

Une Démonstration de la Conjecture de Baum-Connes pour le groupe p -adique $GL(n)$

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Résumé

Nous donnons une démonstration de la conjecture de Baum-Connes pour le groupe p -adique $GL(n)$.

A Proof of the Baum-Connes Conjecture for p -adic $GL(n)$

Abstract

We give a proof of the Baum-Connes conjecture for p -adic $GL(n)$.

Version Francaise Abrégée

Soit F un corps local non archimédien de caractéristique 0. Soit $GL(n, F)$ le groupe des matrices inversibles $n \times n$ à coefficients dans F ; puisque F est un corps localement compact, $G = GL(n, F)$ est un groupe topologique localement compact. Nous désignons par βG l'immeuble affine de G et par $C_r^*(G)$ la C^* -algèbre réduite de G . L'action de G sur βG est propre et en fait βG est un G -espace propre universel. La conjecture de Baum-Connes affirme que l'application indice $\mu : K_*^G(\beta G) \longrightarrow K_*(C_r^*(G))$ ($j = 0, 1$) est un isomorphisme des groupes abéliens, où $K_j^G(\beta G)$ désigne la K -homologie équivariante de βG , au sens de Kasparov, et $K_j(C_r^*(G))$ est la K -théorie de la C^* -algèbre $C_r^*(G)$. Dans cette note nous donnons un aperçu de notre démonstration de la conjecture lorsque $G = GL(n, F)$.

Soit K un sous-groupe compact ouvert de G , soit $\mathcal{H}(G, K)$ l'algèbre de convolution des fonctions sur G à support compact et K -bi-invariantes. L'algèbre de Hecke est $\mathcal{H}(G) = \cup_K \mathcal{H}(G, K)$. Afin d'éviter une analyse des limites inductives, nous définissons

$$h_*(\mathcal{H}(G)) = \varinjlim HP_*(\mathcal{H}(G, K))$$

(où $HP_*(\mathcal{H}(G, K))$ est l'homologie périodique cyclique de l'algèbre unitale $\mathcal{H}(G, K)$ [11]). De l'autre côté, soit $\mathcal{S}(G, K)$ l'algèbre de Fréchet des fonctions sur G à décroissance rapide et K -bi-invariantes. L'algèbre de Schwartz [8] est $\mathcal{S}(G) = \cup_K \mathcal{S}(G, K)$. Nous définissons

$$h_*(\mathcal{S}(G)) = \varinjlim HP_*(\mathcal{S}(G, K))$$

où $HP_*(\mathcal{S}(G, K))$ est l'homologie cyclique périodique pour l'algèbre de Fréchet $\mathcal{S}(G, K)$ [7].

THÉORÈME 1.— L'inclusion $i : \mathcal{H}(G) \longrightarrow \mathcal{S}(G)$ induit un isomorphisme

$$i_* : h_j(\mathcal{H}(G)) \longrightarrow h_j(\mathcal{S}(G))$$

($j = 0, 1$).

Nous indiquons une démonstration du Théorème 1 dans le paragraphe 4 qui rappuie fortement sur des résultats nouveaux en théorie des représentations concernant la structure d'algèbre $\mathcal{H}(G)$ [4] [5] [6].

Pour déduire du Théorème 1 une démonstration de la conjecture de Baum-Connes nous construisons un diagramme commutatif dans lequel chaque flèche verticale est le caractère de Chern et devient un isomorphisme après tensorisation sur \mathbb{Z} de sa source avec \mathbb{C} .

$$\begin{array}{ccc} K_*^G(\beta G) & \xrightarrow{\mu} & K_*(C_r^*(G)) \\ ch \downarrow & & \downarrow ch \\ h_*(\mathcal{H}(G)) & \longrightarrow & h_*(\mathcal{S}(G)) \end{array}$$

Kasparov et Skandalis [10] ont démontré que $\mu : K_j^G(\beta G) \longrightarrow K_j C_r^*(G)$ est injective et que $Image(\mu)$ est un facteur direct de $K_j C_r^*(G)$. On déduit de [14] que $K_j(C_r^*(G))$ est un groupe abélien libre. Le Théorème 1 et la commutativité du diagramme implique que

$$1 \otimes \mu : \mathbb{C} \otimes_{\mathbb{Z}} K_j^G(\beta G) \longrightarrow \mathbb{C} \otimes_{\mathbb{Z}} K_j C_r^*(G)$$

est un isomorphisme. Nous en déduisons que μ est surjective. Nous avons donc démontré le résultat suivant :

Théorème Principal. Soit F un corps local non archimédien de caractéristique 0 et soit $GL(n, F)$ le groupe linéaire de F . Alors la conjecture de Baum-Connes est vraie pour $GL(n, F)$.

