift Term

Let u_t be a solution of the K-heat equation (it is a family of functions on the plane).

Define the *center of mass* of u_t to be

$$\begin{aligned}\mathrm{cm}(u_t) &= \big(\mathrm{cm}_x(u_t), \mathrm{cm}_y(u_t)\big) \\ &= \Big(\int\int u_t(x,y)\, x\, dx\, dy, \int\int u_t(x,y)\, y\, dx\, dy\Big)\end{aligned}$$

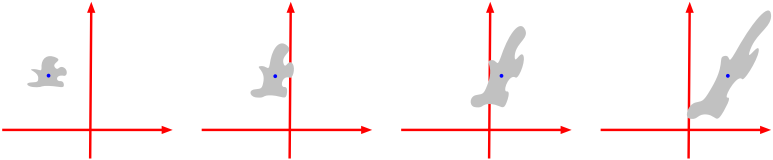
By differentiating under the integral sign, we find that

$$\frac{d}{dt}\mathrm{cm}(u_t) = (\mathrm{cm}_y(u_t), 0).$$

If the term $y\partial_x$ was removed from K, then the derivative would be zero. The drift is entirely attributable to $y\partial_x$.

The Drift Term

Here is a cartoon of what happens, showing where a solution of the K-heat equation is concentrated as t increases.



- If u_0 was concentrated higher, the drift would be faster.
- If u_0 was concentrated lower, the drift would be to the left.

The Concentration Property for the Heat Kernel

The heat operators for Laplacian (on the circle or elsewhere) have the following well-known property of concentration along the diagonal.

Proposition

Let σ_1 and σ_0 be smooth functions on $\mathbb{T}$ with disjoint supports. There is a positive constant k such that

$$\|\sigma_1 \exp(-t\Delta)\sigma_0\| = \mathcal{O}(e^{-k/t})$$

as $t \to 0$ [this is true for any reasonable norm on the left].

This adapts very nicely to incorporate the drift phenomenon we've seen. If φ is a smooth function on the circle, define

$$\varphi_t(x, y) = \varphi(x - t^{-1}by)$$

Then . . .

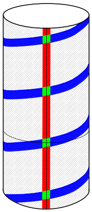
The Heat Kernel for the Hypoelliptic Laplacian

Proposition

If φ and ψ are smooth functions on $\mathbb{T}$ with disjoint supports, then there is a positive constant k such that

$$\left\|\varphi_t \exp(-tL_b)\psi_0\right\| \leq \mathcal{O}(e^{-kb^2})$$

for any fixed t as $b \to \infty$.



So the L_b-heat kernel concentrates on a drifted diagonal, which leads to

$$\exp\left(-tL_b\right)(p,p) \approx 0 \quad \text{unless } p \text{ is near } \mathbb{T}\times t^{-1}b\mathbb{Z}.$$

That is, the heat trace for the hypoelliptic Laplacian concentrates on geodesic bands.

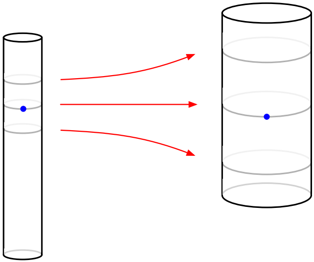
Rescaling Argument

What about when p is near $\mathbb{T}\times t^{-1}b\mathbb{Z}$, say near $(0, t^{-1}bn)$?

I'll set $t = 1$ for simplicity.

Identify $\mathbb{T}\times\mathbb{R}$ with $b^2\mathbb{T}\times\mathbb{R}$ via

$$(u, v) = (b^2x, b(y-bn))$$



Then the (scalar version of the) hypoelliptic Laplacian transforms as follows:

$$L_b = \frac{1}{2}\partial_v^2 + v\partial_u + \frac{1}{2}n^2 + b^2 n\partial_u + \mathcal{O}(b^{-1})$$

Rescaling Argument

The right-hand side in

$$L_b = \tfrac{1}{2}\partial_v^2 + v\partial_u + \tfrac{1}{2}n^2 + b^2 n\partial_u + \mathcal{O}(b^{-1})$$

does not approach a limit as $b \to \infty$ because of the highlighted term.

But the highlighted term commutes with all the others, and exponentiates to the identity operator.

We obtain (taking into account all the terms in full operator L_b on forms, and reinstating t):

$$\begin{aligned}\operatorname{str}(\exp(-t\mathsf{L}_b))\big((0, w{+}bn), (0, w{+}bn)\big)& \\ \approx b \cdot e^{-n^2/2t} \exp(-tK)&((0, bw), (0, bw)\big)\end{aligned}$$

as $b \to \infty$, where $K = -\partial_u^2 + y\partial v$.

Explicit Formulas

There is an explicit formula for the K-heat kernel, found by Kolmogorov:

$$\exp(-tK)\big((u_1, v_2), (u_2, v_2)\big) = \frac{\sqrt{3}}{\pi t^2} \exp\Big(-\frac{1}{2t}\Big(v_1 - v_2\Big)^2 - \frac{6}{t^3}\Big(u_2 - u_1 - \frac{(v_1+v_2)t}{2}\Big)^2\Big)$$

(We only need the case where $u_1 = u_2$ and $v_1 = v_2$.)

We obtain

$$\lim_{b\to\infty} \mathrm{STr}(\exp(-t\mathsf{L}_b)) = \sum_{n\in\mathbb{Z}} e^{-n^2/2t} \int_{-\infty}^{\infty} \exp(-tK)\big((0, v), (0, v)\big)\, dv = \frac{1}{\sqrt{2\pi t}} \sum_{n\in\mathbb{Z}} e^{-n^2/2t}$$

which gives the “Selberg trace formula on the circle."

Thank you!