REPRESENTATION THEORY OF P-ADIC GROUPS: A VIEW FROM OPERATOR ALGEBRAS

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In memory of Harish-Chandra

1. INTRODUCTION

Over the past several years, operator algebraists have become increasingly interested in the problem of calculating the K-theory of group C^*-algebras. The focal point of research in this area is the Baum-Connes Conjecture [BCH], which proposes a description of K-theory for the C^*-algebra of a group in terms of homology and the representation theory of compact subgroups. Although the main applications of the Baum-Connes Conjecture are to issues in geometry and topology, the conjecture also appears to be of interest from the point of view of harmonic analysis. Whereas for applications to topology one is concerned with discrete groups G (arising as the fundamental groups of manifolds), the conjecture's links with harmonic analysis appear to be the strongest for reductive Lie groups and p-adic groups.

The purpose of these notes is to convey to a reasonably broad audience some byproducts of the authors' research into the C^*-algebra K-theory of the p-adic group $GL(N)$, which culminated in a proof of the Baum-Connes Conjecture in this case [BHP2]. Along the way to the proof a number of interesting issues came to light which we feel deserve some exposure, even though our understanding of them is far from complete, and is indeed mostly very tentative.

Much of what follows is focused on what we call here *chamber homology*, which is a type of equivariant homology associated to the action of a reductive p-adic group on its Bruhat-Tits affine building. The problem of computing chamber homology can be approached from a number of different directions. An especially interesting problem is to reconcile chamber homology with the *Bernstein decomposition* for representations of reductive p-adic groups [Be, BD]. This appears to be a far from trivial matter, even in comparatively simple cases. In Section 5 we formulate two very general conjectures which give a broad description of a Bernstein decomposition in chamber homology (perhaps we should call our conjectures questions, since the evidence we have gathered in their favor is not overwhelming).

Since chamber homology is about the representations of compact open subgroups, it is obviously quite likely that a proper account of the correspondence between the Bernstein decomposition and chamber homology for the group $GL(N)$ will revolve around the Bushnell-Kutzko theory of types [BK4]. In Sections 6 and 7 we outline some calculations to support this expectation, along with some conjectural descriptions of chamber homology, at the level of cycles, deriving from the Bushnell-Kutzko theory. One might optimistically view a description of chamber homology along these lines as a primitive foundation for a theory of types.

Our proof of the Baum-Connes conjecture for $GL(N)$ invoked a good deal of powerful and detailed representation theory (both smooth and tempered), some of it very new (and all of it due to the experts in representation theory, not ourselves). Sections 7 and 8 develop most of the relevant ideas, while at the same time adding some perspective, we hope, to the conjectures in earlier sections. A central tool throughout is cyclic homology, and at the end of Section 8 we formulate purely in cyclic homology a conjecture parallel to the Baum-Connes conjecture in K-theory. The Baum-Connes conjecture itself is formulated in Section 9.

Our own heavy dependence on the machinery of representation theory makes all the more remarkable the recent work of V. Lafforgue [La1,La2], who has proved the Baum-Connes conjecture not only for $GL(N)$ but for *all* reductive p-adic groups (and for a lot more besides, in the realms of discrete groups and Lie groups). Lafforgue's argument is developed around the geometric structure of the affine building and foundational properties of Harish Chandra's Schwartz space (essentially, that convolution by a Schwartz class function is a bounded operator on L^2; see [La3]). At the present time it is not clear how to translate Lafforgue's work into assertions and theorems in the language of representation theory. This is obviously an interesting topic for future research.

Our purpose in writing the present notes has been to outline to representation theorists some problems suggested by operator algebra theory which we feel might be of some interest within representation theory. At the same time, we have tried to make the paper a reasonable point of entry for operator algebraists who may want to venture into the beautiful and fascinating world of p-adic representation theory. These dual goals have resulted in a rather long paper, and even at this length we have included essentially no detailed arguments. Lafforgue's work suggests at least the possibility that in the future the interaction between operator algebra theory and representation theory will be considerably strengthened — we certainly hope so! If the present paper contributes even in a modest way to the development of such an interaction we shall be very pleased.

2. Chamber Homology

Let X be a finite-dimensional simplicial, or polysimplicial, complex and suppose that a discrete or totally disconnected group G acts simplicially on X, in such a way that the stabilizer of each vertex in X is a compact and open subgroup of G (thus X is a *proper* G-simplicial, or G-polysimplicial, complex). We shall also suppose

that if an element g of G maps a simplex σ into itself then g fixes all the vertices of σ. The main example we have in mind is the action of a reductive p-adic group on its affine Bruhat-Tits building [Ti].

We are going to define equivariant homology groups $H_*(G;X)$ in such a way that:

- If G is discrete, and if the action of G on X is free, then $H_*(G;X)$ identifies with the ordinary homology $H_*(X/G)$ of the quotient space X/G, with complex coefficients. In particular, if G is trivial then $H_*(G;X)$ is the ordinary homology of X with complex coefficients.
- At the other extreme, if X is a one-point space and if G is a profinite group then $H_0(G;X)$ is the space of locally constant class functions on G, while $H_p(G;X) = 0$, for $p > 0$. Note that by character theory, the space of locally constant class functions on G identifies with $R(G) \otimes_{\mathbb{Z}} \mathbb{C}$, the tensor product of the representation ring of the compact group G with $\mathbb{C}$.

This type of equivariant homology was treated in [BCH]. See also [HN] for another account of its construction, on which the discussion below is based. For another, more general, approach see [Sc2].

For simplicity we shall assume that X is a simplicial, as opposed to polysimplicial, complex and that X is *oriented,* which means the vertices of each simplex are linearly ordered (two orderings are regarded as the same if they differ by an even permutation). The ordering need not be done with any regard to the inclusion relations among simplices. But for simplicity again, we shall assume that G acts in an orientation-preserving manner on X.

If σ is a simplex in X then denote by G_σ its isotropy group in G. It is a compact and open subgroup of G. Denote by $\mathcal{H}(G_\sigma)$ the vector space of locally constant, complex-valued functions on G_σ and form the vector space

$$C_p(G;X) = \bigoplus_{\sigma \in X^p} \mathcal{H}(G_\sigma),$$

where the direct sum is over the set X^p of p-simplices in X. We shall write elements of $C_p(G;X)$ as finite formal sums

$$\sum_{\sigma \in X^p} \varphi_\sigma [\sigma],$$

where $\varphi_\sigma \in \mathcal{H}(G_\sigma)$. Note that if G is the trivial group then φ_σ is simply a complex number, and $C_p(G;X)$ is the space of simplicial p-chains in X (with complex coefficients).

If σ and η are simplices in X, and if $\eta \subset \sigma$, then G_σ is an open subgroup of G_η and every locally constant function on G_σ extends by zero to a locally constant function on G_η. Hence

$$\eta \subset \sigma \quad \Rightarrow \quad \mathcal{H}(G_\sigma) \subset \mathcal{H}(G_\eta).$$

If σ is a simplex in X with vertices $v_0, v_1, \ldots, v_p$ (written in order) and if η is a codimension-one face of σ with vertices $v_0, \ldots, \widehat{v_i}, \ldots, v_p$ then the *incidence number*

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$(-1)^{\langle\eta:\sigma\rangle}$ is $+1$ if this listing of the vertices of η agrees with the given orientation of η, and -1 otherwise. Now define linear maps

$$\partial\colon C_p(G;X) \to C_{p-1}(G;X)$$

by the formula

$$\partial(\varphi_\sigma[\sigma]) = \sum_{\substack{\eta \subset \sigma \\ \eta \in X^{p-1}}} (-1)^{\langle\eta:\sigma\rangle}\varphi_\sigma[\eta],$$

where the sum is over the codimension one faces of σ. The maps ∂ constitute the differentials in a chain complex:

$$C_0(G;X) \xleftarrow{\partial} C_1(G;X) \xleftarrow{\partial} C_2(G;X) \xleftarrow{\partial} \cdots.$$

The group G acts on this complex in the following way:

$$g \sum_{\sigma \in X^p} \varphi_\sigma[\sigma] = \sum_{\sigma \in X^p} g\varphi_\sigma[g\sigma],$$

where $g\varphi_\sigma(\gamma) = \varphi_\sigma(g^{-1}\gamma g)$ (we note that $g\varphi_\sigma \in \mathcal{H}(G_{g\sigma})$, as is required by the definitions, since $G_{g\sigma} = gG_\sigma g^{-1}$).

From each vector space $C_p(G;X)$ we now form the vector space of coinvariants $C_p(G;X)_G$, which is the quotient of $C_p(G;X)$ by the vector subspace spanned by all elements of the form $g(a) - a$, with $g \in G$ and $a \in C_p(G;X)$. Alternatively, if $\mathcal{H}(G)$ denotes the convolution algebra of locally constant and compactly supported functions on G, then $C_p(G;X)_G$ is the tensor product

$$C_p(G;X)_G = C_p(G;X) \underset{\mathcal{H}(G)}{\otimes} \mathbb{C}$$

(the algebra $\mathcal{H}(G)$ will be discussed further in the next section).

2.1. Definition. We define the *chamber homology of X*, denoted $H_*(G;X)$, to be the homology of the complex of coinvariants

$$C_0(G;X)_G \xleftarrow{\partial} C_1(G;X)_G \xleftarrow{\partial} C_2(G;X)_G \xleftarrow{\partial} \cdots.$$

If σ is a simplex in X then denote by $\mathcal{C}\ell(G_\sigma)$ the vector space of locally constant, complex-valued class functions on G_σ (a class function is a function which is constant on conjugacy classes). Form the subspace

$$C_p'(G;X) = \bigoplus_{\sigma \in X^p} \mathcal{C}\ell(G_\sigma) \subset \bigoplus_{\sigma \in X^p} \mathcal{H}(G_\sigma) = C_p(G;X).$$

It has the property that

$$C_p'(G;X)_G = C_p(G;X)_G,$$

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so the chamber homology groups $H_*(G; X)$ may in principle be computed from the spaces $C_p'(G; X)$ instead of the spaces $C_p(G; X)$. To actually do so requires a formula for the differentials ∂ which is adapted to the subspaces $C_p'(G; X)$ (note that extending a class function on G_σ to a larger group G_η by zero will not in general produce a class function on G_η, so our previous formula for ∂ is unsuitable). The correct definition of the differential in this context is

$$\partial'(\varphi_\sigma[\sigma]) = \sum_{\substack{\eta \subset \sigma \\ \eta \in X^{p-1}}} (-1)^{\langle \eta : \sigma \rangle} \operatorname{Ind}_{G_\sigma}^{G_\eta}(\varphi_\sigma)[\eta],$$

where $\operatorname{Ind}_{G_\sigma}^{G_\eta} : \mathcal{C}\ell(G_\sigma) \to \mathcal{C}\ell(G_\eta)$ is *induction*, defined by

$$\operatorname{Ind}_{G_\sigma}^{G_\eta}(\varphi)(g) = \frac{\operatorname{vol}(G_\eta)}{\operatorname{vol}(G_\sigma)} \int_{G_\sigma} \varphi(\gamma g \gamma^{-1}) \, d\gamma.$$

To explain the terminology, we note that the map which assigns to each finite-dimensional representation its character produces an isomorphism

$$R(G_\sigma) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong \mathcal{C}\ell(G_\sigma),$$

where $R(G_\sigma)$ is the ring of finite-dimensional complex linear representations of G_σ. Under the above isomorphism, the map $\operatorname{Ind}_{G_\sigma}^{G_\eta} : \mathcal{C}\ell(G_\sigma) \to \mathcal{C}\ell(G_\eta)$ corresponds to the usual induction operation on representations.

Let us turn now to some examples.

2.2. Example. If G is trivial then the complex which computes chamber homology is just the complex which computes the simplicial homology of X. More generally, if G acts freely on X then of course $\mathcal{C}\ell(G_\sigma) = \mathcal{H}(G_\sigma) = \mathbb{C}$, for all σ, and upon taking coinvariants we obtain a complex which computes the ordinary homology of X/G. Observe that in general (whether or not the action is free), the homology groups $H_p(G; X)$ vanish above the dimension of the simplicial complex X.

2.3. Example. Suppose that $\triangle \subset X$ is a subcomplex which is a fundamental domain for the action of G on X in the sense that the G-orbit of any simplex in X contains precisely one simplex from $\triangle$. Then the complex of coinvariants

$$C_0(G; X)_G \xleftarrow{\partial} C_1(G; X)_G \xleftarrow{\partial} C_2(G; X)_G \xleftarrow{\partial} \cdots$$

identifies with the complex

$$C_0'(G; \triangle) \xleftarrow{\partial'} C_1'(G; \triangle) \xleftarrow{\partial'} C_2'(G; \triangle) \xleftarrow{\partial'} \cdots,$$

where $C_p'(G; \triangle)$ denotes the direct sum

$$C_p'(G; \triangle) = \bigoplus_{\sigma \in \triangle^p} \mathcal{C}\ell(G_\sigma)$$

over the p-simplices in $\triangle$ (not X). The differential ∂' is defined exactly as above. Note that the bottom complex does not involve coinvariants—indeed G does not act on the spaces $C_p'(G; \triangle)$.