1. The Baum-Connes Conjecture

Let F be a non-Archimedean local field of characteristic 0. As usual $GL(n, F)$ is the group of all $n \times n$ invertible matrices with entries in F ; since F is a locally compact field, $G = GL(n, F)$ is a locally compact topological group. We denote by βG the affine building of G , and by $C_r^*(G)$ the reduced C^* -algebra of G . The action of G on βG is proper, and in fact βG is a universal proper G -space. The Baum-Connes conjecture [2] asserts that the index map $\mu: K_*^G(\beta G) \rightarrow K_*(C_r^*(G))$ ($j = 0, 1$) is an isomorphism of abelian groups, where $K_j^G(\beta G)$ is the equivariant K -homology of βG , in the sense of Kasparov, and $K_j(C_r^*(G))$ is the K -theory of the C^* -algebra $C_r^*(G)$. In this note we shall outline our proof that the Baum-Connes conjecture is valid for $G = GL(n, F)$.

We thank C. Bushnell, P. Kutzko and P. Schneider for helpful and enlightening conversations. An alternative proof of Theorem 1, based on an idea of D. Kazhdan, is being developed by V. Nistor and P. Schneider.

2. Periodic Cyclic Homology

Let K be a compact open subgroup of G and let $\mathcal{H}(G, K)$ be the convolution algebra of compactly supported, K -bi-invariant functions on G . The Hecke algebra is $\mathcal{H}(G) = \cup_K \mathcal{H}(G, K)$. To avoid an analysis of direct limits, we define

$$h_*(\mathcal{H}(G)) = \varinjlim HP_*(\mathcal{H}(G, K))$$

(on the right-hand-side is the standard periodic cyclic homology of the unital algebras $\mathcal{H}(G, K)$ [11]). Similarly let $\mathcal{S}(G, K)$ be the Fréchet algebra of rapidly decreasing K -bi-invariant functions on G . The Schwartz algebra [8] is $\mathcal{S}(G) = \cup_K \mathcal{S}(G, K)$. We define

$$h_*(\mathcal{S}(G)) = \varinjlim HP_*(\mathcal{S}(G, K))$$

where on the right-hand-side is periodic cyclic homology for Fréchet algebras [7].

We shall reduce the proof of the Baum-Connes conjecture for G to the following result.

THEOREM 1.— The inclusion $i: \mathcal{H}(G) \longrightarrow \mathcal{S}(G)$ yields an isomorphism

$$i_*: h_j(\mathcal{H}(G)) \longrightarrow h_j(\mathcal{S}(G))$$

with $j = 0, 1$.

We shall outline a proof of theorem 1 in Section 4, which relies heavily on recent advances in representation theory concerning the structure of the algebra $\mathcal{H}(G)$ [4] [5] [6].

3. Chern character

To show that theorem 1 implies validity of the Baum-Connes conjecture of G we construct a commutative diagram

$$\begin{array}{ccc} K_*^G(\beta G) & \xrightarrow{\mu} & K_*(C_r^*(G)) \\ ch \downarrow & & \downarrow ch \\ h_*(\mathcal{H}(G)) & \longrightarrow & h_*(\mathcal{S}(G)) \end{array}$$

in which each vertical arrow is the relevant Chern character and becomes an isomorphism upon tensoring (over \mathbb{Z}) its source with \mathbb{C} .

The Chern character $ch : K_j^G(\beta G) \longrightarrow h_j(\mathcal{H}(G))$ is constructed as follows. It is convenient to replace βG with the affine building for $SL(N+1, F)$, which is a simplicial complex on which G acts in a type-preserving manner (we shall continue to use the notation βG , since the change has no effect on K -homology). Kasparov and Skandalis [10] associate a G - C^* -algebra $A_{\beta G}$ to βG , which is dual to βG in the sense that

$$K_j^G(\beta G) \cong K_j(A_{\beta G} \rtimes G)$$

($j = 0, 1$). The C^* -algebra $A_{\beta G}$ is a subalgebra of $C_0(\mathbb{R}^{2N}, \mathfrak{K}(\ell^2(\beta G)))$, where $\ell^2(\beta G)$ denotes the ℓ^2 -space on the set of all simplices in βG . It has a natural dense and holomorphically closed subalgebra $\mathcal{A}_{\beta G}$, comprised of the smooth and compactly supported functions with values in finite matrices. The algebraic crossed product $\mathcal{A}_{\beta G} \rtimes G$ is dense and holomorphically closed in $A_{\beta G} \rtimes G$. Following Moryoshi's approach [13] there is a Chern character

$$K_j(\mathcal{A}_{\beta G} \rtimes G) \longrightarrow \oplus_l H_{j+2l}^G(\beta G; \mathbb{C})$$

($j = 0, 1$), where $H_*^G(\beta G; \mathbb{C})$ is the equivariant homology of βG , in the sense of [2]. A Mayer-Vietoris argument shows that the Chern character is an isomorphism after tensoring with \mathbb{C} . But it follows from [9] and [16] that

$$\oplus H_{j+2l}^G(\beta G; \mathbb{C}) \cong h_j(\mathcal{H}(G))$$

from which we obtain the required Chern character isomorphism on the left hand side of our diagram.

