# Equivariant Homology for $S L(2)$ of a $p$-adic Field 

PAUL BAUM, NIGEL HIGSON AND ROGER PLYMEN

Let $F$ be a $p$-adic field and let $G=S L(2)$ be the group of unimodular $2 \times$ 2 matrices over $F$. The aim of this paper is to calculate certain equivariant homology groups attached to the action of $G$ on its tree. They arise in connection with a theorem of M. Pimsner on the $K$-theory of the $C^{*}$-algebra of $G[\mathbf{1 2}]$, and our purpose is to explore the representation theoretic content of Pimsner's result.

The outcomes of our calculations are given in Theorems 5.4 and 6.1. In Sections 8 and 9 of the paper we re-examine Pimsner's theorem in the light of these new results.

The first author and A. Connes have formulated a very general conjecture [1] describing the $K$-theory of the reduced $C^{*}$-algebra of any locally compact group. For a semisimple group over a $p$-adic field it asserts, roughly speaking, that the cohomology of the space of tempered representations of $G$ is isomorphic to the equivariant homology of the affine Bruhat-Tits building of $G$. For $S L(2)$ and other split rank one groups the conjecture amounts to Pimsner's theorem, but for groups of higher rank the conjecture is not yet proved. In a sequel to this article we shall study the representation theoretic aspects of the conjecture for $p$-adic groups (we note that the arguments in Sections 5 and 8 readily extend to this general case).

Our homology groups are very closely related to the cyclic homology groups of the convolution algebra of smooth compactly supported functions on $G$, and the results of our calculations are similar to some of P. Blanc and J-L. Brylinski in [3]. But the methods we employ are different, and we hope they complement rather than duplicate those of Blanc and Brylinski. The connection between the two will be explored elsewhere.

[^0]
## 1. Local Fields

We review some terminology. See [23] for further details.
A local field is a locally compact, non-discrete topological field. There is a natural norm function $|\cdot|$ given by the formula

$$
|x| d t=d(x t)
$$

where $d t$ is a Haar measure on the additive group of the field. Restricted to the multiplicative group the norm is a homomorphism into the positive real numbers, and the field is non-Archimedean if the range of $|\cdot|$ is discrete. In this case the norm satisfies the inequality

$$
|x+y| \leq \max (|x|,|y|)
$$

If $F$ is a non-Archimedean local field then the subset

$$
\mathcal{O}=\{x \in F:|x| \leq 1\}
$$

is a principal ideal domain, from which $F$ can be recovered as the field of fractions. It has a unique prime ideal, a generator of which we shall denote by $\varpi$.

We shall use the term $p$-adic field for a non-Archimedean local field of characteristic zero. In concrete terms $F$ is a finite extension of some $\mathbb{Q}_{p}$.

## 2. $S L(2)$ and its Tree

Let $G=S L(2)$ be the group of unimodular $2 \times 2$ matrices with entries in a $p$-adic field $F$. It is a locally compact, totally disconnected topological group [6] [20].

Let

$$
I=\left\{\left.\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right) \in S L(2) \right\rvert\, a_{21} \in \varpi \mathcal{O}\right\}
$$

or, more schematically,

$$
I=\left(\begin{array}{cc}
\mathcal{O} & \mathcal{O} \\
\varpi \mathcal{O} & \mathcal{O}
\end{array}\right) \cap S L(2)
$$

This is a compact open subgroup of $G$, called the Iwahori subgroup.
Let

$$
w_{0}=\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right) \quad \text { and } \quad w_{1}=\left(\begin{array}{cc}
0 & -\varpi^{-1} \\
\varpi & 0
\end{array}\right)
$$

These elements appear in the Tits system associated to $G$, which plays an important role in what follows. Let $\tilde{W}$ be the group generated by $w_{0}$ and $w_{1}$, and denote by $W$ the quotient of $\tilde{W}$ by the normal subgroup consisting of $\pm 1$. The Tits system may be viewed as an elaboration of the Bruhat decomposition

$$
\begin{equation*}
G=\bigcup_{w \in W} I w I \tag{2.1}
\end{equation*}
$$

in which multiplication in $W$ is related to multiplication of the double cosets in $I \backslash G / I$. See [4] or [17] for details.

Let

$$
K_{0}=I \cup I w_{0} I=\left(\begin{array}{cc}
\mathcal{O} & \mathcal{O} \\
\mathcal{O} & \mathcal{O}
\end{array}\right) \cap S L(2)
$$

and

$$
K_{1}=I \cup I w_{1} I=\left(\begin{array}{cc}
\mathcal{O} & \varpi^{-1} \mathcal{O} \\
\varpi \mathcal{O} & \mathcal{O}
\end{array}\right) \cap S L(2),
$$

which are compact open subgroups of $G$. Note that $K_{0} \cap K_{1}=I$. The tree for $G=S L(2)$ is the graph $\beta G$ constructed as follows:
(1) To each left coset of $I$ in $G$ associate an oriented edge with one black and one white vertex.
(2) Join the edges $g I$ and $g^{\prime} I$ at the black vertex if $g I$ and $g^{\prime} I$ lie in a common left coset of $K_{0}$. Join them at the white vertex if $g I$ and $g^{\prime} I$ lie in a common left coset of $K_{1}$.
The fact that $\beta G$ is a tree is related, via the axioms for the Tits system, to the geometry of $W$ in the word length metric for the generators $w_{0}$ and $w_{1}[4]$.

The group $G$ acts on $\beta G$ by multiplication on the left. Note that $I$ is the stabilizer of the fundamental edge (labelled by $I$ ), and that $K_{0}$ and $K_{1}$ are the stabilizers of the black and white vertices of this edge, respectively.

## 3. Homology

Let $K$ be a compact, totally disconnected group. Denote by $R(K)$ the representation ring of $K$. (Its multiplicative structure is irrelevant to us: view $R(K)$ as an abelian group.) Denote by $C_{i n v}^{\infty}(K)$ the vector space of smooth (meaning locally constant), complex valued functions on $K$ which are invariant under the adjoint action.

Lemma. The map which associates to a representation its character induces an isomorphism from $R(K) \otimes_{\mathbb{Z}} \mathbb{C}$ to $C_{\text {inv }}^{\infty}(K)$.

Proof. A finite dimensional representation of $K$ factors through a finite quotient $K / N[\mathbf{1 4}]$. So the representation ring $R(K)$ is the direct limit of the representation rings of the finite groups $K / N$, where $N$ varies amongst the open normal subgroups of $K$. The lemma follows from the corresponding result for finite groups.

Let $H$ be an open subgroup of $K$. Since it is of finite index, induction of representations gives a homomorphism

$$
\operatorname{Ind}_{H}^{K}: R(H) \rightarrow R(K),
$$

and a corresponding linear map

$$
\operatorname{Ind}_{H}^{K}: C_{i n v}^{\infty}(H) \rightarrow C_{i n v}^{\infty}(K)
$$

The latter is given by the formula

$$
\begin{equation*}
\operatorname{Ind}_{H}^{K} \phi(g)=\sum_{k \in H \backslash K} \phi\left(k g k^{-1}\right), \tag{3.1}
\end{equation*}
$$

where the sum is over representatives of the right cosets of $H$ in $K$, and where we extend $\phi$ to a function on $K$ by setting it equal to zero outside of $H$.