2.4. Example. The previous example applies to the action of the p-adic group $SL(N)$ on its affine Bruhat-Tits building X, for which we may take $\triangle$ to be any chamber of the building. The case of $SL(2)$ is particularly simple. Here X is a tree and $\triangle$ is any edge in the tree. The isotropy group of an edge is an Iwahori subgroup I of G, and the isotropy groups of the two vertices of the edge are the two maximal compact subgroups of G which contain I. The chamber homology $H_*(G; X)$ is therefore computed from the complex

$$\mathcal{Cl}(K_0) \oplus \mathcal{Cl}(K_1) \xleftarrow{\partial} \mathcal{Cl}(I),$$

where the differential is induction into $\mathcal{Cl}(K_0)$ and the negative of induction into $\mathcal{Cl}(K_1)$ (or the other way round, depending on the choices of orientation). The homology of even this very small complex is challenging to compute. For example a cycle for the group $H_1(G; X)$ consists of a class function on I which induces to zero on both K_0 and K_1. Thus if ρ and ρ' are distinct representations of I which induce to the same representation on K_0, as well as on K_1, then the difference of the characters of ρ and ρ' is a cycle for $H_1(G; X)$. In [BHP1] the authors construct a basis for $H_1(G; X)$ comprised of cycles of this type. Every representation of either K_0 or K_1 determines a cycle for $H_0(G; X)$; in [BHP1] the authors characterize the boundaries among the cycles in terms of invariant distribution theory on G.

2.5. Example. If $G = SL(3)$ then the complex in Example 2.3 is built around a 2-simplex $\triangle$. The isotropy groups of $\triangle$ and its faces form a so-called triangle of groups, with an Iwahori subgroup I attached to the triangle itself, the three distinct maximal compact subgroups K_i which contain I attached to the vertices, and intermediate subgroups $J_{ij} = K_i \cap K_j$ attached to the three sides of the triangle. See Figure 1. The structure of cycles for $H_0(G; X)$ and $H_2(G; X)$ is rather similar to the structure of the highest and lowest dimensional cycles in the $SL(2)$ case (we have not however obtained a complete analysis in the $SL(3)$ case). An investigation of cycles for $H_1(G; X)$ in the same spirit presents new and as yet unexplored challenges.
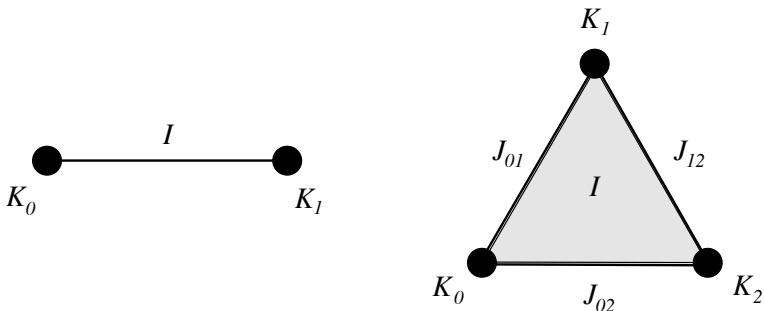

FIG. 1. Compact open subgroup data to compute $H_*(G; X)$ for the groups $G = SL(2)$ and $G = SL(3)$. Here X is the affine building of G.

2.6. Example. The affine building of the p-adic group $G = GL(N)$ has the structure of a polysimplicial, as opposed to simplicial, complex. But since it is simply the product $X \times \mathbb{R}$ of the building for $SL(N)$ and the real line, it is not difficult to construct a complex to compute its chamber homology along the lines of the previous example.[1] The action of $SL(N)$ on its affine building extends to an action of $GL(N)$, and the restriction of this action to the group $^\circ G = {}^\circ GL(N)$ of matrices T for which $\det(T)$ and $\det(T^{-1})$ are p-adic integers is a proper action, for which any chamber $\triangle$ in X is a fundamental domain. The chamber homology for this action of $^\circ G$ may be computed, or at least presented, exactly as we did for $SL(N)$ in the previous example. The group G itself is the semidirect product $^\circ G \rtimes \mathbb{Z}$ associated to the action of the matrix

$$
\Pi = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
. & . & . & & . \\
. & . & . & & . \\
. & . & . & & . \\
0 & 0 & 0 & \ldots & 1 \\
\pi & 0 & 0 & \ldots & 0
\end{pmatrix}
$$

on $^\circ GL(N)$ by conjugation. Here π is the prime generator of the maximal ideal within the ring of integers in our p-adic field. In line with this semidirect product description, the complex to compute chamber homology for $GL(N)$ may be written as a 'mapping torus' complex, namely it is the totalization of the complex

$$
\begin{array}{ccccccc}
C'_0(^\circ G; \triangle) & \xleftarrow{\partial'} & C'_1(^\circ G; \triangle) & \xleftarrow{\partial'} & C'_2(^\circ G; \triangle) & \xleftarrow{\partial'} & \ldots \\
\mathrm{Ad}_\Pi \downarrow & & \mathrm{Ad}_\Pi \downarrow & & \mathrm{Ad}_\Pi \downarrow & & \\
C'_0(^\circ G; \triangle) & \xleftarrow{\partial'} & C'_1(^\circ G; \triangle) & \xleftarrow{\partial'} & C'_2(^\circ G; \triangle) & \xleftarrow{\partial'} & \ldots
\end{array}
$$

Here $\triangle$ is chosen to be the chamber in X which is stabilized by the standard Iwahori subgroup of G (we shall say more about the Iwahori subgroup in Section 6). The chamber $\triangle$ is mapped into itself by the action of Π on the building X, so that there is therefore an induced action of Π on the complex $C'_*(^\circ G; \triangle)$. The reader might compare this description of chamber homology with the fact that the quotient $[X \times \mathbb{R}]/G$ identifies with the mapping torus of an automorphism of a chamber in X, acting by cyclically permuting the vertices of the chamber. The situation for $GL(3)$ is illustrated in Figure 2.

In both of the last two examples it should be emphasized that while setting up a complex to compute chamber homology may make it clearer how $H_*(G; X)$ combines the representation theory and combinatorics of the parahoric subgroups of G (that is, the isotropy subgroups of simplices in the building), having done so

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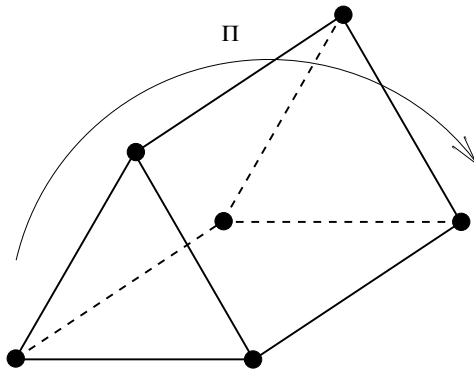[1]We could also barycentrically subdivide the building to obtain a simplicial complex, and then proceed from there.

FIG. 2. A prism in the affine building for $GL(3)$. The group element $\Pi \in GL(3)$ maps the front face $\triangle$ of the prism to the back, with a rotation of $2\pi/3$.

we have moved hardly any distance toward *computing* what the chamber homology groups actually are. Even in the 'simple' case of $SL(2)$ the chain groups $C'_*(G;X)$ involve the representation rings of quite nontrivial profinite groups, a direct computation of which is challenging, to say the least. In the following sections we shall proceed in a very indirect manner to obtain some insight into chamber homology.

We conclude this section with a remark concerning the functoriality of $H_*(G;X)$. The chamber homology of a proper G-simplicial (or polysimplicial) complex X is invariant under barycentric subdivision of X, and also under (simplicial) equivariant homotopy. It follows that $H_*(G;X)$ is an invariant of the equivariant homotopy type of X. Now if G is a reductive p-adic group (our main concern here) and if X is the affine Bruhat-Tits building of G then it is noted in [BCH] that X is a *universal* proper G-space: it has the property that every proper G-space Z has a unique-up-to-G-homotopy equivariant map into X. Since universality clearly characterizes X, up to G-homotopy, it follows that $H_*(G;X)$ is actually intrinsically associated to the topological group G (namely it is the chamber homology of the universal proper G-space).

We shall return to the notion of universal proper G-space in Section 9 of this paper.

3. Cyclic Homology of the Hecke Algebra

Let G be a totally disconnected group and denote by $\mathcal{H}(G)$ the convolution algebra of locally constant, compactly supported, complex-valued functions on G. This is the *Hecke algebra* of G. The construction of $\mathcal{H}(G)$ requires a choice of Haar measure on G, but perhaps it is worth pointing out that integration theory for locally constant functions on a totally disconnected group is purely algebraic in nature, so that no analysis is really involved in the construction of $\mathcal{H}(G)$.

The purpose of this section is to review a calculation which connects the Hecke algebra $\mathcal{H}(G)$ to chamber homology through cyclic homology [HN,Sc1]. We shall

begin by saying a few very brief words about cyclic homology. See [Lo] for a complete treatment of the subject.

If A is an associative algebra over the complex numbers $\mathbb{C}$ then the cyclic homology groups $HC_j(A)$, for $j = 0, 1, 2, \ldots$, are the homology groups of the totalization of the *cyclic bicomplex* shown in Figure 3. The differentials are described in [Lo]. The operator t is essentially a cyclic permutation of the multiple tensor product $A \otimes \cdots \otimes A$. In fact

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1},$$

and it is from this that the term 'cyclic homology' derives.

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\Big\downarrow b & & \Big\downarrow b' & & \Big\downarrow b & & \\
A \otimes A \otimes A & \xleftarrow{1-t} & A \otimes A \otimes A & \xleftarrow{N} & A \otimes A \otimes A & \xleftarrow{1-t} & \cdots \\
\Big\downarrow b & & \Big\downarrow b' & & \Big\downarrow b & & \\
A \otimes A & \xleftarrow{1-t} & A \otimes A & \xleftarrow{N} & A \otimes A & \xleftarrow{1-t} & \cdots \\
\Big\downarrow b & & \Big\downarrow b' & & \Big\downarrow b & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & \cdots
\end{array}
$$

FIG. 3. The cyclic bicomplex.

The cyclic bicomplex has a built-in 2-periodic structure: if we delete the first two columns then what is left is simply a copy of the entire bicomplex. Associated to this there are natural maps

$$S \colon HC_n(A) \longrightarrow HC_{n-2}(A).$$

There is version of cyclic homology which emphasizes this periodicity. To a first degree of approximation this *periodic cyclic homology* is the inverse limit

$$HP_*(A) = \varprojlim_{S} HC_*(A).$$

The formal definition of $HP_*(A)$ involves extending the cyclic complex infinitely to the left, preserving its 2-periodicity, and then totalizing by taking the direct *product* of chain groups with a given total degree. The periodic cyclic homology groups $HP_*(A)$ are then related to the inverse limit $\varprojlim HC_*(A)$ by a short-exact sequence

$$0 \longrightarrow \varprojlim{}^1 HC_{*-1}(A) \longrightarrow HP_*(A) \longrightarrow \varprojlim HC_*(A) \longrightarrow 0,$$

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of a type familiar in homological algebra whenever inverse limits arise. In favorable circumstances, such as those we shall be considering in this paper, the sequence simplifies: $\varprojlim^1 HC_*(A) = 0$ and $HP_*(A) \cong \varprojlim HC_*(A)$.

In practice the computation of cyclic homology is often approached through Hochschild homology theory, the Hochschild homology of an algebra being in this context the homology of the first two columns in the complex of Figure 3. Associated to the process of deletion of the first two columns from the cyclic complex is a long exact sequence,

$$\cdots \longrightarrow HH_{n+2}(A) \xrightarrow{I} HC_{n+2}(A) \xrightarrow{S} HC_n(A) \xrightarrow{B} HH_{n+1}(A) \xrightarrow{I} \cdots,$$

called the Connes-Tsygan exact sequence, which connects Hochschild homology with the periodization map in cyclic homology. The model result here is the following theorem of Hochschild, Kostant and Rosenberg [HKR][2] (see also [LQ]):

3.1. Theorem. *Let X be a nonsingular complex affine variety and denote by $\mathcal{O}(X)$ the algebra of regular functions on X. There are isomorphisms*

$$HH_n(\mathcal{O}(X)) \cong \Omega^n(X),$$

and

$$HP_{\mathrm{ev/odd}}(\mathcal{O}(X)) \cong H^{\mathrm{ev/odd}}(X),$$

where $\Omega^n(X)$ denotes the space of algebraic differential forms on X and $H^{\mathrm{ev/odd}}(X)$ denotes the periodized cohomology (see the remark below) of the de Rham complex of algebraic differential forms.