The Chern character $ch : K_j(C_r^*(G)) \longrightarrow h_j(\mathcal{S}(G))$ is constructed using the method of Connes [7]. Let $C_r^*(G, K)$ be the closure of $\mathcal{H}(G, K)$ in $C_r^*(G)$. Since $\mathcal{S}(G, K)$ is a dense and holomorphically closed subalgebra of $C_r^*(G, K)$, there is a Chern character

$$ch : K_j(C_r^*(G, K)) \cong K_j(\mathcal{S}(G, K)) \longrightarrow HP_j(\mathcal{S}(G, K)).$$

The proof that this map becomes an isomorphism upon tensoring with \mathbb{C} is an elaboration of Theorem 46 of Connes [7], and relies on the detailed description of the algebras $\mathcal{S}(G, K)$ given in [12]. Using the notation of [12], the Fourier transform maps $\mathcal{S}(G, K)$ isomorphically onto the Fréchet algebra of smooth, Weyl-group-invariant, matrix-valued functions on the compact manifolds $\cup E_2(M : K \cap M)$ (the Weyl groups act on the matrices, via normalized intertwining operators, but with trivial R -groups). The isomorphism now follows, using [17].

4. The Bernstein Decomposition

Key to our proof of theorem 1 is the *Bernstein decomposition* of $\mathcal{H}(G)$. As in Bushnell-Kutzko [6] let $\mathcal{B}(G)$ be the *Bernstein spectrum* of G . By definition $\mathcal{B}(G)$ is the set of equivalence classes of pairs (M, σ) , where M is a Levi subgroup and σ is an irreducible supercuspidal representation of M . Two such pairs (M_i, σ_i) , $i = 1, 2$, are equivalent if $M_2 = g^{-1}M_1g$ for some $g \in G$, and the conjugated representation σ_1^g is equivalent to $\sigma_2 \otimes \chi$ for some unramified quasicharacter χ of M_2 . We have the *Bernstein decomposition* [3]

$$\mathcal{H}(G) = \bigoplus \mathcal{H}(G)^s$$

of the Hecke algebra into a direct sum, over $s \in \mathcal{B}(G)$, of two-sided ideals.

Choose one Levi subgroup M in each G -conjugacy class, and let $(M, \sigma) \in s$. Let $\Psi(M)$ be the group of unramified quasicharacters of M , and let $\mathcal{O}(s)$ be the orbit of σ , so that $\mathcal{O}(s) = \{\chi \otimes \sigma : \chi \in \Psi(M)\}$. The orbit $\mathcal{O}(s)$ has the structure of a complex torus $(\mathbb{C}^\times)^{d(s)}$. Let $W(s) = \{w \in W(M) : w\sigma \in \mathcal{O}(s)\}$.

As in [5](p.4), we can think of s as a vector (τ_1, \dots, τ_r) of irreducible supercuspidal representations of smaller general linear groups, the entries of this vector being only determined up to tensoring with unramified quasicharacters and permutation. If the vector is equivalent to $(\sigma_1, \dots, \sigma_1, \dots, \sigma_r, \dots, \sigma_r)$ with σ_j repeated e_j times, $1 \leq j \leq r$, and $\sigma_1, \dots, \sigma_r$ are pairwise distinct, then we say that s has *exponents* e_1, \dots, e_r . Note that $e_1 + \dots + e_r = d(s) = \dim_{\mathbb{C}} \mathcal{O}(s)$.

If s has exponents e_1, \dots, e_r then $W(s)$ is a product of symmetric groups:

$$W(s) = S_{e_1} \times \dots \times S_{e_r}$$

Form the semidirect product $\mathbb{Z}^{d(s)} \rtimes W(s)$.

THEOREM 2.— Let $s \in \mathcal{B}(G)$. Then we have

$$h_*(\mathcal{H}(G)^s) \cong HP_*(\mathbb{C}[\mathbb{Z}^{d(s)} \rtimes W(s)])$$

where we define $h_*(\mathcal{H}(G)^s) = \varinjlim HP_*(\mathcal{H}(G, K) \cap \mathcal{H}(G)^s)$.