Let $I, K_{0}$ and $K_{1}$ be the compact subgroups of $G=S L(2)$ defined in the previous section. Form the "complex"

$$
\begin{equation*}
0 \rightarrow C_{i n v}^{\infty}(I) \xrightarrow{\operatorname{Ind}_{I}^{K_{0}} \oplus-\operatorname{Ind}_{I}^{K_{1}}} C_{i n v}^{\infty}\left(K_{0}\right) \oplus C_{i n v}^{\infty}\left(K_{1}\right) \rightarrow 0 \tag{3.2}
\end{equation*}
$$

and denote by $H_{*}(G, \mathbb{C})$ its homology, so that

$$
H_{0}(G ; \mathbb{C})=\text { cokernel }\left(C_{i n v}^{\infty}(I) \xrightarrow{\operatorname{Ind}_{I}^{K_{0}} \oplus-\operatorname{Ind}_{I}^{K_{1}}} C_{i n v}^{\infty}\left(K_{0}\right) \oplus C_{i n v}^{\infty}\left(K_{1}\right)\right)
$$

and

$$
H_{1}(G ; \mathbb{C})=\operatorname{kernel}\left(C_{i n v}^{\infty}(I) \xrightarrow{\operatorname{Ind}_{I}^{K_{0}} \oplus-\operatorname{Ind}_{I}^{K_{1}}} C_{i n v}^{\infty}\left(K_{0}\right) \oplus C_{i n v}^{\infty}\left(K_{1}\right)\right)
$$

These may be viewed as equivariant homology groups for the action of $G$ on $\beta G$. We shall not go into this here, except to note that the quotient space $\beta G / G$ can be identified with the fundamental edge in $\beta G$, and if we attach to this edge and its vertices the invariant functions on the corresponding stabilizer subgroups of $G$, we obtain a co-sheaf on $\beta G / G$, of which $H_{j}(G, \mathbb{C})$ is the homology.

We can also form an "integral" complex

$$
\begin{equation*}
0 \rightarrow R(I) \xrightarrow{\operatorname{Ind}_{I}^{K_{0}} \oplus-\operatorname{Ind}_{I}^{K_{1}}} R\left(K_{0}\right) \oplus R\left(K_{1}\right) \rightarrow 0 \tag{3.3}
\end{equation*}
$$

The groups $H_{j}(G ; \mathbb{C})$ are obtained from the homology groups $H_{j}(G)$ of this complex by tensoring with $\mathbb{C}$ (hence the notation).

## 4. Pimsner's Theorem

Let $K$ be a compact, totally disconnected group. Choose a Haar measure and view $C^{\infty}(K)$ as an algebra with respect to convolution multiplication.

Let $V$ be an irreducible complex linear representation of $K$, and denote by $\chi$ its character. The function

$$
E_{V}(g)=\frac{\operatorname{dim}(V)}{\operatorname{vol}(K)} \chi(g)
$$

is a central idempotent in $C^{\infty}(K)$. (Our formula for $E_{V}$ gives the projection onto the isotypical component of $L^{2}(K)$ labelled by the conjugate representation $\bar{V}$.)

Proposition 4.1. (Compare [18].) The algebra $C^{\infty}(K)$ decomposes as a direct sum

$$
C^{\infty}(K)=\bigoplus_{V} E_{V} C^{\infty}(K)
$$

over representatives of the isomorphism classes of irreducible complex linear representations. Each summand $E_{V} C^{\infty}(K)$ is isomorphic to a complex matrix algebra. If $v \in V$ is a unit vector, with respect to an invariant inner product, then the function

$$
\begin{equation*}
e_{v}(g)=\frac{\operatorname{dim}(V)}{\operatorname{vol}(K)}(g v, v) \tag{4.1}
\end{equation*}
$$

is a minimal idempotent in $E_{V} C^{\infty}(K)$.
Let $A$ be an algebra (perhaps without unit) and denote by $\mathbf{K}_{0}(A)$ its $K$ theory, as in [11] or [2]. Let $\Phi: A \rightarrow V$ be a trace with values in a vector space $V$, in other words a linear map such that

$$
\Phi\left(a a^{\prime}\right)=\Phi\left(a^{\prime} a\right), \quad \text { for all } a, a^{\prime} \in A
$$

There is an induced map $\Phi_{*}: \mathbf{K}_{0}(A) \rightarrow V$. If $A$ has a unit this is defined by the formula

$$
\begin{equation*}
\Phi_{*}\left(\left[p_{i j}\right]\right)=\sum_{i} \Phi\left(p_{i i}\right) \tag{4.2}
\end{equation*}
$$

where $\left[p_{i j}\right]$ is an idempotent in a matrix algebra over $A$. If $A$ has no unit then we extend $\Phi$ to the unitalization $\tilde{A}$ of $A$ by setting $\Phi(1)=0$, and then use (4.2).

We define

$$
\begin{gathered}
\Phi: C^{\infty}(K) \longrightarrow C_{i n v}^{\infty}(K), \\
\Phi(\phi)(g)=\int_{K} \phi\left(k^{-1} g k\right) d k .
\end{gathered}
$$

It is a trace, and we obtain a map

$$
\begin{equation*}
\Phi_{*}: \mathbf{K}_{0}\left(C^{\infty}(K)\right) \rightarrow C_{i n v}^{\infty}(K) \tag{4.3}
\end{equation*}
$$

Lemma 4.2. The map $\Phi_{*}: \mathbf{K}_{0}\left(C^{\infty}(K)\right) \rightarrow C_{i n v}^{\infty}(K)$ is an isomorphism onto the subgroup $R(K)$ of $C_{i n v}^{\infty}(K)$. We have

$$
\begin{equation*}
\Phi_{*}\left(\left[e_{v}\right]\right)=\text { character of } V \tag{4.4}
\end{equation*}
$$

where $e_{v}$ is defined in (4.1).
Proof. This follows immediately from Proposition 4.1.
Let $I$ be an open subgroup of $K$ and restrict the Haar measure on $K$ to a Haar measure on $I$. Then $C^{\infty}(I)$ is a subalgebra of $C^{\infty}(K)$. Checking the definitions we see:

Lemma 4.3. The diagram

$$
\begin{array}{cc}
\mathbf{K}_{0}\left(C^{\infty}(I)\right) & \stackrel{\text { induced }}{\text { by inclusion }} \\
\mathbf{K}_{0}\left(C^{\infty}(K)\right) \\
\Phi_{*} \downarrow & \downarrow^{\Phi_{*}} \\
C_{\text {inv }}^{\infty}(I) & \underset{\operatorname{Ind}_{I}^{K}}{ }
\end{array} C_{i n v}^{\infty}(K)
$$

commutes.
Fix a Haar measure on $G=S L(2)$. The algebra $C_{c}^{\infty}(G)$ acts on $L^{2}(G)$ by convolution on the left (the regular representation), and the $C^{*}$-algebra of Hilbert space operators generated by $C_{c}^{\infty}(G)$ is the reduced $C^{*}$-algebra of $G$, denoted $C_{r}^{*}(G)$.