Remark. In the theorem we use the notation $H^{\mathrm{ev/odd}}$ for the direct sum of either the even or odd degree cohomology groups: the theorem asserts that $HP_n(\mathcal{O}(X))$ is the direct sum of the groups $H^{2p}(X)$ if n is even and the direct sum of the groups $H^{2p+1}(X)$ if n is odd. We will use similar notation later in the paper.

The idea of the proof of Theorem 3.1 is to use homological techniques (for example, projection resolutions) to identify Hochschild homology, which fits very naturally into the ordinary scheme of homological algebra. Then one identifies the de Rham differential using the maps I and B in the Connes-Tsygan exact sequence to prove the second part of the theorem.

We are now in a position to state the relevant result for $\mathcal{H}(G)$:

3.2. Theorem. [HN,Sc1] *Let G be a totally disconnected group acting properly on an affine Bruhat-Tits building X. There are isomorphisms*

$$HP_{\mathrm{ev/odd}}(\mathcal{H}(G)) \cong H_{\mathrm{ev/odd}}(G; X).$$

In other words, the chamber homology of the G-space X identifies, after periodization, with the periodic cyclic homology of the Hecke algebra $\mathcal{H}(G)$.

Remark. This theorem parallels a substantial collection of prior results for discrete groups, begun in [Bur] and summarized in [Lo]. For instance if a discrete group Γ

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[2]They did not originally formulate it in the language of cyclic theory.

acts properly on a space such as an affine building then it follows from these earlier results that the periodic cyclic homology of the group algebra $\mathbb{C}[\Gamma]$ identifies with $H_{\mathrm{ev/odd}}(G; X)$.

Here is a sketch of the proof of Theorem 3.2. We begin with a simple geometrical fact: the set G_{c} of elements in G each of which fixes some point of X is open in G (it is the union of all the compact open subgroups of G), as is its complement G_{nc}, comprised of elements which fix no point of X. Next, associated to a partition of G, such as

$$G = G_{\mathrm{c}} \cup G_{\mathrm{nc}},$$

into open, conjugation invariant subsets, there is a corresponding direct sum decomposition

$$HH_n(\mathcal{H}(G)) = HH_n(\mathcal{H}(G))_{\mathrm{c}} \oplus HH_n(\mathcal{H}(G))_{\mathrm{nc}},$$

and a similar decomposition in cyclic theory. The two components in the decomposition can now be treated separately.

First, one can show that the periodicity map

$$HC_n(\mathcal{H}(G))_{\mathrm{nc}} \xrightarrow{S} HC_{n-2}(\mathcal{H}(G))_{\mathrm{nc}}$$

is zero, from which it follows that $HP_*(\mathcal{H}(G))_{\mathrm{nc}} = 0$.

Next, the G_{c}-component of cyclic homology is computed through a sequence of isomorphisms

$$H_*(G; X) \cong H_*(G, \mathcal{H}(G_{\mathrm{c}})) \cong HH_*(\mathcal{H}(G))_{\mathrm{c}},$$

in which $H_*(G, \mathcal{H}(G)_{\mathrm{c}})$ denotes the smooth homology of G with coefficients in the smooth G-module of compactly supported, locally constant functions on G_{c} (on which G acts by conjugation). The G_{c}-component of the Connes-Tsygan sequence degenerates, producing isomorphisms

$$HC_n(\mathcal{H}(G))_{\mathrm{c}} \cong H_n(G; X) \oplus H_{n-2}(G; X) \oplus H_{n-4}(G; X) \oplus \cdots$$

(the direct sum ends with either $H_0(G; X)$ or $H_1(G; X)$, depending on the parity of n). This gives the proof of Theorem 3.2, and in fact proves a little more:

3.3. Theorem. *Let G be a totally disconnected group acting properly on an affine building X. The Connes-Tsygan exact sequence produces an isomorphism of the chamber homology group $H_n(G; X)$ with the intersection of the spaces*

$$\mathrm{Kernel}[S\colon HC_n(\mathcal{H}(G)) \to HC_{n-2}(\mathcal{H}(G))]$$

and

$$\mathrm{Image}[S\colon HC_{n+2}(\mathcal{H}(G)) \to HC_n(\mathcal{H}(G))]$$

inside the cyclic homology group $HC_n(\mathcal{H}(G))$. $\square$

We remark that a more detailed account of Theorem 3.1 reveals that a similar assertion can be made, recovering the de Rham cohomology of the complex of algebraic forms on a variety X from the cyclic homology of $\mathcal{O}(X)$.

We conclude this section with a few comments concerning *topological* algebras. For these it is appropriate to place the fundamental bicomplex of Figure 3 into the context of topological vector space theory [Bou] and form *completed* tensor products $A \otimes \cdots \otimes A$. The theory of topological tensor products is quite elaborate, and in order to chart a reasonably straight course through it we shall make the assumption that the topological vector space underlying the algebra A is *nuclear* and *Fréchet*. In this situation every separately continuous bilinear map on A is jointly continuous (so that there is only one natural notion of continuous multiplication operation on A) and furthermore there is an essentially unique topology on the n-fold tensor product of A with itself. See [Gr]. Completing with respect to this topology we obtain the *the topological tensor product*, and using this tensor product in Figure 3 we define cyclic and periodic cyclic theory for the nuclear Fréchet algebra A.

The central result concerning cyclic homology for topological algebras is the following theorem of Connes [Co1]:

3.4. Theorem. *Let M be a smooth closed manifold and denote by $C^\infty(M)$ the algebra of smooth complex-valued functions on M (with the topology of uniform convergence of derivatives of all orders). The periodic cyclic homology of $C^\infty(M)$ identifies with the periodized ordinary cohomology of the manifold M (with complex coefficients):*

$$HP_{\mathrm{ev/odd}}(C^\infty(M)) \cong H^{\mathrm{ev/odd}}(M). \qquad \square$$

As with Theorem 3.1, a more thorough account of the matter reveals that the individual cohomology groups $H^n(M)$ may be recovered from the cyclic homology of $C^\infty(M)$ via the formula in Theorem 3.3. We will return in Section 8 to the obvious analogy between Theorems 3.1 and 3.4.

4. The Bernstein Decomposition

The purpose of this section is to review a very beautiful construction of Bernstein [Be,BD], which associates to a reductive p-adic group G a commutative, associative algebra $\mathfrak{Z}(G)$ analogous to both the center of the universal enveloping algebra of a real reductive group and the center of the group algebra of a finite group.

A linear representation of a totally disconnected group G on a complex vector space V is *smooth* if the isotropy group of each vector in V is an open subgroup of G. By integration of the representation, we see that a smooth representation is the same thing as a $\mathcal{H}(G)$-module V for which $\mathcal{H}(G) \cdot V = V$. The *smooth dual* of G is the set $\mathrm{Irr}(G)$ of linear equivalence classes of irreducible, smooth representations of G.

Now let G be a reductive p-adic group. We aim to make somewhat more precise the following result:

4.1. Decomposition Theorem (First version). *The smooth dual $\mathrm{Irr}(G)$ maps finite-to-one onto a Hausdorff topological space $\Omega(G)$ (which is a disjoint union of affine algebraic varieties) and there is a decomposition*

$$\mathcal{H}(G) = \bigoplus_\Omega \mathcal{H}(G)_\Omega,$$

parametrized by the connected components of the space $\Omega(G)$. On the right hand side above is the algebraic direct sum of a countable family of pairwise orthogonal ideals $\mathcal{H}(G)_\Omega$ of the algebra $\mathcal{H}(G)$.

This is the *Bernstein decomposition* of the Hecke algebra $\mathcal{H}(G)$. The details of the Decomposition Theorem are not all essential for what follows and so some readers may wish to skip from here to the next section. The main point is that the decomposition is determined by the representation theory of G, as understood in terms of induction from parabolic subgroups (in the case of $GL(N)$ this means induction from block upper triangular subgroups). In the next section we will investigate how the Bernstein decomposition is related to the representation theory of *compact* subgroups, and in particular how it is related to chamber homology.

Here then is a brief account of the Decomposition Theorem, borrowing heavily from [BD]. Let $\mathcal{M}(G)$ be the category of smooth representations of G. We fix a minimal parabolic subgroup $P_0 \subset G$ and its Levi decomposition $P_0 = M_0 \cdot U_0$. By a *standard Levi subgroup of G* we mean a subgroup M which contains M_0 and which is a Levi component of the parabolic subgroup $P = MP_0$ (notation: $M < G$). For any standard Levi subgroup $M < G$ we have the induction-type functor $i_{GM} : \mathcal{M}(M) \to \mathcal{M}(G)$, obtained by extending a representation from M to P, then inducing to G.

Let $\Psi(G) \subset \{\psi : G \to \mathbb{C}^\times\}$ be the group of unramified characters of G. It acts naturally on $\mathrm{Irr}(G)$ by $\psi : \pi \mapsto \psi\pi$. The group $\Psi(G)$ has a natural structure of *complex algebraic group* and is isomorphic to a product of copies of $\mathbb{C}^\times$.

If $G = GL(N)$, and if P_0 is chosen to be the upper triangular matrices, then the Levi subgroups are the block-diagonal subgroups of G. The group $\Psi(G)$ is comprised of the characters $T \mapsto z^{\mathrm{val\,det}(T)}$, for $z \in \mathbb{C}^\times$, where val denotes the valuation on our p-adic field.

We refer the reader to say [Be] for the key notion of supercuspidal representation which appears in the definition below. The irreducible supercuspidal representations are in a sense the fundamental building blocks for the representation theory of a reductive group G, since an irreducible smooth representation of G is either supercuspidal or a subquotient of the representation obtained by parabolically inducing a supercuspidal representation of a Levi factor. The problem of *classifying* supercuspidal representations is thus fundamental to the representation theory of p-adic groups.

4.2. Definition. A *supercuspidal pair* for G is a pair (M, ρ) where $M < G$ is a standard Levi subgroup and $\rho \in \mathrm{Irr}\, M$ is an irreducible supercuspidal representation of M. We denote by $\Omega(G)$ the set of all supercuspidal pairs up to conjugation by G. This is the *Bernstein variety* of the group G. For any supercuspidal pair (M, ρ) we shall call the image of the map $\Psi(M) \to \Omega(G)$, given by[3] $\psi \mapsto (M, \psi\rho)$, a *Bernstein component* of $\Omega(G)$.

The Bernstein components $\Omega \subset \Omega(G)$ are complex affine algebraic varieties: the map $\Psi(M) \to \Omega(G)$ in the definition identifies Ω with the quotient of $\Psi(M)$ by the

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[3]One can show that if ρ is supercuspidal then so is $\psi\rho$; thus the map is well-defined.

action of a finite group. In the case where $M = G$ this finite group is simply the subgroup of $\Psi(G)$ comprised of characters ψ for which $\psi\rho$ is equivalent to ρ.

As we noted above, if $\pi \colon G \to \mathrm{Aut}(V)$ is a smooth representation then there exists a supercuspidal pair (M, ρ), such that π is a sub-quotient of $i_{GM}(\rho)$. This pair is uniquely defined up to conjugation in G and hence defines a point in $\Omega(G)$, which is called the *infinitesimal character* of the representation and denoted $\mathrm{inf.\,ch.}_V$. The map $\mathrm{inf.\,ch.} \colon \mathrm{Irr}(G) \to \Omega(G)$ is onto and finite-to-one.

4.3. Definition. The *central algebra* $\mathfrak{Z}(G)$ is the direct product $\mathfrak{Z}(G) = \Pi_\Omega \mathcal{O}(\Omega)$ of the algebras of regular functions on Bernstein components of $\Omega(G)$.

4.4. Theorem. [BD] *On each smooth G-module E there exists a natural action of $\mathfrak{Z}(G)$ such that*

  (i) *$z : E \to E$ is a morphism of smooth G-modules for each $z \in \mathfrak{Z}(G)$.*
  (ii) *Each G-module morphism $\alpha : E \to E'$ is a $\mathfrak{Z}(G)$-morphism.*
  (iii) *On each irreducible smooth G-module V the action of $z \in \mathfrak{Z}(G)$ is given by*

$$z = \mathrm{inf.\,ch.}_V(z) \qquad \square$$

Remark. The system of actions of $\mathfrak{Z}(G)$ on the class of all smooth G-modules is uniquely determined by properties (i)–(iii).