PROOF.— Let $\mathcal{H}(e, q)$ denote the affine Hecke algebra associated to the affine Weyl group $\mathbb{Z}^e \rtimes S_e$. For K small enough we have compatible Morita equivalences

$$\mathcal{H}(G, K) \cap \mathcal{H}(G)^s \sim \mathcal{H}(e_1, q_1) \otimes \dots \otimes \mathcal{H}(e_r, q_r)$$

where the natural numbers q_1, \dots, q_r are specified by [5] (1.4) and [4](5.6.6). This result is due to Bushnell and Kutzko [4], [5], [6], especially [4](4.2.4), [4](5.6.6), [5](1.4), [6](4.3)(2.14). Using the Künneth formula the calculation of cyclic homology is reduced to that of the Iwahori Hecke algebra $\mathcal{H}(e, q)$, for which a separate argument establishes the result (note that when $q = 1$ the Hecke algebra $\mathcal{H}(e, q)$ specializes to the complex group algebra of the affine Weyl group $\mathbb{Z}^e \rtimes S_e$). \square

Let S_n be the n th symmetric group, and let S_n act on $S^1 \times S^1 \times \dots \times S^1$ (n factors) by permuting the entries of each n -tuple. Let $\mathbb{T}^n = S^1 \times S^1 \times \dots \times S^1$ and, following [1], set $\widehat{\mathbb{T}^n} = \{(\phi, p) \in S_n \times \mathbb{T}^n \mid \phi p = p\}$. Now S_n acts on $\widehat{\mathbb{T}^n}$ by

$$g(\phi, p) = (g\phi g^{-1}, gp)$$

with $g, \phi \in S_n, p \in \mathbb{T}^n$. Then $\widehat{\mathbb{T}^n}/S_n$ is the quotient space.

THEOREM 3.— Let $s \in \mathcal{B}(G)$. Then we have

$$HP_*(\mathbb{C}[\mathbb{Z}^{d(s)} \rtimes W(s)]) \cong H_*(\widehat{\mathbb{T}^{d(s)}}/W(s); \mathbb{C})$$

PROOF.— This is a standard calculation in cyclic homology. \square

The Schwartz algebra $\mathcal{S}(G)$ also decomposes as a direct sum (i.e. a purely algebraic direct sum) of two-sided ideals

$$\mathcal{S}(G) = \bigoplus \mathcal{S}(G)^s.$$

THEOREM 4.— For each $s \in \mathcal{B}(G)$ we have

$$h_*(\mathcal{S}(G)^s) \cong H^*(\widehat{\mathbb{T}^{d(s)}}/W(s); \mathbb{C})$$

PROOF.— Using the notation of [12] again, $\mathcal{S}(G, K) \cap \mathcal{S}(G)^s$ is isomorphic to the algebra of smooth, Weyl-group-invariant, matrix-valued functions on

the components of $\cup_M E_2(M : K \cap M)$ associated to s . The theorem now follows from Bernstein's description of the discrete series of $GL(n)$ in terms of segments [18], the Harish-Chandra Plancherel formula [8], and an argument modelled on [15]. \square

It follows from Theorems 2–4 that, for each $s \in \mathcal{B}(G)$, we have

$$\dim_{\mathbb{C}} h_j(\mathcal{H}(G)^s) = \dim_{\mathbb{C}} h_j(\mathcal{S}(G)^s)$$

But Kasparov & Skandalis [10] prove that $\mu : K_j^G(\beta G) \longrightarrow K_j(C_r^*(G))$ is injective. Therefore $i_* : h_j(\mathcal{H}(G)) \longrightarrow h_j(\mathcal{S}(G))$ is injective, so Theorem 1 is proved.

5. Concluding Argument

Kasparov & Skandalis in fact prove that $\mu : K_j^G(\beta(G)) \longrightarrow K_j C_r^*(G)$ is injective and that $Image(\mu)$ is a direct summand of $K_j C_r^*(G)$. From [14] it follows that $K_j C_r^*(G)$ is a free abelian group. Theorem 1 plus the commutative diagram imply that

$$1 \otimes \mu : \mathbb{C} \otimes_{\mathbb{Z}} K_j^G(\beta(G)) \longrightarrow \mathbb{C} \otimes_{\mathbb{Z}} K_j C_r^*(G)$$

is an isomorphism. If μ were not surjective this would be impossible. Hence we have proved the following result.

MAIN THEOREM.— Let F be a local non-Archimedean field of characteristic 0 and let $GL(n, F)$ be the general linear group over F . Then the Baum-Connes conjecture is true for $GL(n, F)$.

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