The inclusions of $K_{0}$ and $K_{1}$ into $G$ induce maps $\mathbf{K}_{0}\left(C^{\infty}\left(K_{i}\right)\right) \rightarrow \mathbf{K}_{0}\left(C_{r}^{*}(G)\right)$ and, in view of Lemmas 4.2 and 4.3, we obtain from them a map

$$
\mu_{0}: H_{0}(G) \rightarrow \mathbf{K}_{0}\left(C_{r}^{*}(G)\right)
$$

There is also a map

$$
\mu_{1}: H_{1}(G) \rightarrow \mathbf{K}_{1}\left(C_{r}^{*}(G)\right)
$$

(on the right is the topological $K$-theory group, as defined in [2]). A class in the homology group $H_{1}(G)$ may be viewed as a class in $\mathbf{K}_{0}\left(C^{\infty}(I)\right)$ which maps to zero in $\mathbf{K}_{0}\left(C^{\infty}\left(K_{0}\right)\right)$ and $\mathbf{K}_{0}\left(C^{\infty}\left(K_{1}\right)\right)$. It can be represented as $[p]-[q]$, where $p$ and $q$ are idempotents in $C^{\infty}(I)$ (or in some matrix algebra over it) such that

$$
p=u_{0} v_{0}=u_{1} v_{1} \quad \text { and } \quad q=v_{0} u_{0}=v_{1} u_{1}
$$

for some $u_{0}, v_{0} \in C^{\infty}\left(K_{0}\right)$ and some $u_{1}, v_{1} \in C^{\infty}\left(K_{1}\right)$. The quantity $u_{0} v_{1}+1-p$ is an invertible element in the algebra obtained by adjoining a unit to $C_{r}^{*}(G)$. We define

$$
\mu_{1}([p]-[q])=\left[u_{0} v_{1}+1-p\right] .
$$

Pimsner's result, applied to the situation we are interested in, is as follows.
Pimsner's Theorem. The maps

$$
\mu_{i}: H_{i}(G) \rightarrow \mathbf{K}_{i}\left(C_{r}^{*}(G)\right) \quad(i=0,1)
$$

are isomorphisms of abelian groups.

## 5. Calculation of $H_{0}$

A distribution on a totally disconnected space $X$ is a linear functional $F$ on the space $C_{c}^{\infty}(X)$ of compactly supported smooth functions (there are no continuity requirements on $F$ ). If $Y$ is an open subset of $X$ then since $C_{c}^{\infty}(Y) \subset C_{c}^{\infty}(X)$ any distribution on $X$ restricts to one on $Y$.

Lemma 5.1. Let $\mathcal{U}$ be an open cover of $X$, and for each $U \in \mathcal{U}$ let $F_{U}$ be a distribution on $U$. Suppose that for all $x \in X$, and all $U, U^{\prime} \in \mathcal{U}$ containing $x$, the distributions $F_{U}$ and $F_{U^{\prime}}$ agree on some neighbourhood of $x \in U \cap U^{\prime}$. Then there is a unique distribution $F$ on $X$ such that $\left.F\right|_{U}=F_{U}$ for all $U \in \mathcal{U}$.

Proof. This follows from the existence of smooth partitions of unity.
A distribution on a totally disconnected group is invariant if it is fixed under the adjoint action. Equivalently it is a complex valued trace on the convolution algebra $C_{c}^{\infty}(G)$. Note that if the group is compact then an invariant distribution is the same thing as a linear functional on the invariant smooth functions.

Let $\left(\phi_{0}, \phi_{1}\right) \in C_{i n v}^{\infty}\left(K_{0}\right) \oplus C_{i n v}^{\infty}\left(K_{1}\right)$ represent a non-zero element of $H_{0}(G ; \mathbb{C})$. There is a linear functional on $C_{i n v}^{\infty}\left(K_{0}\right) \oplus C_{i n v}^{\infty}\left(K_{1}\right)$ which vanishes on the image of the boundary map in our complex (3.2), but which is non-zero on $\left(\phi_{0}, \phi_{1}\right)$. In other words there are invariant distributions $F_{0}$ and $F_{1}$ on $K_{0}$ and $K_{1}$ such that

$$
F_{0}\left(\phi_{0}\right)+F_{1}\left(\phi_{1}\right) \neq 0
$$

and

$$
\begin{equation*}
F_{0}\left(\operatorname{Ind}_{I}^{K_{0}} \psi\right)-F_{1}\left(\operatorname{Ind}_{I}^{K_{1}} \psi\right)=0, \quad \text { for all } \psi \in C_{i n v}^{\infty}(I) \tag{5.1}
\end{equation*}
$$

Lemma (Frobenius Reciprocity). Let I be an open subgroup of a compact totally disconnected group $K$, let $F$ be an invariant distribution on $K$, and let $\psi \in C_{\text {inv }}^{\infty}(I)$. Then

$$
F\left(\operatorname{Ind}_{I}^{K} \psi\right)=[K: I] F(\psi)
$$

Proof. This follows immediately from the formula (3.1) for induction.
Corollary. If $F_{0}$ and $F_{1}$ are invariant distributions on $K_{0}$ and $K_{1}$ satisfying (5.1) then they have a common restriction to $I$.

Proposition 5.2. Let $F_{0}$ and $F_{1}$ be invariant distributions on $K_{0}$ and $K_{1}$ which have a common restriction to $I$. There is an invariant distribution $F$ on $G$ such that $\left.F\right|_{K_{0}}=F_{0}$ and $\left.F\right|_{K_{1}}=F_{1}$.

Proof. We shall construct $F$ using Lemma 5.1. Let $G_{c}$ be the subset of compact elements in $G$, and let $G_{n c}$ be its complement. They are both open sets. Cover $G$ by $G_{n c}$ and all the conjugates in $G$ of $K_{0}$ and $K_{1}$ (these cover $\left.G_{c}\right)$. We define $F_{G_{n c}}$ to be zero and $F_{g K_{i} g^{-1}}$ to be the conjugate of $F_{i}$ by $g$.

Denote by $\approx$ the equivalence relation on $K_{0} \cup K_{1}$ generated as follows: if $k$ and $k^{\prime}$ are both elements of $K_{0}$, or are both elements of $K_{1}$, and if they are conjugate within that group (not merely within $G$ ), then $k \approx k^{\prime}$. The hypotheses of Lemma 5.1 follow from the assertion that if $k, k^{\prime} \in K_{0} \cup K_{1}$, and if $k^{\prime}=g k g^{-1}$ for some $g \in G$, then $k \approx k^{\prime}$.

Case 1: $k$ and $k^{\prime}$ are not equivalent to elements of $I$. The fixed point set of any element of $G$ acting on the tree $\beta G$ is connected. In addition, an element of $K_{0}$ or $K_{1}$ is conjugate (in $K_{0}$ or $K_{1}$ respectively) to an element of $I$ if and only if it fixes an edge. It follows that $k$ and $k^{\prime}$ fix no edge in the tree, and since their
fixed point sets are connected, they each consist of a single vertex, say $X$ and $X^{\prime}$, respectively. From $k^{\prime}=g k g^{-1}$ we see that $X=X^{\prime}$ and $g X=X$.

Case 2: $k$ and $k^{\prime}$ are equivalent to elements of $I$. We may assume that $k, k^{\prime} \in I$, and using the Bruhat decomposition (2.1) we may assume that $g$ is an element of the group $\tilde{W}$ generated by $w_{0}$ and $w_{1}$. We proceed by induction on the word length of $g$ in the quotient group $W$. If it is zero then there is nothing to prove. Otherwise write $g=w g^{\prime}$, where $w$ is one of $w_{0}, w_{1}$ and where length $\left(g^{\prime}\right)<$ length $(g)$. The key step in the proof is to show that $g^{\prime} k g^{\prime-1} \in I$. This is done in Lemma 3, p. 92 of [ $\mathbf{1 7}]$. By induction,

$$
k \approx g^{\prime} k g^{\prime-1} \approx w g^{\prime} k g^{\prime-1} w^{-1}=k^{\prime}
$$

The content of the proposition is that $H_{0}(G ; \mathbb{C})$ injects into the dual of the space of invariant distributions on $G$, via the pairing

$$
\left(\phi_{0}, \phi_{1}\right) \mapsto F\left(\phi_{0}\right)+F\left(\phi_{1}\right)
$$

(which is well defined by Frobenius Reciprocity). We shall calculate $H_{0}(G ; \mathbb{C})$ on the basis of this, borrowing several results from distribution theory (see especially [19]).