4.5. Decomposition Theorem (Second Version)**.** *Let E be a smooth G-module. Then for each Bernstein component $\Omega \subset \Omega(G)$ the characteristic function $1_\Omega \in \mathfrak{Z}(G)$ acts on E as a projector onto a G-submodule E_Ω, and $E = \bigoplus_\Omega E_\Omega$.* $\square$

Now consider $\mathcal{H}(G)$ as a smooth G-module with respect to the left action of G. Then the corresponding action of $\mathfrak{Z}(G)$ on $\mathcal{H}(G)$ identifies $\mathfrak{Z}(G)$ with the algebra of all endomorphisms of $\mathcal{H}(G)$, invariant with respect to the left and right actions of G (see [BD] Section 1). The above theorem implies that the Hecke algebra $\mathcal{H}(G)$ can be decomposed as a direct sum of two-sided ideals $\mathcal{H}(G)_\Omega$, as in our first version of the Decomposition Theorem.

5. The Bernstein Decomposition for Chamber Homology

Let us now combine some ideas from the previous three sections. We have seen that there is a very natural "chamber" homology theory associated to the action of a reductive p-adic group G on its affine building X, which encodes aspects of the representation theory of the compact open subgroups of G. At the same time the representation theory of G itself decomposes according to the components Ω of the Bernstein variety $\Omega(G)$. This involves not compact open subgroups but Levi subgroups and supercuspidal representations thereon. What is the relationship between the Bernstein decomposition and chamber homology?

We shall attempt to formulate an answer by approaching the question through cyclic homology.

5.1. Lemma. *Associated to the Bernstein decomposition*

$$\mathcal{H}(G) = \oplus_\Omega \mathcal{H}(G)_\Omega,$$

of the Hecke algebra of G there is a decomposition

$$HC_*(\mathcal{H}(G)) = \oplus_\Omega HC_*(\mathcal{H}(G)_\Omega)$$

in cyclic homology. □

This sort of additivity follows almost immediately from the definition of cyclic homology. In Section 3 we noted that the chamber homology of G may be placed into the context of cyclic homology. In fact Theorem 3.3 gives a precise formula for $H_n(G; X)$ in terms of cyclic theory, and using it we define

$$H_n(G; X)_\Omega = \text{Kernel}\,[S\colon HC_n(\mathcal{H}(G)_\Omega) \to HC_{n-2}(\mathcal{H}(G)_\Omega)]$$
$$\cap \text{Image}\,[S\colon HC_{n+2}(\mathcal{H}(G)_\Omega) \to HC_n(\mathcal{H}(G)_\Omega)]$$

Combining the above lemma with Theorems 3.2 and 3.3 we obtain a decomposition

$$H_*(G; X) = \oplus_\Omega H_*(G; X)_\Omega.$$

It is not a simple matter to trace through the argument summarized in Section 3 to produce a more explicit description of the 'Bernstein components' $H_*(G; X)_\Omega$ of chamber homology. In fact the best we can manage at the present time is a series of guesses about a more definite description of $H_*(G; X)_\Omega$, substantiated in part by one or two calculations.

We begin by describing the simplest means of attempting to construct a complex to compute $H_*(G; X)_\Omega$. Consider first the complex

$$C_0(G; X) \xleftarrow{\partial} C_1(G; X) \xleftarrow{\partial} C_2(G; X) \xleftarrow{\partial} \cdots$$

introduced in Section 2, in which

$$C_p(G; X) = \bigoplus_{\sigma \in X^p} \mathcal{H}(G_\sigma).$$

Recall that we defined chamber homology to be the homology of the complex of coinvariants associated to $C_*(G; X)$. Now any element of $\mathcal{H}(G_\sigma)$, that is, any locally constant function on the compact open subgroup $G_\sigma \subset G$, may be extended by zero to become a locally constant and compactly supported function on G. In this way we obtain an inclusion $\mathcal{H}(G_\sigma) \subset \mathcal{H}(G)$ and we may define

$$\mathcal{H}(G_\sigma)_\Omega = \mathcal{H}(G_\sigma) \cap \mathcal{H}(G)_\Omega.$$

5.2. Definition. Denote by $C_p(\Omega; X)$ the direct sum

$$C_p(\Omega; X) = \bigoplus_{\sigma \in X^p} \mathcal{H}(G_\sigma)_\Omega.$$

The spaces $C_p(\Omega; X)$ assemble to form a G-subcomplex

$$C_0(\Omega; X) \xleftarrow{\partial} C_1(\Omega; X) \xleftarrow{\partial} C_2(\Omega; X) \xleftarrow{\partial} \cdots$$

of the complex $C_*(G; X)$ introduced in Section 2; denote by

$$C_*(\Omega; X)_G = C_*(\Omega; X) \otimes_{\mathcal{H}(G)} \mathbb{C}$$

the associated complex of coinvariants, as in Section 2, and denote by $H_*(\Omega; X)$ the homology of the complex $C_*(\Omega; X)_G$.

The inclusion of $C_*(\Omega; X)$ into $C_*(G; X)$ induces a map from $H_*(\Omega; X)$ into $H_*(G; X)_\Omega$.

5.3. Conjecture. *The inclusions of the subcomplexes $C_*(\Omega; X)$ into $C_*(G; X)$ induce an isomorphism*

$$\oplus_\Omega H_*(\Omega; X) \cong H_*(G; X).$$

Despite the fact that $\mathcal{H}(G) = \oplus_\Omega \mathcal{H}(G)_\Omega$, it does not of course follow that $\mathcal{H}(G_\sigma) = \oplus_\Omega \mathcal{H}(G_\sigma)_\Omega$. Thus while the direct sum $\oplus_\Omega C_*(\Omega; X)$ injects into the complex $C_*(G; X)$, the map is far from surjective. The same is true after taking coinvariants, and so the relation between the direct sum of the homology groups $H_*(\Omega; X)$ and $H_*(G; X)$ is far from clear.

Let us continue with a second assertion:

5.4. Conjecture. *For every Bernstein component Ω the complex of coinvariants*

$$C_0(\Omega; X)_G \xleftarrow{\partial} C_1(\Omega; X)_G \xleftarrow{\partial} C_2(\Omega; X)_G \xleftarrow{\partial} \cdots$$

is finite-dimensional.

In order to better understand what this conjecture is about, let us consider the example of $G = SL(N)$, for which, as we noted in Section 2, the complex computing chamber homology has a more simple and concrete appearance. The same is true for $H_*(\Omega; X)$: if $\triangle$ is any chamber in the building for G then the complex of coinvariants $C_*(\Omega; X)_G$ identifies with the complex

$$\bigoplus_{\sigma \in \triangle^0} \mathcal{Cl}(G_\sigma)_\Omega \xleftarrow{\partial'} \bigoplus_{\sigma \in \triangle^1} \mathcal{Cl}(G_\sigma)_\Omega \xleftarrow{\partial'} \bigoplus_{\sigma \in \triangle^2} \mathcal{Cl}(G_\sigma)_\Omega \xleftarrow{\partial'} \cdots,$$

in which the direct sums are over the finitely many faces of $\triangle$ of dimension p, and $\mathcal{Cl}(G_\sigma)_\Omega$ denotes the space of those locally constant, complex-valued class functions on G_σ which lie in $\mathcal{H}(G)_\Omega$, when extended by zero to be functions on G. The differentials ∂' are given by induction, as in Section 2.

The vector space $\mathcal{C}\ell(G_\sigma)_\Omega$ has a natural basis, comprised of the characters of those irreducible representations V of G_σ which are associated only to the Bernstein component Ω, in the sense that if E is any smooth representation of G then the G_σ-representation V occurs only in the Bernstein component $E_\Omega \subseteq E$. Or to put it another way, $\mathcal{C}\ell(G_\sigma)_\Omega$ is generated by the characters of those irreducible representations V of G_σ for which[4] $[\text{Ind}_{G_\sigma}^{G} V]_\Omega = \text{Ind}_{G_\sigma}^{G} V$. In Conjecture 5.4 we are asserting that for every simplex σ in the affine building X this class of representations is finite-dimensional.

Taken together, Conjectures 5.3 and 5.4 propose a definite link between the Bernstein decomposition for G and the structure of representations of compact open subgroups, the bridging concept being the geometry of the action of G on its affine building. The conjectures assert that to each Bernstein component Ω of any reductive group G there is associated a finite-up-to-conjugacy collection of representations of compact open subgroups, which is organized into a complex and which carries that part of the chamber homology $H_*(G; X)$ associated to Ω. This has some resemblance to a theory of types, in the sense of Bushnell and Kutzko [BK4], a point to which we shall return in Section 7.

6. Calculations in Chamber Homology

In this section we shall outline a few calculations which we hope will add a little substance to the conjectures of the previous section (and along the way we will generate more questions). In the next section we will try to place our calculations within a more systematic framework.

We will concentrate on the p-adic group $G = GL(3)$. As before, let X be the affine building for G. Our aim is to construct certain classes in the homology groups $H_*(\Omega; X)$ which we defined in the previous section, and indeed to arrive at a reasonable understanding of the complex

$$C_0(\Omega; X)_G \xleftarrow{\partial} C_1(\Omega; X)_G \xleftarrow{\partial} C_2(\Omega; X)_G \xleftarrow{\partial} \dots,$$

which computes $H_*(\Omega; X)$.

As we have already noted, it is possible to write the above complex in a more concrete form. If Ω is a Bernstein component for $GL(3)$ then by the *local complex* for Ω we shall mean the complex

$$C_0'(^\circ G; \triangle)_\Omega \xleftarrow{\partial'} C_1'(^\circ G; \triangle)_\Omega \xleftarrow{\partial'} C_2'(^\circ G; \triangle)_\Omega,$$

where $\triangle$ is the chamber in the affine building for $SL(3)$ which is fixed by the standard Iwahori subgroup $I \subset GL(3)$:

$$I = \begin{bmatrix} \mathcal{O}^\times & \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O}^\times & \mathcal{O} \\ \pi\mathcal{O} & \pi\mathcal{O} & \mathcal{O}^\times \end{bmatrix}.$$

---

[4]The appropriate notion of induction here is smooth, compact induction, so that $\text{Ind}_{G_\sigma}^{G} V$ is comprised of G_σ-covariant, compactly supported and locally constant functions from G into V.

Here $C'_p(^\circ G; \triangle)_\Omega$ is the direct sum

$$C'_p(^\circ G; \triangle)_\Omega = \bigoplus_{\sigma \in \triangle^p} \mathcal{C}\ell(G_\sigma)_\Omega$$

introduced in the previous section, the sum being over the p-dimensional faces of the chamber $\triangle$. Conjecture 5.4 in the previous section asserts that this complex is finite-dimensional. Conjecture 5.3 asserts that if Π is the matrix

$$\Pi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \pi & 0 & 0 \end{bmatrix}$$

then the *local mapping torus complex*

$$
\begin{array}{ccccc}
C'_0(^\circ G; \triangle)_\Omega & \xleftarrow{\ \partial'\ } & C'_1(^\circ G; \triangle)_\Omega & \xleftarrow{\ \partial'\ } & C'_2(^\circ G; \triangle)_\Omega \\
\Big\downarrow{\scriptstyle \mathrm{Ad}_\Pi} & & \Big\downarrow{\scriptstyle \mathrm{Ad}_\Pi} & & \Big\downarrow{\scriptstyle \mathrm{Ad}_\Pi} \\
C'_0(^\circ G; \triangle)_\Omega & \xleftarrow{\ \partial'\ } & C'_1(^\circ G; \triangle)_\Omega & \xleftarrow{\ \partial'\ } & C'_2(^\circ G; \triangle)_\Omega
\end{array}
$$

computes the Ω-component $H_*(G; X)_\Omega$ of the chamber homology of $GL(3)$.

A full account of the Ω-component of chamber homology would require these parts:

(1) the determination of all representations of the compact groups G_σ which, when induced to G, lie within the Ω-component of the smooth dual of G (this amounts to determining the vector spaces in the local complex for Ω);

(2) the computation of how the representations in (1) induce from one G_σ to another (this amounts to determining the differentials ∂' in the local complex);

(3) the computation of the action of Ad_Π on the local complex (so as to determine the local mapping torus complex); and

(4) the verification that the local mapping torus complex really does carry the Ω-component of chamber homology.

Unfortunately we have not carried out this program in full for any Ω. What we have done, for certain Ω, is obtain *some* but not perhaps *all* of the representations required by (1), and with these, which assemble to form an Ad_Π-invariant subcomplex of the local complex, we have gone on to compute ∂', the action of Ad_Π, and the homology of the associated local mapping torus complex. The computations support the assertion that we have located *all* the representations in (1). We will formulate a precise and more general conjecture along these lines in the next section. In the present section we shall focus on the computational aspects (2) and (3) of the above program.