Let $T$ be a maximal torus in $G=S L(2)$. (Up to conjugacy there are only finitely many: apart from the subgroup of diagonal matrices, $T$, or its transpose, is of the form

$$
T=\left\{\left.\left(\begin{array}{cc}
a & b  \tag{5.1}\\
b x & a
\end{array}\right) \right\rvert\, a^{2}-b^{2} x=1\right\}
$$

where $x \in F^{\times}$is a non-square. The conjugacy class depends only on the residue class of $x$ in $F^{\times} / F^{\times 2}$.) The quotient space $G / T$ admits an invariant measure. It depends on choices of Haar measure on $G$ and $T$, and is characterized by the formula

$$
\begin{equation*}
\int_{G} \phi(g) d g=\int_{G / T}\left(\int_{T} \phi(g t) d t\right) d g \tag{5.2}
\end{equation*}
$$

(the inner integral is a $T$-invariant function on $G$, and so may be regarded as a function on $G / T)$.

Denote by $T^{\text {reg }}$ the regular elements in $T$ (for $S L(2)$ these are all except plus or minus the identity matrix). Fix $t \in T^{r e g}$, and for $\phi \in C_{c}^{\infty}(G)$ define

$$
\begin{equation*}
F_{t}^{T}(\phi)=\int_{G / T} \phi\left(g t g^{-1}\right) d g \tag{5.3}
\end{equation*}
$$

The integral converges and defines an invariant distribution on $G$.

Proposition 5.3. Let $\phi \in C_{c}^{\infty}(G)$ and suppose that $F_{t}^{T}(\phi)=0$ for every maximal torus $T$ and every $t \in T^{\text {reg }}$. Then $F(\phi)=0$ for every invariant distribution $F$ on $G$.

Proof. See Theorem 0 of [8] or Section 11 of [21], which deal with the general case. For $G=S L(2)$ the proposition follows easily from the arguments in [19].

Fix $\phi \in C_{c}^{\infty}(G)$ and view $F_{t}^{T}(\phi)$ as a function of $t$. It is smooth and invariant under the action of the Weyl group $W_{T}$ (the normalizer of $T$ in $G$, divided by $T)$. We obtain maps

$$
\begin{gathered}
F^{T}: H_{0}(G ; \mathbb{C}) \rightarrow C^{\infty}\left(T^{r e g}\right)^{W_{T}} \\
\left(\phi_{0}, \phi_{1}\right) \mapsto F_{t}^{T}\left(\phi_{0}\right)+F_{t}^{T}\left(+\phi_{1}\right)
\end{gathered}
$$

Since conjugate maximal tori give essentially the same map, we select one representative from each conjugacy class, and form the direct sum

$$
\begin{equation*}
H_{0}(G ; \mathbb{C}) \xrightarrow{\oplus F^{T}} \oplus C^{\infty}\left(T^{\text {reg }}\right)^{W_{T}} . \tag{5.4}
\end{equation*}
$$

Putting Propositions 5.1 and 5.3 together we see that (5.4) is injective.
Let $w \in G$ be either a unipotent element or minus a unipotent. The centralizer $G(w)$ of $w$ is unimodular the quotient space $G / G(w)$ is equipped with an invariant measure as in (5.2). The integral

$$
F_{w}(\phi)=\int_{G / G(w)} \phi(g) d g
$$

converges for any $\phi \in C_{c}^{\infty}(G)$ (see [19], Section 1.2). It defines an invariant distribution on $G$, depending only the conjugacy class of $w \in G$. These are

$$
\langle w\rangle=\left\langle\left(\begin{array}{cc} 
\pm 1 & a \\
0 & \pm 1
\end{array}\right)\right\rangle, \quad\left(a=0 \quad \text { or } \quad a \in F^{\times} / F^{\times^{2}}\right) .
$$

In particular, there are only finitely many conjugacy classes.
Denote by $T_{c}^{\text {reg }}$ the compact elements in $T^{\text {reg }}$ (those which lie in a compact subgroup of $G$ ). With the exception of the diagonal subgroup, we have $T_{c}^{\text {reg }}=$ $T^{\text {reg }}$, and for the diagonal subgroup $T_{c}^{\text {reg }}$ is comprised of those elements of $T^{\text {reg }}$ with entries in the ring of integers $\mathcal{O}$.

Theorem 5.4. The map $\oplus F^{T}: H_{0}(G ; \mathbb{C}) \rightarrow \oplus C^{\infty}\left(T^{\text {reg }}\right)^{W_{T}}$ gives rise to an exact sequence

$$
0 \rightarrow \oplus C_{c}^{\infty}\left(T_{c}^{r e g}\right)^{W_{T}} \rightarrow H_{0}(G ; \mathbb{C}) \xrightarrow{\oplus F_{w}} \oplus_{w} \mathbb{C} \rightarrow 0,
$$

where the second direct sum is over the conjugacy classes of elements $w$, such that $\pm w$ is unipotent.

Proof. If $\left(\phi_{0}, \phi_{1}\right) \in C_{i n v}^{\infty}\left(K_{0}\right) \oplus C_{i n v}^{\infty}\left(K_{1}\right)$ then it is clear that $F_{t}^{T}\left(\phi_{0}+\phi_{1}\right)$ is zero for all non-compact $t$ : hence the restriction to $T_{c}^{\text {reg }}$. On the other hand it is easily checked that $\oplus C_{c}^{\infty}\left(T_{c}^{r e g}\right)^{W_{T}}$ is contained in the range of the map $\oplus F^{T}$.

According to the theory of Shalika germs [19], if $F_{w}(\phi)=0$ for every unipotent $w$ then $F_{t}^{T}(\phi)=0$ for all $t$ near 1 . Multiplying by the central element $-1 \in G$ we see that if $F_{-w}(\phi)=0$ for every unipotent $w$, then $F_{t}^{T}(\phi)$ vanishes for all $t$ near -1 . This proves exactness in the middle of the sequence. Finally, one verifies that $\oplus F_{w}$ is surjective.

## 6. Calculation of $H_{1}$

If $\phi$ is a function on the Iwahori subgroup $I$ then define a function $\tilde{\phi}$ on the group $\mathcal{O}^{\times}$of units in $\mathcal{O}$ by the formula

$$
\tilde{\phi}(a)=\phi\left(\left(\begin{array}{cc}
a & 0 \\
o & a^{-1}
\end{array}\right)\right) .
$$

We shall prove the following result.
THEOREM 6.1. The map $\phi \mapsto \tilde{\phi}$ is an isomorphism from $H_{1}(G ; \mathbb{C})$ to the space of locally constant functions $\tilde{\phi}$ on $\mathcal{O}^{\times}$such that $\tilde{\phi}\left(a^{-1}\right)=-\tilde{\phi}(a)$.

The proof uses a few elementary facts in representation theory. Let $K$ be any compact group and let $H$ be an open subgroup. For $x \in K$ let

$$
H_{x}=H \cap x^{-1} H x
$$

If $\phi$ is a function on $H$ then define a function $\phi^{x}$ on $H_{x}$ by

$$
\phi^{x}(g)=\phi\left(x g x^{-1}\right)
$$

Define an inner product on $C_{i n v}^{\infty}(K)$ by the formula

$$
(\phi, \psi)_{K}=\frac{1}{\operatorname{vol}(K)} \int_{K} \phi(k) \overline{\psi(k)} d k
$$

Lemma (Mackey Formula). If $\phi, \psi \in C_{i n v}^{\infty}(H)$ then

$$
\left(\operatorname{Ind}_{H}^{K} \phi, \operatorname{Ind}_{H}^{K} \psi\right)_{K}=\sum_{x \in H \backslash K / H}\left(\phi^{x}, \psi\right)_{H_{x}}
$$

(The notation " $x \in H \backslash K / H$ " means that the sum is taken over representatives for the double coset space $H \backslash K / H$.)