We shall need to carry out some induction computations in the representation theory of compact groups, and for these *Frobenius Reciprocity* and the *Mackey formula* will prove very useful. Suppose that J is an open subgroup of a compact

group K. If ρ and π are finite-dimensional, complex-linear representations of J and K, respectively then Frobenius reciprocity provides a vector space isomorphism

$$\operatorname{Hom}_J(\rho, \operatorname{Res}^K_J \pi) \cong \operatorname{Hom}_K(\operatorname{Ind}^K_J \rho, \pi).$$

If ρ_1 and ρ_2 are finite-dimensional, complex-linear representations of open subgroups J_1 and J_2 of K, and if $x \in K$ then we define the *intertwining vector space* to be

$$I_x(\rho_1, \rho_2) = \operatorname{Hom}_{xJ_1 x^{-1} \cap J_2}(\rho_1^x, \rho_2),$$

where $\rho_1^x(y) = \rho_1(x^{-1}yx)$. The Mackey formula provides a vector-space isomorphism

$$\operatorname{Hom}_K(\operatorname{Ind}^K_{J_1} \rho_1, \operatorname{Ind}^K_{J_2} \rho_2) \cong \bigoplus_{x \in J_1 \backslash K / J_2} I_x(\rho_1, \rho_2).$$

Setting $\rho = \rho_1 = \rho_2$ we get

$$\operatorname{End}_K(\operatorname{Ind}^K_J \rho) \cong \bigoplus_{x \in J \backslash K / J} I_x(\rho, \rho).$$

The formula can, for example, be used to determine the irreducibility of induced representations, since $\pi = \operatorname{Ind}^K_J \rho$ is irreducible if and only if $\operatorname{End}_K(\pi) \cong \mathbb{C}$.

To apply the Mackey formula we shall need to know the double-coset decompositions for the isotropy groups G_σ appearing in the local complexes for $GL(3)$. If we set

$$s_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad s_1 = \begin{bmatrix} 0 & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ \pi & 0 & 0 \end{bmatrix} \qquad s_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then the I-double cosets for the isotropy groups of the vertices and edges of $\triangle$ are represented by the elements in various groups generated by subsets of $\{s_0, s_1, s_2\}$, as illustrated in Figure 4. Thus for instance the edge isotropy groups are

$$J_{ij} = I \cup I s_k I,$$

in the notation of Figure 1, where i, j and k are distinct indices from $\{0, 1, 2\}$. It follows that the vertex stabilizers K_i have double coset decompositions

$$K_i = J \cup J s_k J,$$

where $J = J_{ij}$ and again i, j and k are distinct indices.

With these preliminaries in hand, let us turn to an analysis of the local complex for some Bernstein components Ω.

6.1. The Borel component of the Bernstein variety. Let T be the diagonal subgroup of G and let σ be the trivial representation of T. Then (T, σ) is a supercuspidal pair in the sense of Definition 4.2. It determines a Bernstein component
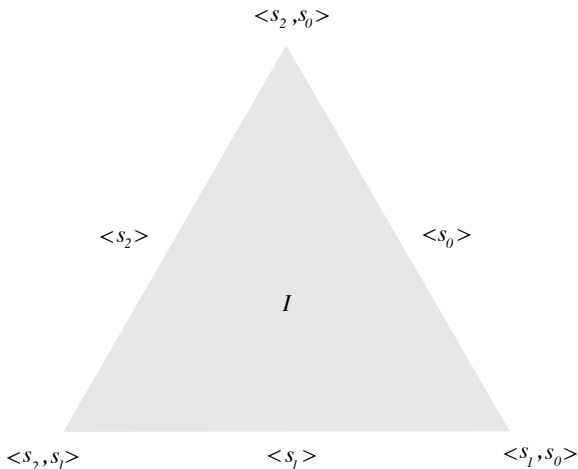
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Fig. 4. I-double coset representatives for the isotropy groups of the faces of the Iwahori subgroup I. For example the notation $\langle s_2, s_1 \rangle$ means that the group generated by s_2 and s_1 forms a complete set of I-double coset representatives for the isotropy group K_0 of the lower left vertex.

Ω which was first analyzed (for a general G, not just $GL(3)$) by Borel [Bor], who showed that a smooth representation[5] satisfies $E = E_\Omega$ if and only it is generated as a representation of G by those of its vectors fixed by the Iwahori subgroup I.

Borel's result implies that the trivial representation ι of the Iwahori subgroup I belongs to the local complex for the *Borel component* Ω, as indeed does every irreducible component of every representation induced from ι.

6.2. Conjecture. *The irreducible components of the representations induced from the trivial representation of the Iwahori subgroup generate the whole of the local complex for the Borel component of the Bernstein variety.*

Remark. In other terms, the conjecture asserts that if an irreducible representation π of a parahoric subgroup never occurs in any representation of G which has no I-fixed vectors then π is a component of a representation induced from the trivial representation of I. Thus the conjecture adds to Borel's theorem a sort of uniqueness assertion for the trivial representation of the Iwahori subgroup.

What are the irreducible components of the representations induced from the trivial representation ι of the Iwahori subgroup? If ι is induced to the isotropy subgroup of an edge of $\triangle$ then according to the Mackey formula,

$$\mathrm{End}_{J_{ij}}(\mathrm{Ind}_I^{J_{ij}} \iota) \cong I_e(\iota, \iota) \oplus I_{s_k}(\iota, \iota),$$

from which we see right away that $\mathrm{End}_{J_{ij}}(\mathrm{Ind}_I^{J_{ij}} \iota) \cong \mathbb{C} \oplus \mathbb{C}$, so that the induced representation is a direct sum of two distinct irreducible constituents. By Frobenius

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[5]Borel worked with smooth and admissible representations, but the result is the same for general smooth representations.

Reciprocity one of these constituents is the trivial representation of J_{ij}, so that

$$\operatorname{Ind}_I^{J_{ij}} \iota = \iota \oplus \sigma.$$

The Mackey formula shows that both of the representations σ and ι on J_{ij}, when induced to a vertex stabilizer, decompose as a direct sum of at most two (necessarily distinct) irreducible representations. In addition, when the trivial representation of I is induced to a vertex stabilizer K_i then $\operatorname{End}_{K_i}(\operatorname{Ind}_I^{K_i} \iota)$ is six-dimensional; by Frobenius Reciprocity the trivial representation of K_i occurs with multiplicity one in the induced representation.

A moment's thought shows that the only possibility here is this: the two representations ι and σ of each J_{ij} decompose into two irreducible representations upon induction to a vertex isotropy group, and that of the four representations of each vertex isotropy group obtained in this way, two are isomorphic to one another, netting three distinct irreducible representations, ι, τ and τ', of each vertex group. The local complex for the Borel component Ω can now be described as in Figure 5 (assuming a postive resolution of Conjecture 6.2). Each dot represents a representation (giving a linear generator of the local complex) of the face-group next to which it is placed. The arrows indicate how these representations decompose under induction. The action of Ad_Π is given by the obvious rotational symmetry of the diagram. It is now a simple matter to compute the homology of the local complex and of the local mapping torus complex, the result for the latter being

$$H_0 = \mathbb{C}^3, \quad H_1 = \mathbb{C}^4, \quad H_2 = \mathbb{C}, \quad H_3 = 0.$$

As we have already mentioned, we shall take a second look at these calculations in Section 7.

6.3. Tame Characters. Let $\lambda : F^\times \to \mathbb{C}^\times$ be a *tame* character of $F^\times$ (we mean that the restriction of the character to $\mathcal{O}^\times$ factors through a multiplicative character of the residue field $\mathcal{O}/\pi\mathcal{O}$; we are also assuming in this subsection that this restriction is nontrivial). Let T be the diagonal subgroup of $GL(3)$ and define $\lambda : T \to \mathbb{C}^\times$ by

$$\lambda \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \lambda(c).$$

Then (T, λ) is a supercuspidal pair, and so it determines a component Ω in the Bernstein variety.

There are three one-dimensional representations of the Iwahori subgroup I which are naturally associated to λ, namely

$$\nu_0 \begin{pmatrix} a & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = \lambda(a), \quad \nu_1 \begin{pmatrix} * & * & * \\ * & b & * \\ * & * & * \end{pmatrix} = \lambda(b), \quad \nu_2 \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & c \end{pmatrix} = \lambda(c).$$

Parallel to Borel's result on the trivial representation of the Iwahori is the assertion that a representation E of G satisfies $E = E_\Omega$ if and only if for any j it is generated
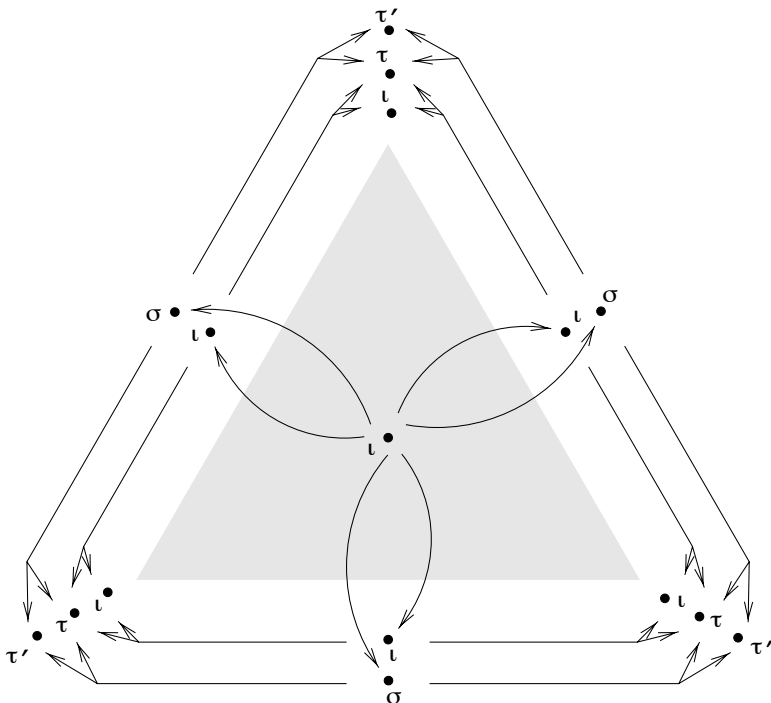
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FIG. 5. The conjectural local complex for the Borel component of
the Bernstein variety. The trivial representation of I induces to a
direct sum of two irreducible representations at each edge isotropy
group. Each of these in turn induces to a two-fold direct sum at
each vertex-isotropy group, with one coincidence among the four
representations. The inner dot at each face represents the trivial
representation of the face-isotropy group.

by its ν_j-isotypical vectors. It follows in particular that each of the representations
ν_0, ν_1 and ν_2 belongs to the local complex attached to Ω. Note that conjugation
with Π cyclically permutes ν_0, ν_1 and ν_2, so that the complex generated from the
representations ν_j by induction to the various face-isotropy groups for $\triangle$ is Ad_Π
invariant. As in the previous subsection, we believe that these representations
generate the full local complex; as in the previous section we shall compute the
local complex, presuming this to be so. In what follows we shall use the notation
for isotropy groups indicated in Figure 1.

6.4. Lemma. *If i, j, and k are distinct then the representation $\mathrm{Ind}_I^{J_{ij}} \nu_k$ is a direct
sum of two distinct irreducible representations. If i, j, and k are not all distinct
then $\mathrm{Ind}_I^{J_{ij}} \nu_k$ is irreducible.*

Proof. Recall that the unique nontrivial double coset of I in J_{ij} is generated by s_k
(where i, j and k are distinct indices from $\{0, 1, 2\}$). One checks that $I_{s_0}(\tau, \tau) = 0$

and also $I_{s_2}(\tau, \tau) = 0$, whereas $I_{s_1}(\tau, \tau) = \mathbb{C}$ (since τ is a one-dimensional representation these computations amount to determining whether or not $\tau^{s_j} = \tau$ on $s_j I s_j \cap I$). The lemma follows from this and the Mackey formula. $\quad\square$

If i, j and k are distinct then the one-dimensional representation ν_k of I extends to I_{ij} (to take $k = 2$, for example, the group J_{01} is comprised of matrices which are block upper triangular, modulo π, with a 1×1 block in the lower right corner, so that applying λ to this entry we still obtain a group homomorphism). It follows from Frobenius Reciprocity that in the decomposition

$$\operatorname{Ind}_I^{J_{ij}} \nu_k = \beta \oplus \beta'$$

given by Lemma 6.4, one of the representations, say β', is this extended one-dimensional representation.

6.5. Lemma. *For every i and j, the representation $\operatorname{Ind}_I^{K_i} \nu_j$ is a direct sum of two irreducible representations.*

Proof. We'll do the calculation when $i \neq j$ and leave the other case to the reader. The double coset space $I \backslash K_i / I$ identifies with the permutation group $\langle s_j, s_k \rangle$ (where as usual i, j and k are distinct). If $x \in \langle s_j, s_k \rangle$ then

$$\nu_j^x = \nu_j \quad \Leftrightarrow \quad x = 1, s_j.$$

Therefore by the Mackey formula we have

$$\operatorname{End}_{K_i}(\operatorname{Ind}_I^{K_i} \nu_j) = I_1(\nu_j) \oplus I_{s_j}(\nu_j) = \mathbb{C} \oplus \mathbb{C}$$

as required. $\quad\square$

It follows from this that the representations β and β' of the edge-isotropy groups discussed above induce irreducibly to representations of the vertex-isotropy groups.

6.6. Lemma. *For any i, j and k (not necessarily distinct) there is an equivalence of representations*
$$\operatorname{Ind}_I^{K_i} \nu_j \cong \operatorname{Ind}_I^{K_i} \nu_k.$$

Proof. It is enough to prove that

$$\operatorname{End}_{K_i}(\operatorname{Ind}_I^{K_i} \nu_j, \operatorname{Ind}_I^{K_i} \nu_k) = \mathbb{C} \oplus \mathbb{C}.$$

Now the left-hand side is $\oplus I_x(\nu_j, \nu_k)$ with $x \in I \backslash K_i / I$. Once again, let us analyze the case where i, j and k are distinct, and leave the remaining cases to the reader. We take $x \in \langle s_j, s_k \rangle$ and calculate that

$$I_x(\nu_j, \nu_k) \neq 0 \quad \Leftrightarrow \quad x = s_k s_j, s_j s_k s_j.$$

The result now follows from the Mackey formula. $\quad\square$

One can be a little more precise: the extended one-dimensional representations β' of any two edge isotropy groups induce to the same irreducible representation γ of the isotropy group of the vertex shared by the two edges. This also follows from the Mackey formula.

The various observations that we have made now justify the rather intricate diagram presented in Figure 6, which summarizes the structure of the local complex in the case we are considering (the diagram should be read in the same way as Figure 5). The automorphism Ad_Π acts via the obvious rotational symmetry of the diagram, and the homology of the local mapping torus complex is as follows:

$$H_0 = \mathbb{C}^2, \quad H_1 = \mathbb{C}^4, \quad H_2 = \mathbb{C}^2, \quad H_3 = 0.$$

6.7. Supercuspidal Representations. Every irreducible supercuspidal representation of the group $G = GL(3)$ defines a component of the Bernstein variety $\Omega(G)$. Our previous examples involved the minimal Levi subgroup of G, whereas we are now looking at the maximal Levi subgroup — G itself. So by turning our attention to supercuspidal representations we are in some sense focusing on the opposite end of the Bernstein variety $\Omega(G)$.

How do we expect the local complex to present itself in the case of a component Ω in the Bernstein variety associated to a supercuspidal representation of G? It is a well-known conjecture that every irreducible supercuspidal representation of a reductive p-adic group G with compact center should be induced from a representation of a compact-modulo-center open subgroup. For $G = SL(N)$ and $G = GL(N)$ this conjecture has in fact been proved by Bushnell and Kutzko [BK1,BK2]. The answer to our question appears to be related to this *induction conjecture,* and since the presence of a noncompact center in G is a complicating factor let us begin with an account of $SL(N)$. Here the simplest and most attractive counterpart in chamber homology to the induction conjecture would be this:

6.8. Conjecture. *The local complex associated to the component Ω of the Bernstein variety determined by an irreducible supercuspidal representation of $G = SL(N)$ identifies with the simplicial homology complex associated to some nonempty face $\triangle_\Omega$ of the fundamental chamber $\triangle$ in the affine building for G. In other words the spaces $\mathcal{Cl}(G_\sigma)_\Omega$ are either zero or are spanned by the character of a single irreducible representation of G, and the set of faces σ of $\triangle$ for which such a representation exists[6] is precisely the set of faces of $\triangle_\Omega \subset \triangle$.*

We emphasize that in the conjecture the term 'face' does not mean 'codimension one face.' Faces of all dimensions, from a vertex to $\triangle$ itself, are allowed. Figure 7 shows a potential local complex under the scheme of the conjecture. Note that for $G = SL(3)$ the local complex is formed exactly as for $GL(3)$, here there is no automorphism Ad_Π, and so the local complex, as opposed to a mapping torus complex, is itself the complex $C_*(\Omega; X)_G$. Conjecture 6.8 is consistent with the fact

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[6]Once again, the defining property of such a representation is that when induced from G_σ to G the resulting representation lies wholly in the Ω-component of the smooth dual.
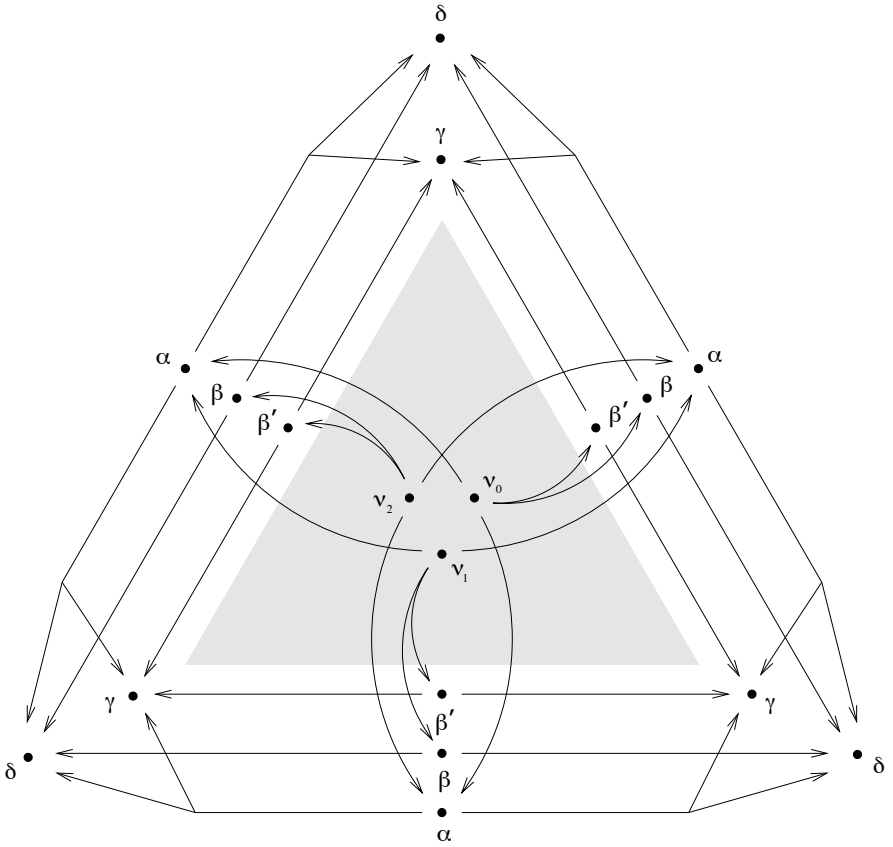
FIG. 6. The conjectural local complex for the component of the Bernstein variety associated to a tame character of $F^\times$. The automorphism Ad_Π acts 'by rotation' on the diagram, so that for instance all the representations labelled β lie in a single orbit under the action.

that the *homology* of the local complex should be the homology of a point (we'll consider homology groups, as opposed to the complexes themselves, in the next section).

Membership of the character of a representation π in $\mathcal{Cl}(G_\sigma)_\Omega$ is very close to the property that π induces to the given supercuspidal representation (certainly it is implied by it). So Conjecture 6.8 comprises both an existence part (namely $\triangle_\Omega \neq \emptyset$) and a uniqueness part: for instance if two representations of distinct vertex-isotropy groups both induce to the given supercuspidal representation then it is because the two representations are both induced from a common representation of a smaller group.
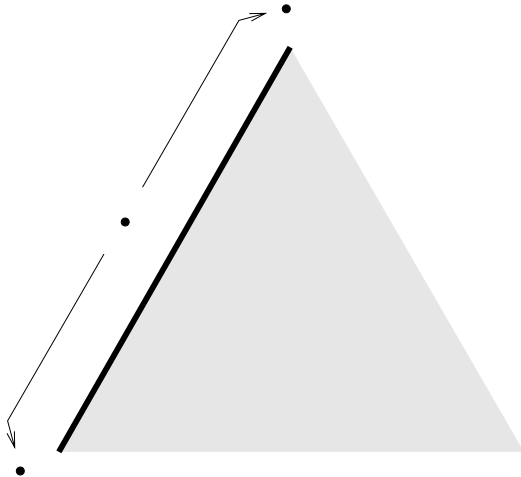
FIG. 7. A conjectural local complex associated to a supercuspidal representation of $G = SL(3)$. The face $\triangle_\Omega \subset \triangle$ is shown in black.

The case of $G = GL(N)$ is more complicated, since we must now take into account the action of Ad_Π on the local complex. For reasons to explained in the next section the homology of the local mapping torus complex should be the homology of a circle. There seem to be two simple ways of arranging this. The first is to proceed as in the above conjecture, but require that $\triangle_\Omega = \triangle$ (this is forced by the Ad_Π-invariance of the local complex since the only Ad_Π-invariant face of $\triangle$ is $\triangle$ itself). The second, which is new to $GL(N)$, is to assert that the local complex is concentrated in degree zero, that is on the vertex isotropy groups, with one representation of each vertex-isotropy group in the local complex and no representations of the isotropy groups of higher-dimensional faces. Both arrangements produce

$$H_0 = \mathbb{C}, \quad H_1 = \mathbb{C},$$

as required. One could modify Conjecture 6.8 to suit $G = GL(N)$ by deleting the requirement that $\triangle_\Omega$ be a simplex and replacing it by the requirement that $\triangle_\Omega$ be a disjoint union of simplices on which Ad_Π acts transitively. This new conjecture would cover both of the above cases, but it clearly lacks the elegance of the assertion for $SL(N)$, and we are not certain that the new formulation is fully adequate.

7. STRUCTURE OF THE HECKE ALGEBRA

In the previous section we focused on the complexes $C_*(\Omega; X)_G$ and their homology groups $H_*(\Omega; X)$. Conjecturally, the homology groups $H_*(\Omega; X)$ identify with the Bernstein components $H_*(G; X)_\Omega$ of chamber homology. The purpose of this section is to illuminate this conjecture just a little by studying in more detail these Bernstein components.

Recall from Section 5 that we decomposed chamber homology into Bernstein components by means of the Bernstein decomposition

$$\mathcal{H}(G) = \oplus_\Omega \mathcal{H}(G)_\Omega$$

of chamber homology and the computation in Section 3 of the cyclic homology of the Hecke algebra. By definition,

$$H_n(G; X)_\Omega = \text{Kernel}\,[S \colon HC_n(\mathcal{H}(G)_\Omega) \to HC_{n-2}(\mathcal{H}(G)_\Omega)]$$
$$\cap \text{Image}\,[S \colon HC_{n+2}(\mathcal{H}(G)_\Omega) \to HC_n(\mathcal{H}(G)_\Omega)]$$

From the above lemma we obtain a decomposition

$$H_*(G; X) = \oplus_\Omega H_*(G; X)_\Omega.$$

Thus in order to better understand the Bernstein components of chamber homology we must in effect compute the cyclic homology of the ideals $H(G)_\Omega$ in the Hecke algebra of G.

7.1. The Borel component. Let G be a reductive p-adic group. If Ω is the Borel component of the Bernstein variety $\Omega(G)$ (corresponding to the trivial representation of the minimal Levi subgroup of G) then Borel's work, to which we referred earlier, implies that

$$\mathcal{H}_\Omega = \mathcal{H}e\mathcal{H},$$

where

$$e(g) = \begin{cases} \dfrac{1}{\text{vol}\,(I)} & \text{if } g \in I \\[2mm] 0 & \text{if } g \notin I, \end{cases}$$

and where we have used the abbreviation $\mathcal{H} = \mathcal{H}(G)$. The element e is an idempotent in $\mathcal{H}$. It has the characteristic property that in any smooth representation of G the operator e is I-equivariant and its range is precisely the space of I-fixed vectors.