For a proof see $[\mathbf{1 8}]$.
Corollary. $\operatorname{Ind}_{H}^{K} \phi=0$ if and only if

$$
\sum_{x \in H \backslash K / H}\left(\phi^{x}, \phi\right)_{H_{x}}=0,
$$

in which case

$$
\sum_{x \in H \backslash K / H}\left(\phi^{x}, \psi\right)_{H_{x}}=0
$$

for every $\psi \in C_{i n v}^{\infty}(H)$.
We shall also use the following facts about the range of the induction map in the case where $K$ is totally disconnected.

Lemma 6.2. The range of $\operatorname{Ind}_{H}^{K}: C_{i n v}^{\infty}(H) \rightarrow C_{i n v}^{\infty}(K)$ is equal to the range of the composition

$$
C_{i n v}^{\infty}(K) \xrightarrow{\text { restriction }} C_{i n v}^{\infty}(H) \xrightarrow{\operatorname{Ind}_{H}^{K}} C_{i n v}^{\infty}(K) .
$$

It consists of those invariant functions on $K$ which vanish on all conjugacy classes disjoint from $H$.

Let $I$ and $K_{0}$ be the subgroups defined in Section 3. Let

$$
I(k)=\left(\begin{array}{cc}
\mathcal{O} & \mathcal{O} \\
\varpi^{k} \mathcal{O} & \mathcal{O}
\end{array}\right) \cap S L(2)
$$

Lemma 6.3.

$$
K_{0}=I \cup I(k) w_{0} I(k)
$$

Proof. Since $K_{0}=I \cup I w_{0} I$ it suffices to show that $I w_{0} I=I(k) w_{0} I(k)$. There are left and right $I(k)$-coset representatives in $I$ of the form $\left(\begin{array}{ll}1 & 0 \\ a & 1\end{array}\right)$. From

$$
w_{0}\left(\begin{array}{ll}
1 & 0 \\
a & 1
\end{array}\right)=\left(\begin{array}{cc}
1 & -a \\
0 & 1
\end{array}\right) w_{0} \in I w_{0}
$$

we get $I w_{0} I=I w_{0} I(k)$. From

$$
\left(\begin{array}{cc}
1 & 0 \\
a & 1
\end{array}\right) w_{0}=w_{0}\left(\begin{array}{cc}
1 & -a \\
0 & 1
\end{array}\right) \in w_{0} I(k)
$$

we get $I w_{0} I(k)=I(k) w_{0} I(k)$.
Consider now the subgroup

$$
I(k)_{w_{0}}=I(k) \cap w_{0}^{-1} I(k) w_{0}=\left(\begin{array}{cc}
\mathcal{O} & \varpi^{k} \mathcal{O} \\
\varpi^{k} \mathcal{O} & \mathcal{O}
\end{array}\right) \cap S L(2) .
$$

Lemma 6.4. For every positive integer $k$, the space $H_{1}(G ; \mathbb{C}) \subset C_{\text {inv }}^{\infty}(I)$ lies in the image of the induction map $\operatorname{Ind}_{I(k)_{w_{0}}}^{I}: C_{i n v}^{\infty}\left(I(k)_{w_{0}}\right) \rightarrow C_{\text {inv }}^{\infty}(I)$.

Proof. Conjugation with the matrix $\left(\begin{array}{cc}0 & 1 \\ \varpi & 0\end{array}\right)$ gives an automorphism

$$
\alpha: S L(2) \rightarrow S L(2), \quad \alpha:\left(\begin{array}{cc}
a & b \\
c & d
\end{array}\right) \mapsto\left(\begin{array}{cc}
d & \varpi^{-1} c \\
\varpi b & a
\end{array}\right) .
$$

It is of period two, restricts to an automorphism of $I$, and exchanges the groups $K_{0}$ and $K_{1}$. For an invariant function $\phi$ on $I$ let $\phi^{\alpha}(g)=\phi(\alpha(g))$, and note that $\phi$ induces to zero on $K_{1}$ (resp. $K_{0}$ ) if and only if $\phi^{\alpha}$ induces to zero on $K_{0}$ (resp. $K_{1}$ ). Therefore

$$
\begin{equation*}
\phi \in H_{1}(G ; \mathbb{C}) \Longleftrightarrow \phi^{\alpha} \in H_{1}(G ; \mathbb{C}) \tag{6.1}
\end{equation*}
$$

Note also that

$$
\begin{equation*}
\alpha\left[I(k)_{w_{0}}\right] \subset I(k+1) . \tag{6.2}
\end{equation*}
$$

Combining (6.1) and (6.2) with an induction argument, it suffices to show that if $\phi \in H_{1}(G ; \mathbb{C})$ is induced from $I(k)$ then $\phi^{\alpha}$ is induced from $I(k+1)_{w_{0}}$.

If $\phi \in H_{1}(G ; \mathbb{C})$ is induced from $I(k)$ then, as noted in Lemma 6.2, we may choose a $\psi \in C_{\text {inv }}^{\infty}(I(k))$ which is the restriction of an invariant function on $I$, such that $\phi=\operatorname{Ind}_{I(k)}^{I} \psi$. We have that

$$
\operatorname{Ind}_{I(k)}^{K_{0}} \psi=\operatorname{Ind}_{I}^{K_{0}} \operatorname{Ind}_{I(k)}^{I} \psi=\operatorname{Ind}_{I}^{K_{0}} \phi=0
$$

and so according to the Mackey criterion

$$
\sum_{x \in I(k) \backslash K_{0} / I(k)}\left(\psi^{x}, \theta\right)_{I(k)_{x}}=0
$$

for every $\theta \in C_{i n v}^{\infty}(I(k))$. All but one of the double cosets lies within $I$, and since $\psi$ is $I$-invariant we obtain

$$
\begin{equation*}
\sum_{x \neq w_{0}}(\psi, \theta)_{I(k)_{x}}+\left(\psi^{w_{0}}, \theta\right)_{I(k)_{w_{0}}}=0 \tag{6.3}
\end{equation*}
$$

Let

$$
\theta(g)=\left\{\begin{array}{rl}
\psi(g) & \begin{array}{l}
\text { if } g \text { is not conjugate in } I(k) \text { to an } \\
\text { element of } I(k)_{w_{0}}
\end{array} \\
0 & \text { otherwise }
\end{array} .\right.
$$

The last term in (6.3) is then zero and all the remaining terms are non-negative (see the formula for the inner product). Therefore all the terms are zero. But the term corresponding to the coset containing the identity is a multiple of the integral of $|\psi|^{2}$ over the conjugacy classes disjoint from $I(k)_{w_{0}}$, and so we see that $\psi$ vanishes on all such conjugacy classes. Returning to our invariant function $\phi$ on $I$, we see that $\phi$ vanishes on all conjugacy classes in $I$ which are disjoint from $I(k)_{w_{0}}$, and therefore $\phi^{\alpha}$ vanishes on all conjugacy classes disjoint from $\alpha\left[I(k)_{w_{0}}\right]$.

Proof of Theorem 6.1. Let $\phi \in H_{1}(G ; \mathbb{C})$ and let $g \in I$. If $\phi(g) \neq 0$ then according to Lemma 6.4 the conjugacy class of $g$ intersects each $I(k)_{w_{0}}$. Since this conjugacy class is compact, it intersects the intersection of all the $I(k)_{w_{0}}$, which is the diagonal subgroup of $I$. This shows that the map $\phi \mapsto \tilde{\phi}$ is injective.

To prove that $\tilde{\phi}\left(a^{-1}\right)=-\tilde{\phi}(a)$ we use the automorphism $\alpha$ introduced in Lemma 6.4. It acts as an involution on $H_{1}(G ; \mathbb{C})$, which accordingly decomposes as a direct sum of $\pm 1$ eigenspaces. Since $\alpha(g)=g^{-1}$, if $g$ is diagonal, it suffices to show that the +1 eigenspace is trivial. Note that $\widetilde{\phi^{w_{0}}}=\widetilde{\phi^{\alpha}}$, and therefore $\phi^{\alpha}=\phi^{w_{0}}$. Applying Mackey's formula for $\operatorname{Ind}_{I}^{K_{0}} \phi$ we get

$$
(\phi, \phi)_{I}+\left(\phi^{\alpha}, \phi\right)_{I_{w_{0}}}
$$

Thus if $\phi^{\alpha}=\phi$ then $\phi=0$.
It remains to show that $\phi \mapsto \tilde{\phi}$ is surjective. The map $a \mapsto\left(\begin{array}{cc}a & 0 \\ 0 & a^{-1}\end{array}\right)$ passes to a homeomorphism of $\mathcal{O}^{\times}$onto an open and closed subset of the space of conjugacy classes in $I$. Consequently each locally constant function $\psi$ on $\mathcal{O}^{\times}$
gives an invariant function $\phi$ on $I$, supported on the conjugacy classes of diagonal matrices, such that $\tilde{\phi}=\psi$. If $\psi\left(a^{-1}\right)=-\psi(a)$ for all $a \in \mathcal{O}^{\times}$then it follows easily from the formula for induction that $\phi \in H_{1}(G ; \mathbb{C})$.

## 7. The Tempered Dual of $S L(2)$

The structure of the $C^{*}$-algebra $C_{r}^{*}(G)$ is worked out in [13] and [9]. There is a decomposition

$$
\begin{equation*}
C_{r}^{*}(G) \cong C_{\text {principal }}^{*}(G) \oplus C_{\text {discrete }}^{*}(G) \tag{7.1}
\end{equation*}
$$

corresponding to the decomposition of the tempered dual of $G$ into principal and discrete series $[\mathbf{5}][\mathbf{1 6}]$. We shall describe each component in turn.

The principal series representations are parametrized by the set $E_{2} M$ of unitary characters of the diagonal subgroup $M$ of $G$ ( $M$ is of course isomorphic to the multiplicative group of $F)$. Each character $\tau$ of $M$ extends to a character on the subgroup $B$ of upper triangular matrices, and the principal series representation $\pi(\tau)$ is obtained by unitarily inducing $\tau$ from $B$ to $G$.

Denote by $\mathcal{H}_{\pi(\tau)}$ the Hilbert space on which $\pi(\tau)$ acts. The spaces $\mathcal{H}_{\pi(\tau)}$ combine to form a bundle $\mathcal{H}$ of Hilbert spaces on $E_{2} M$. The representations $\pi(\tau)$ and $\pi(\bar{\tau})$ are unitarily equivalent and there are "normalized intertwining operators"

$$
\mathcal{A}(\tau): \mathcal{H}_{\pi(\tau)} \rightarrow \mathcal{H}_{\pi(\bar{\tau})}
$$

implementing this equivalence. They give $\mathcal{H}$ the structure of a $\mathbb{Z} / 2$-equivariant vector bundle. See [9].

We define

$$
C_{\text {principal }}^{*}(G)=C_{0}\left(E_{2} M, \mathcal{K}(\mathcal{H})\right)^{\mathbb{Z} / 2}
$$

Thus $C_{\text {principal }}^{*}(G)$ is the $C^{*}$-algebra of continuous, $\mathbb{Z} / 2$-equivariant, compact operator-valued endomorphisms of $\mathcal{H}$, which vanish at infinity. The representation of $C_{c}^{\infty}(G)$ on each $\mathcal{H}_{\pi(\tau)}$ determines a homomorphism of $C^{*}$-algebras

$$
\begin{equation*}
C_{r}^{*}(G) \rightarrow C_{\text {principal }}^{*}(G) \tag{7.2}
\end{equation*}
$$

We define

$$
C_{\text {discrete }}^{*}(G)=\oplus_{\pi \in E_{2} G} \mathcal{K}\left(\mathcal{H}_{\pi}\right)
$$

where the sum is over the discrete series of $G$. The representation of $C_{c}^{\infty}(G)$ on each $\mathcal{H}_{\pi}$ gives rise to a $C^{*}$-algebra homomorphism from $C_{r}^{*}(G)$ onto $C_{\text {discrete }}^{*}(G)$. This map, combined with (7.2), gives the isomorphism (7.1).

## 8. Pimsner's Isomorphism for $K_{0}$

Using the results of Section 5 we shall exhibit an inverse to Pimsner's map

$$
\mu_{0}: H_{0}(G) \rightarrow \mathbf{K}_{0}\left(C_{r}^{*}(G)\right)
$$

Let $\mathcal{S}(G)$ be the Harish-Chandra algebra of $G$, comprised of uniformly locally constant functions on $G$ of rapid decay [20] [21]. It is a subalgebra of $C_{r}^{*}(G)$ which contains $C_{c}^{\infty}(G)$.

Theorem 8.1.
(1) The map

$$
\mathbf{K}_{0}(\mathcal{S}(G)) \xrightarrow[\text { inclusion }]{\text { induced by }} \mathbf{K}_{0}\left(C_{r}^{*}(G)\right)
$$

is an isomorphism of abelian groups.
(2) Each of the orbital integrals $F_{t}^{T}$, as in (5.3), extends to a trace on $\mathcal{S}(G)$.

Proof. See [21] and [22].
It follows that each orbital integral defines a functional

$$
F_{t *}^{T}: \mathbf{K}_{0}\left(C_{r}^{*}(G)\right) \rightarrow \mathbb{C}
$$

Lemma 8.2. Normalize the Haar measure on $G$ so that $\operatorname{vol}\left(K_{0}\right)=\operatorname{vol}\left(K_{1}\right)=$ 1. The diagram

commutes.
Proof. Let $V$ be an irreducible unitary representation of one of $K_{0}$ or $K_{1}$. Its character determines an element of $H_{0}(G ; \mathbb{C})$, and it suffices to check commutativity for this. Let $\left\{v_{1}, \ldots, v_{n}\right\}$ be an orthonormal basis for $V$, and define $e_{v_{i}}$ as in (4.1). Note that the character of $V$ is

$$
\chi_{V}=\frac{1}{n}\left(e_{v_{1}}+\cdots+e_{v_{n}}\right)
$$

Applying first $\mu_{0}$, and then $F_{t *}^{T}$, we obtain $F_{t}^{T}\left(e_{v_{1}}\right)$. Applying $F_{t}^{T}$ first we obtain

$$
F_{t}^{T}\left(\chi_{V}\right)=\frac{1}{n}\left(F_{t}^{T}\left(e_{v_{1}}\right)+\cdots+F_{t}^{T}\left(e_{v_{n}}\right)\right)
$$

But $F_{t}^{T}\left(e_{v_{i}}\right)=F_{t}^{T}\left(e_{v_{j}}\right)$, since $e_{v_{i}}$ and $e_{v_{j}}$ are equivalent projections and $F_{t}^{T}$ is a trace. Hence $F_{t}^{T}\left(\chi_{V}\right)=F_{t}^{T}\left(e_{v_{1}}\right)$.