There is a 'Morita equivalence' of algebras[7]

$$\mathcal{H}e\mathcal{H} \underset{\text{Morita}}{\sim} e\mathcal{H}e,$$

and since cyclic homology is Morita invariant, the problem of computing the Borel component of chamber homology reduces to that of computing the cyclic homology of the unital algebra $e\mathcal{H}e$. But $e\mathcal{H}e$ is precisely the convolution algebra of I-bi-invariant, compactly supported functions on G, and its structure is very well known, thanks to the work of Iwahori and Matsumoto [IM]. Associated to G there

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[7]Since $\mathcal{H}e\mathcal{H}$ is not a unital algebra, we are not quite using the ordinary notion of Morita equivalence from algebra. But for instance $\mathcal{H}e\mathcal{H}$ is a direct limit of unital algebras, each equivalent to $e\mathcal{H}e$, and the associated system of bimodules is compatible with the inclusion maps in the directed system.

is an 'affine Weyl group' $\widetilde{W}$ containing the Weyl group W of G. The I-double cosets in G are parametrized by $\widetilde{W}$ and so as a linear space the *Iwahori Hecke algebra* $e\mathcal{H}e$ has a natural basis $\{T_a : a \in \widetilde{W}\}$. If we fix a generating system $S \subset W$ then, as an algebra, $e\mathcal{H}e$ has the following presentation:

$$\begin{cases} T_a T_b = T_{ab} & \text{if } \ell(a) + \ell(b) = \ell(ab), \\ T_s^2 = (q-1)T_s + q & \text{if } s \in S. \end{cases}$$

See [Lu]. Here q (a prime power) is the cardinality of the residue field of our p-adic field. Of course the same relations determine an algebra $\mathcal{H}(\tilde{W}, q)$ for any value of q; note that if $q = 1$ then $\mathcal{H}(\tilde{W}, q)$ reduces to the complex group algebra of $\widetilde{W}$.

Let us now specialize to $G = GL(N)$. Here the affine Weyl group $\widetilde{W}$ is the semidirect product $\mathbb{Z}^N \rtimes S_N$ associated to the permutation action of the symmetric group on the free abelian group $\mathbb{Z}^N$, and in this case we shall use the notation $\mathcal{H}(N, q)$ for the Iwahori Hecke algebra.

One can show that the *periodic* cyclic homology of the affine Hecke algebra is actually independent of q:

7.2. Theorem. $HP_*(\mathcal{H}(N, q)) \cong HP_*(\mathcal{H}(N, 1)) = HP_*(\mathbb{C}[\mathbb{Z}^N \rtimes S_N])$

This gives an effective means of computing cyclic homology for the Bernstein component $\mathcal{H}_\Omega$ since the problem of computing cyclic homology for group algebras has been studied in some depth, and with considerable success. One very suggestive approach is this. The group $\widetilde{W} = \mathbb{Z}^N \rtimes S_N$ acts in the obvious way on the euclidean space $V = \mathbb{R}^N$. This action is actually a product of two actions, one on the quotient $^\circ V$ of V by the line spanned by $(1, \ldots, 1)$, and one on this line itself. If we define

$$^\circ \mathbb{Z}^N = \{(k_1, \ldots, k_N) \in \mathbb{Z}^N : k_1 + \cdots + k_N = 0\}$$

then $^0 V$ tessellates as the Coxeter complex associated to the Coxeter group $^\circ \widetilde{W} = {}^\circ \mathbb{Z}^N \rtimes S_N$, and the product of this with the standard simplicial structure on a line gives a polysimplicial structure on V which is preserved by the action of $\widetilde{W}$.

The situation when $N = 3$ is illustrated in Figure 8. It should be clear that the action of $\widetilde{W}$ on V models the action of $GL(3)$ on its affine building (in fact V is simply an apartment in the building). The cyclic homology of the group algebra $\mathbb{C}[\widetilde{W}]$ identifies with the chamber homology for the model action, exactly as in Section 3. But now an interesting point arises: the complex to compute chamber homology in this case, which may be viewed as a mapping torus complex just as in Sections 2 and 6, identifies *at the level of complexes* with the complex we generated in Paragraph 6.1 from the trivial representation of the Iwahori subgroup. This is because the way in which the trivial representation of the Iwahori subgroup decomposes into irreducibles upon induction to higher parahoric subgroups is governed by intertwining algebras which are in fact finite Hecke algebras $\mathcal{H}(W[S'], q)$, where $S' \subset S$. These are finite semisimple algebras, and are hence rigid under deformation of the parameter q (through real values). Setting $q = 1$ we obtain the intertwining
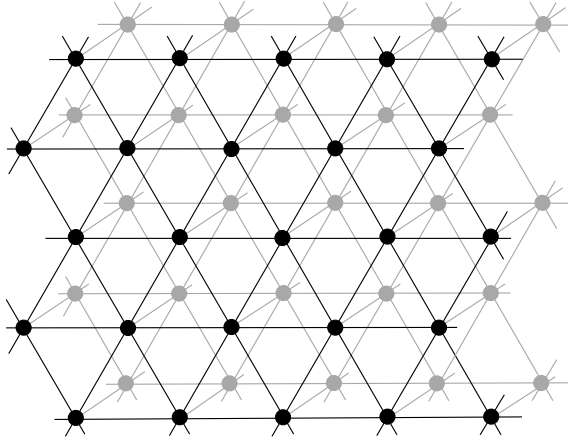
FIG. 8. The polysimplicial subdivision of $V = \mathbb{R}^3$ for the extended Coxeter group $\widetilde{W} = \mathbb{Z}^3 \rtimes S_3$. The Coxeter subgroup $°\widetilde{W} = °\mathbb{Z}^3 \rtimes S_3$ acts on the 2-dimensional Coxeter complex $°V$ shown in black. This action extends to $\widetilde{W}$. The action of $\widetilde{W}$ on V is a product of actions on $°V$ and $\mathbb{R}$.

algebras appropriate to the computation of chamber homology for the action of $\widetilde{W}$ on V. This provides, we hope, a good conceptual reason to believe in the validity of Conjecture 6.2. It also shows that the conjecture has a very interesting geometric foundation in the structure of an apartment in the affine building.

In concluding this discussion we must note that Theorem 7.2 computes the *periodic* cyclic homology for the affine Hecke algebra, and hence for the Borel component $\mathcal{H}(G)_\Omega$ for the Hecke algebra of $GL(N)$, but not the nonperiodic cyclic homology. So strictly speaking we have only identified the homology considered in Paragraph 6.1 with the groups of $H_*(G; X)_\Omega$ after periodization (meaning the passage to $H_{\text{ev/odd}}$). Presumably there is a slightly stronger version of Theorem 7.2, implying that the spaces

$$\text{Kernel}\,[S \colon HC_n(\mathcal{H}(N, q) \to HC_{n-2}(\mathcal{H}(N, q))]$$
$$\cap \operatorname{Image}[S \colon HC_{n+2}(\mathcal{H}(N, q)) \to HC_n(\mathcal{H}(N, q))]$$

are independent of q, and presumably the same holds for the Iwahori Hecke algebra of any G.

7.3. Extended Quotients. Let us record for later discussion another aspect of the problem of computing (periodic) cyclic homology for the group algebra $\mathbb{C}[\widetilde{W}]$. Suppose that a group Γ acts properly and (for simplicity) cocompactly on a space X. The *extended quotient* associated to this action is the quotient space $\widehat{X}/\Gamma$, where

$$\widehat{X} = \{\,(\gamma, x) \in \Gamma \times X \,:\, \gamma x = x\,\}$$

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The group action on $\widehat{X}$ is $g \cdot (\gamma, x) = (g \gamma g^{-1}, gx)$. We are interested in the action of $\widetilde{W}$ on V, as discussed above.

7.4. Theorem. (See [BC].) *The periodic cyclic homology of the group algebra $\mathbb{C}[\widetilde{W}]$ is isomorphic to the periodized ordinary cohomology (with complex coefficients) of the extended quotient $\widehat{V}/\widetilde{W}$:*

$$HP_{\mathrm{ev/odd}}(\mathbb{C}[\widetilde{W}]) \cong H^{\mathrm{ev/odd}}(\widehat{V}/\widetilde{W}).$$

Observe that in our case (where $\widetilde{W}$ is the affine Weyl group for $GL(N)$) the extended quotient $\widehat{V}/\widetilde{W}$ identifies with the extended quotient for the permutation action of S_N on a real N-torus $\mathbb{T}^N$. The reader can easily check that the homology groups computed in Paragraph 6.1 agree with the cohomology of this torus extended quotient.

7.5. Bernstein Components and Bushnell-Kutzko Types. For the *p*-adic group $GL(N)$ Bushnell and Kutzko have completed a very ambitious program [BK1,BK3,BK4] which is immediately called to mind by the conjectures in Section 5, namely they have parametrized the components of the Bernstein variety, and indeed in effect the entire smooth dual, by representation theoretic data attached to compact open subgroups. Very interesting connections seem to exist between their impressive achievements and our conjectures.

The Bushnell-Kutzko theory is organized around an elegant notion of Ω-type[8] which we will present in a moment. But most relevant to the present discussion is the following detailed and complete account of the structure of the Hecke algebra $\mathcal{H}(G)$, up to Morita equivalence.

Let Ω be the Bernstein component of a supercuspidal pair (M, ρ) for $GL(N)$. Since M is a block-diagonal subgroup we can we can think of Ω as represented by a vector $(\tau_1, \ldots, \tau_s)$ of irreducible supercuspidal representations of the block-diagonal component groups, the entries of this vector being only determined up to tensoring with unramified characters and up to permutation. If the vector is equivalent to $(\sigma_1, \ldots, \sigma_1, \ldots, \sigma_r, \ldots, \sigma_r)$ where σ_j is repeated N_j times, $1 \leq j \leq r$, and where $\sigma_1, \ldots, \sigma_r$ are pairwise distinct, then we say that Ω has *exponents* $\{N_1, \ldots, N_r\}$.

7.6. Examples. The exponent list for the Borel component is $\{N\}$. For the 'tame character' component considered in Paragraph 6.3 it is $\{N-1, 1\}$; and for a component associated to a supercuspidal representation of G it is $\{1\}$.

The following beautiful result is due to Bushnell and Kutzko (see [BK1], [BK3] and [BK4]).

7.7. Theorem. *Let $N_1, \ldots, N_r$ be the exponents of a component Ω of the Bernstein variety for $GL(N)$. There is a Morita equivalence*

$$\mathcal{H}(G)_\Omega \sim \mathcal{H}(N_1, q_1) \otimes \cdots \otimes \mathcal{H}(N_r, q_r)$$

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[8]Bushnell and Kutzko use a different notation for the components of the Bernstein variety, and so speak of $\mathfrak{s}$-types, not Ω-types.

where $q_1, \ldots, q_r$ are natural number invariants attached to Ω.

Remark. The natural numbers $q_1, \ldots, q_r$ are the cardinalities of the residue fields of certain extension fields $E_1/F, \ldots, E_r/F$.

If $M < G$ is a Levi subgroup then denote by $W(M)$ the quotient by M of the normalizer of M in G. If (M, ρ) is a supercuspidal pair and if Ω is the associated Bernstein component then denote by $W(\Omega)$ the subgroup of $W(M)$ which fixes ρ up to equivalence of representations and unramified twists $\rho \mapsto \psi\rho$. If Ω has exponents $\{N_1, \ldots, N_r\}$ then $W(\Omega)$ is a product of symmetric groups,

$$W(\Omega) = S_{N_1} \times \ldots \times S_{N_r},$$

and we can form the semidirect product $\mathbb{Z}^{d(\Omega)} \rtimes W(\Omega)$, where $d(\Omega) = N_1 + \cdots + N_r$. Combining and slightly generalizing Theorems 7.2 and 7.4 we obtain from Theorem 7.7 the following result:

7.8. Theorem. *For every Bernstein component Ω of $GL(N)$ there is an isomorphism*

$$HP_{\mathrm{ev/odd}}(\mathcal{H}(G)_\Omega) \cong H^{\mathrm{ev/odd}}(\widehat{\mathbb{T}^{d(\Omega)}}/W(\Omega)).$$

In other words, thanks to the structure theorem of Bushnell and Kutzko it is possible to completely compute the periodic cyclic homology of the Hecke algebra.