Viewing $t \in T^{\text {reg }}$ as a variable, and summing over representatives of the conjugacy classes of maximal tori, we obtain a commutative diagram

$$
\begin{array}{ccc}
H_{0}(G ; \mathbb{C}) & \xrightarrow{\mu_{0}} & \mathbf{K}_{0}\left(C_{r}^{*}(G)\right) \\
\oplus F^{T} \downarrow & & \downarrow \oplus F_{*}^{T} \\
\oplus C^{\infty}\left(T^{r e g}\right)^{W_{T}} & \\
= & \oplus C^{\infty}\left(T^{r e g}\right)^{W_{T}}
\end{array}
$$

(see [20] for the fact that the right vertical map goes into smooth functions: this also follows from Pimsner's isomorphism). Since the left vertical map is injective we have effectively inverted Pimsner's map $\mu_{0}$.

Note that Pimsner's theorem implies the Selberg Principle of [7]: if $t \in T^{\text {reg }}$ is non-compact then the orbital integral $F_{t}^{T}(a)$ vanishes for every class $a \in$ $\mathbf{K}_{0}(\mathcal{S}(G))$.

It is interesting to look at the map $\oplus F^{T}$ on $\mathbf{K}_{0}\left(C_{r}^{*}(G)\right)$ in the light of the detailed determination of the tempered dual of $G$ in $[\mathbf{5}],[\mathbf{1 6}]$, and the corresponding description of the group $\mathbf{K}_{0}\left(C_{r}^{*}(G)\right)$ [13]. For simplicity let us suppose that the cardinality of the residual field $\mathcal{O} / \varpi \mathcal{O}$ is odd.

The principal series representations of $G$ have already been described, and the $\mathbf{K}_{0}$-group of $C_{\text {principal }}^{*}(G)$ contains one generator for each character of $\mathcal{O}^{\times}$, up to the equivalence $\sigma \sim \bar{\sigma}$. There are three additional generators, associated to the elliptic representations in the principal series (these appear as components of the reducible principal series representations: those labelled by the three non-trivial characters of $M$ of order two).

The group $\mathbf{K}_{0}\left(C_{\text {discrete }}^{*}(G)\right)$ has one generator for each discrete series representation. These consist of the Steinberg representation $S t_{G}$ and the supercuspidal representations. The latter fall naturally into three families, corresponding to the three quadratic extensions of $F$. Each family is parametrized by characters of the norm-one group of the quadratic extension (for a precise statement see [15]).

As indicated in Section 5, the maximal tori of $G$ are, up to conjugacy, the diagonal subgroup and the norm-one groups of the quadratic extensions of $F$ (see [15]; up to a transposition the norm one groups embed naturally into $S L(2)$ ). Based on character theory [5] [15] it is natural to guess that the map $\oplus F^{T}$ admits a description at the level of $K$-theory generators, along the lines suggested by the above remarks, so that $K$-theory generators for the principal series are associated to functions supported on the diagonal subgroup, and so on. We have not, however, checked this.

## 9. Pimsner's Isomorphism for $K_{1}$

The space $E_{2} M$ decomposes as a disjoint union of circles, parametrized by the characters of $\mathcal{O}^{\times}$. This leads to the following description of $\mathbf{K}_{1}\left(C_{r}^{*}(G)\right.$ ) (see [13]).

Proposition 9.1. The abelian group $\mathbf{K}_{1}\left(C_{r}^{*}(G)\right)$ is isomorphic to the free abelian group on the set of unordered pairs of characters $\{\sigma, \bar{\sigma}\}$ of $\mathcal{O}^{\times}$, excluding those characters of order two.

We shall obtain a corresponding description of the integral homology group $H_{1}(G)$.

Fix a character $\sigma: \mathcal{O}^{\times} \rightarrow S^{1}$, not of order two, and denote by $k$ the least positive integer such that

$$
\begin{equation*}
\sigma\left[1+\varpi^{k} \mathcal{O}\right]=1 \tag{9.1}
\end{equation*}
$$

The character $\sigma$ extends to the group $I(k)$ (introduced in Section 6) using the formula

$$
\sigma:\left(\begin{array}{cc}
a & b  \tag{9.2}\\
\varpi^{k} c & d
\end{array}\right) \mapsto \sigma(a)
$$

We define

$$
\phi_{\sigma}=\operatorname{Ind}_{I(k)}^{I} \sigma
$$

Lemma 9.2.
(1) $\phi_{\sigma}$ is an irreducible character.
(2) $\phi_{\sigma}{ }^{\alpha}=\phi_{\bar{\sigma}}$, where $\alpha$ is the automorphism of $S L(2)$ introduced in the proof of Lemma 6.4.

Proof. The irreducible characters constitute an orthonormal basis for $R(I)$ [18], so irreducibility is equivalent to $\left(\phi_{\sigma}, \phi_{\sigma}\right)_{I}=1$. By Mackey's formula,

$$
\left(\phi_{\sigma}, \phi_{\sigma}\right)_{I}=\sum_{x \in I(k) \backslash I / I(k)}\left(\sigma^{x}, \sigma\right)_{I(k)_{x}}
$$

The inner product of two one-dimensional characters is zero if they are distinct, and one if they are equal. Using the fact that $k$ is the least integer satisfying (9.1) it is easily verified that $\sigma^{x} \neq \sigma$ on $I(k)_{x}$ unless $x$ lies in the identity double coset. This proves part (1).

Since both $\phi_{\sigma}{ }^{\alpha}$ and $\phi_{\bar{\sigma}}$ are irreducible characters, to prove (2) it suffices to show that $\left(\phi_{\bar{\sigma}}, \phi_{\sigma}{ }^{\alpha}\right)_{I}=1$. By functoriality of induction,

$$
\phi_{\sigma}{ }^{\alpha}=\left(\operatorname{Ind}_{I(k)}^{I} \sigma\right)^{\alpha}=\operatorname{Ind}_{\alpha[I(k)]}^{I} \sigma^{\alpha} .
$$

We have that

$$
\alpha[I(k)]=\left(\begin{array}{cc}
\mathcal{O} & \varpi^{k-1} \mathcal{O} \\
\varpi \mathcal{O} & \mathcal{O}
\end{array}\right) \cap S L(2)
$$

and

$$
\sigma^{\alpha}:\left(\begin{array}{cc}
a & \varpi^{k-1} \\
\varpi c & d
\end{array}\right) \mapsto \sigma(d)
$$

It follows from (9.1) that $\sigma(d)=\bar{\sigma}(a)$, and so we shall write $\bar{\sigma}$ for $\sigma^{\alpha}$, noting that this notation is consistent with (9.2) on the intersection of $I(k)$ and $\alpha[I(k)]$. Applying the version of Mackey's formula appropriate to representations induced from two different subgroups [18] we obtain

$$
\left(\phi_{\bar{\sigma}}, \phi_{\sigma}{ }^{\alpha}\right)_{I}=\left(\operatorname{Ind}_{I(k)}^{I} \bar{\sigma}, \operatorname{Ind}_{\alpha[I(k)]}^{I} \sigma^{\alpha}\right)_{I}=\sum_{x \in I(k) \backslash I / \alpha[I(k)]}\left(\bar{\sigma}^{x}, \bar{\sigma}\right)_{x^{-1} I(k) x \cap \alpha[I(k)]} .
$$

There is only one double coset here, so the inner product is 1 , as required.
Proposition 9.3. Let $c_{\sigma}=\phi_{\sigma}-\phi_{\bar{\sigma}}$. Then $c_{\sigma} \in H_{1}(G)$, and the cycles $c_{\sigma}$, one selected from each pair of characters $\{\sigma, \bar{\sigma}\}$, consititute a basis for $H_{1}(G)$.

Proof. Using Mackey's formula and Lemma 6.3 one calculates that

$$
\left(\operatorname{Ind}_{I}^{K_{0}} c_{\sigma}, \operatorname{Ind}_{I}^{K_{0}} c_{\sigma}\right)_{K_{0}}=\sum_{x \in I(k) \backslash K_{0} / I(k)}\left(\sigma^{x}-\bar{\sigma}^{x}, \sigma-\bar{\sigma}\right)_{I(k)_{x}}=0
$$

(compare the proof of part (1) of the Lemma 9.2), so that $\operatorname{Ind}_{I}^{K_{0}} c_{\sigma}=0$. By part (2) of Lemma 9.2, and functoriality of induction,

$$
\operatorname{Ind}_{I}^{K_{1}} c_{\sigma}=\left(\operatorname{Ind}_{I}^{K_{0}} c_{\sigma}^{\alpha}\right)^{\alpha}=\left(-\operatorname{Ind}_{I}^{K_{0}} c_{\sigma}\right)^{\alpha}=0
$$

Therefore $c_{\sigma} \in H_{1}(G)$.
From the formula for induction one calculates that

$$
c_{\sigma}:\left(\begin{array}{cc}
a & 0  \tag{9.3}\\
0 & a^{-1}
\end{array}\right) \mapsto \frac{\sigma(a)-\overline{\sigma(a)}}{\left|a-a^{-1}\right|}
$$

So it follows from Theorem 6.1 that the cycles $c_{\sigma}$ constitute a basis for the vector space $H_{1}(G ; \mathbb{C})$. To get the integral result we observe that the inner product of any cycle in $H_{1}(G)$ with any $c_{\sigma}$ must be an even integer. This follows from the fact that $\alpha$ acts as multiplication by -1 on $H_{1}(G)$. The proof is completed by noting that $\left(c_{\sigma}, c_{\sigma}\right)_{I}=2$ and $\left(c_{\sigma}, c_{\sigma^{\prime}}\right)_{I}=0$ if $c_{\sigma^{\prime}} \neq \pm c_{\sigma}$.

We note the resemblance between formula (9.3) for $c_{\sigma}$ and the formula

$$
\theta_{\pi(\tau)}:\left(\begin{array}{cc}
a & 0 \\
0 & a^{-1}
\end{array}\right) \mapsto \frac{\tau(a)+\tau\left(a^{-1}\right)}{\left|a-a^{-1}\right|}
$$

for the characters of the principal series representations $\pi(\tau)$ [5].
We conclude by relating the generators $c_{\sigma}$ of $H_{1}(G)$ to the natural generators of $\mathbf{K}_{1}\left(C_{r}^{*}(G)\right)$. Because the proper setting for the following proof is cyclic homology, which we shall consider elsewhere, we shall only outline an argument.

Proposition 9.4. Pimsner's map $\mu_{1}$ takes $c_{\sigma}$ to a $\mathbf{K}_{1}$-generator attached to the component of the principal series labelled by $\sigma$.

Proof (SKETCH). To calculate $\mu_{1}\left(c_{\sigma}\right)$, note that the idempotents $p$ and $q$ representing $c_{\sigma}$, as in Section 4, may be chosen to be

$$
p=\frac{1}{\operatorname{vol}(I(k))} \bar{\sigma} \quad \text { and } \quad q=\frac{1}{\operatorname{vol}(I(k))} \sigma
$$

(they are viewed first as functions on $I(k)$, then extended by zero to $I$ ). This follows from (4.4) and Lemma 4.3. Another application of Mackey's formula, this time in the context of locally compact groups [10], shows that both $p$ and $q$, viewed as idempotents in $C_{r}^{*}(G)$, give rank one projection valued functions in the component of $C_{\text {principal }}^{*}(G)$ labelled by $\sigma$, and zero elsewhere in $C_{p r i n c i p a l}^{*}(G)$. Thus $\mu_{1}\left(\phi_{\sigma}\right)$ must be a multiple of the generator of $\mathbf{K}_{1}\left(C_{r}^{*}(G)\right)$ labelled by $\sigma$. The fact that the multiple is $\pm 1$ follows from Pimsner's Theorem.

## References

1. P. Baum, N. Higson and A. Connes, Classifying space for proper actions and K-theory of crossed product $C^{*}$-algebras, Preprint (1992).
2. B. Blackadar, K-theory for operator algebras, MSRI Publication Series 5, Springer-Verlag, New York-Heidelberg-Berlin-Tokyo, 1986.
3. P. Blanc and J.L. Brylinski, Cyclic homology and the Selberg principle, Preprint (1991).
4. K. Brown, Buildings, Springer-Verlag, New York, 1988.
5. I.M. Gelfand, M.I. Graev and I.I. Pyatetskii-Shapiro, Representation theory and automorphic functions, Academic Press, New York, 1991.
6. Harish Chandra, Harmonic analysis on reductive p-adic groups, Springer Lecture Notes in Mathematics 162 (1970).
7. P. Julg and A. Valette, L'opérateur de cobord tordu sur un arbre et le principe de Selberg, II, J. Operator Theory 17 (1987), 347-355.
8. D. Kazhdan, Cuspidal geometry of p-adic groups, Journal d'Analyse Mathématique 47 (1986), 1-36.
9. C.W. Leung and R.J. Plymen, $L^{2}$-Fourier transform for reductive p-adic groups, Bull. London Math. Soc. 23 (1991), 146-152.
10. R. Lipsman, Group representations, Springer Lecture Notes in Mathematics 388 (1974).
11. J. Milnor, Introduction to algebraic K-theory, Ann. of Math. Studies, vol 72, Princeton University Press, New Jersey, 1971.
12. M. Pimsner, $K K$-groups of crossed products by groups acting on trees, Invent. Math. 86 (1986), 603-634.
13. R.J. Plymen, $K$-theory of the reduced $C^{*}$-algebra of $S L_{2}\left(\mathbb{Q}_{p}\right)$, Springer Lecture Notes in Mathematics 1132 (1985), 409-420.
14. L. Pontrjagin, Topological Groups, Gordon and Breach, New York, 1966.
15. P. Sally and J.A. Shalika, Characters of the discrete series representations of $S L(2)$ over a local field, Proc. Nat. Acad. Sci. USA 61 (1968), 1231-1237.
16. P. Sally and J.A. Shalika, The Plancherel formula for $S L(2)$ over a local field, Proc. Nat. Acad. Sci. USA 63 (1969), 661-667.
17. J.P. Serre, Trees, Springer-Verlag, Berlin, 1980.
18. J.P. Serre, Linear representations of finite groups, Springer-Verlag, Berlin, 1977.
19. J. A. Shalika, A theorem on semi-simple $\mathcal{P}$-adic groups, Ann. of Math. 95 (1972), 226-242.
20. A. Silberger, Introduction to harmonic analysis on reductive p-adic groups, Mathematical Notes, vol 23, Princeton University Press, New Jersey, 1979.
21. M.F. Vigneras, On formal dimensions for reductive p-adic groups, Israel Mathematical Conference Proceedings, Vol 2: Festschreft in honor of I.I. Piatetski-Shapiro, Weizmann Science Press, Jerusalem, 1990, pp. 225-266.
22. M.F. Vigneras, Principe de Selberg et intégrales orbitales généralisées pour les fonctions à décroissance rapide sur un groupe réductif p-adique, preprint (1992).
23. A. Weil, Basic number theory, Springer-Verlag, Berlin, 1974.

Department of Mathematics, Pennsylvania State University, University Park PA 16802.

E-mail address: baum@math.psu.edu and higson@math.psu.edu

Department of Mathematics, University of Manchester, Manchester M13 9PL.
E-mail address: mbbgsrp@cms.mcc.ac.uk


[^0]:    1991 Mathematics Subject Classification. Primary 46L20; Secondary 22E50.
    Paul Baum and Nigel Higson were partially supported by NSF.
    Nigel Higson is an Alfred P. Sloan Foundation Research Fellow.
    This paper is in final form and no version of it will be submitted for publication elsewhere.