7.9. Examples. Here is the cohomology of the space $\widehat{\mathbb{T}^{d(\Omega)}}/W(\Omega)$ for the components Ω considered in Section 6, namely the Borel component, the component associated to a tame character and the component associated to a supercuspidal representation:

$$\widehat{\mathbb{T}^{d(\Omega)}}/W(\Omega) = \widehat{\mathbb{T}^3}/S_3 \qquad H^0 = \mathbb{C}^3, \quad H^1 = \mathbb{C}^4, \quad H^2 = \mathbb{C}$$

$$\widehat{\mathbb{T}^{d(\Omega)}}/W(\Omega) = \widehat{\mathbb{T}^2 \times \mathbb{T}^1}/S_2 \times S_1 \qquad H^0 = \mathbb{C}^2, \quad H^1 = \mathbb{C}^4, \quad H^2 = \mathbb{C}^2$$

$$\widehat{\mathbb{T}^{d(\Omega)}}/W(\Omega) = \widehat{\mathbb{T}^1}/S_1 \qquad H^0 = \mathbb{C}, \quad H^1 = \mathbb{C}.$$

The unlisted groups are zero. Note that the results support the calculations and conjectures in Sections 5 and 6.

We have presented the basic result 7.7 outside of its natural context: the theory of types. We conclude our discussion of the Bushnell-Kutzko theory by noting how the theory of types suggests a precise description of the complexes $C_*(\Omega; X)_G$ considered in the last section.

7.10. Definition. An Ω-*type* is a pair (J, ρ), comprised of a compact open subgroup J and an irreducible representation ρ of J, with the following property: an irreducible, smooth representation of G belongs to the Ω component of the smooth dual of G if and only if it contains the representation ρ with nonzero multiplicity when it is restricted to J.

7.11. Examples. The trivial representation of the Iwahori subgroup is a type for the Borel component of the Bernstein variety of any reductive group. The representations ν_k of the Iwahori subgroup that we studied in Paragraph 6.3 are all types for the component associated to tame character, as in that paragraph.

The following fact is proved in [BK3]:

7.12. Lemma. *Let (J, ρ) be an Ω-type. Then the character of ρ, extended by zero to be a function on G, belongs to the Bernstein component $\mathcal{H}(G)_\Omega$ of the Hecke algebra.*

We come now to yet another conjecture, which should be viewed as a development of our Conjecture 6.2 about the Borel component of the Bernstein variety of a reductive group. Here we specialize G to $GL(N)$ but make an assertion about every component of the Bernstein variety.

7.13. Conjecture. *Let $G = GL(N)$ and let Ω be any component of the Bernstein variety $\Omega(G)$. Form a subcomplex of the local complex for Ω (see Section 6) as follows: the character of an irreducible representation π of G_σ belongs to the subcomplex if and only if there is an Ω-type (J, ρ) such that $J \subset G_\sigma$ and π is a component of $\mathrm{Ind}_{J_\sigma}^{G_\sigma} \rho$. Then this subcomplex is the entire local complex for Ω.*

In short, the conjecture asserts that the local complex associated to a Bernstein component Ω is 'generated' by the Ω types. The conjecture is clearly related to the issue of *uniqueness* of Ω-types.

7.14. Supercuspidal Representations. In this paragraph we shall quickly summarize Bernstein's Paley-Wiener Theorem [Be], as it applies to the Bernstein components $\Omega \subset \Omega(G)$ associated to supercuspidal representations of $G = GL(N)$.

Suppose that ρ is an irreducible, supercuspidal representation of G. All the twists $\psi\rho$ (where $\psi \in \Psi(G)$ is an unramified character, as in Section 4) may be realized on the same space, say V. Associated to the representation $\psi\rho$ there is a representation of the Hecke algebra $\mathcal{H}(G)$ as endomorphisms of V. So varying ψ over $\Psi(G)$ we obtain a homomorphism of algebras

$$\rho : \mathcal{H}(G) \to \mathcal{F}(\Psi(G), \mathrm{End}(V)),$$

where $\mathcal{F}(\Psi(G), \mathrm{End}(V))$ denotes functions from $\Psi(G)$ into $\mathrm{End}(V)$. This map is injective on $\mathcal{H}(G)_\Omega$ (and it vanishes on the other Bernstein components of $\mathcal{H}(G)$). The Paley-Wiener Theorem characterizes its image.

The map $\psi \mapsto \psi\rho$ identifies Ω with a quotient of $\Psi(G)$ by a finite subgroup $\mathcal{G} \subset \Psi(G)$, comprised of those characters for which $\psi\rho$ is equivalent to ρ. If $G = GL(N)$ then $\mathcal{G}$ is cyclic and the equivalences $\psi\rho \sim \rho$ can be realized by an action of $\mathcal{G}$ on the trivial bundle $\Psi(G) \times V$ (for general groups G there is a similar *projective* action). It is clear that the homomorphism under consideration in fact maps $\mathcal{H}(G)$ into $\mathcal{F}(\Psi(G), \mathrm{End}(V))^\mathcal{G}$.

7.15. Definition. If K is a compact open subgroup of G then denote by $\mathcal{H}(G /\!/ K)$ the subalgebra of $\mathcal{H}(G)$ comprised of K-bi-invariant, compactly supported functions

on G. If Ω is any Bernstein component of $\Omega(G)$ then we write

$$\mathcal{H}(G /\!\!/ K)_\Omega = \mathcal{H}(G /\!\!/ K) \cap \mathcal{H}(G)_\Omega$$

and note that

$$\mathcal{H}(G)_\Omega = \cup_K \mathcal{H}(G /\!\!/ K)_\Omega,$$

the union being over the directed system of compact open subgroups of G.

7.16. Theorem. *Let Ω be the Bernstein component determined by a supercuspidal representation of $G = GL(N)$. If K is any compact open subgroup of G then the image of the homomorphism $\rho \colon \mathcal{H}(G /\!\!/ K) \to \mathcal{F}(\Psi(G), \mathrm{End}(V))^{\mathcal{G}}$ is the algebra $\mathcal{O}(\Psi(G), \mathrm{End}(V_K))^{\mathcal{G}}$ of $\mathcal{G}$-covariant regular functions from the affine variety $\Psi(G)$ into the endomorphisms of the finite-dimensional vector space V_K of K-fixed vectors in V.*

Remark. Essentially the same theorem holds for any reductive group and indeed for any Bernstein component which is *generic*, in the sense that $W(\Omega)$ is trivial. But the fact that in general $\mathcal{G}$ only acts projectively is a complicating factor in what follows.

Each algebra $\mathcal{O}(\Psi(G), \mathrm{End}(V_K))^{\mathcal{G}}$ is Morita equivalent to the algebra $\mathcal{O}(\Omega)$ of regular functions on $\Omega = \Psi(G)/\mathcal{G}$ (this uses the fact that the *a priori* projective action of $\mathcal{G}$ may be altered to a linear action). Taking a direct limit over compact open subgroups, as in Definition 7.15, we obtain[9] from the Morita invariance of cyclic theory an isomorphism

$$HC_*(\mathcal{H}(G)_\Omega) \cong HC_*(\mathcal{O}(\Omega)).$$

For $GL(N)$ the component Ω of a supercuspidal representation identifies with $\mathbb{C}^\times$, and so $\mathcal{O}(\Omega)$ is simply the algebra of complex Laurent polynomials. Its cyclic homology may be readily computed in a number of ways (among them of course an appeal to Theorem 3.1). We obtain the following simple formula for the Bernstein component of chamber homology associated to a supercuspidal representation of $G = GL(N)$:

$$G = GL(N) \quad \Rightarrow \quad H_p(G; X)_\Omega = \begin{cases} \mathbb{C} & \text{if } p \leq 1, \\ 0 & \text{if } p \geq 2. \end{cases}$$

Supercuspidal representations of the group $SL(N)$ may be treated in a similar way; in fact this is much easier, since here $\Psi(G)$ is a one-point space. We conclude that $HC_*(\mathcal{H}(G)_\Omega) \cong HC_*(\mathbb{C})$, from which we obtain

$$G = SL(N) \quad \Rightarrow \quad H_p(G; X)_\Omega = \begin{cases} \mathbb{C} & \text{if } p = 0, \\ 0 & \text{if } p \geq 1. \end{cases}$$

Remark. In fact we could have obtained the same conclusions from the Bushnell-Kutzko theory (note that the Iwahori Hecke algebra $\mathcal{H}(1, q)$ identifies with the algebra of complex Laurent polynomials), but the present account seems conceptually very appealing. Furthermore modulo the projective action issue raised in the remark following Theorem 7.16 the discussion in this section may be applied to any G.

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[9]See also the remark following Conjecture 8.9 below.

8. The Schwartz Algebra

Let $\mathcal{A} = \mathbb{C}[z, z^{-1}]$ be the algebra of complex Laurent polynomials, or in other words the algebra of regular functions on the affine variety $\mathbb{C}^{\times}$. We noted in Section 3 that the cyclic homology of A may be computed from the de Rham complex of algebraic differential forms on $\mathbb{C}^{\times}$. In addition, by Theorem 3.4 the cyclic homology of the Fréchet algebra $\mathcal{B} = C^{\infty}(S^1)$ may be computed from the de Rham complex of smooth differential forms on S^1. But the inclusion of S^1 into $\mathbb{C}^{\times}$ as the real part of the variety $\mathbb{C}^{\times}$ is a homotopy equivalence; so it follows[10] from the forgoing computations of periodic cyclic homology that the inclusion $\mathcal{A} \subset \mathcal{B}$ induces an isomorphism

$$HP_*(\mathcal{A}) \xrightarrow{\cong} HP_*(\mathcal{B}).$$

The purpose of this section is to transport this observation in a rather natural way to the context of reductive p-adic groups and their Hecke algebras.

We are going to think of the smooth dual $\mathrm{Irr}(G)$ as some sort of algebraic variety, and the Hecke algebra $\mathcal{H}(G)$ as some sort of algebra of regular functions on $\mathrm{Irr}(G)$ (our remarks about the Bernstein variety in Section 4 and the Paley-Weiner Theorem in Section 7 should help support this point of view, which has been promoted by many people). There is an natural candidate for the 'real part' of the 'variety' $\mathrm{Irr}(G)$, namely the *tempered dual* $\mathrm{Irr}^{\mathrm{t}}(G)$. In addition there is a natural candidate for the algebra of smooth functions on this 'real part,' namely the *Schwartz algebra* $\mathcal{S}(G)$ of Harish Chandra. This is the algebra of uniformly locally constant[11] complex functions f on G which satisfy the decay condition

$$\nu_n(f) = \sup_{g \in G} |f(g)| \, \Xi(g)^{-1} (1 + \sigma(g))^n < \infty,$$

for every $n \in \mathbb{N}$. Here σ is what one might call a proper length function on G, meaning that $\sigma(g_1 g_2) \leq \sigma(g_1) + \sigma(g_2)$ and in addition $\sigma \colon G \to [0, \infty)$ is a continuous, proper function. Thus apart from the presence of the additional Ξ-term the decay condition is reminiscent of the usual rapid decay condition for a function on $\mathbb{R}^n$.[12] As for the special function Ξ, it plays a central role in Harish Chandra's theory, but has no abelian counterpart. We refer the reader to [HC,Si] for the definition of Ξ, which will not be important here.

8.1. Definition. If K is a compact open subgroup of G and then we shall denote by $\mathcal{S}(G /\!\!/ K)$ the subspace of functions in $\mathcal{S}(G)$ which are constant on K-double cosets.

The space $\mathcal{S}(G /\!\!/ K)$ is nuclear and Fréchet in the topology associated to the norms ν_n. The convolution product extends to a continuous multiplication on $\mathcal{S}(G /\!\!/ K)$, which is thereby given the structure of a nuclear Fréchet algebra.

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[10]The paper [Ha] gives a very complete treatment of the issue here.

[11]A function f on G is uniformly locally constant if there is an open subgroup K of G such that $f(k_1 g k_2) = f(g)$, for every $g \in G$ and all $k_1, k_2 \in K$.

[12]Recall that a smooth function on $\mathbb{R}^n$ lies in the Schwartz class if it and all its derivatives are of rapid decay. In the context of reductive p-adic groups the condition on derivatives is replaced by the condition of uniform local constancy.

The topological vector space and algebra structure of the Schwartz space $\mathcal{S}(G)$ itself is more complicated, but of course

$$\mathcal{S}(G) = \cup_K \mathcal{S}(G /\!\!/ K),$$

and it therefore seems reasonable to approach the Schwartz space through the directed system of unital nuclear Fréchet subalgebras $\mathcal{S}(G /\!\!/ K)$ (see [BP] for a further discussion of this point). Thus we are going to *define*

$$HP_*(\mathcal{S}(G)) = \varinjlim_K HP_*(\mathcal{S}(G /\!\!/ K)),$$

where the right-hand-side is periodic cyclic homology for Fréchet algebras. We are going to address two issues: the computation of $HP_*(\mathcal{S}(G))$; and the comparison of $HP_*(\mathcal{S}(G))$ with $HP_*(\mathcal{H}(G))$. For both we will mostly specialize to the group $GL(N)$.

Our computation of $HP_*(\mathcal{S}(G))$ depends on a complete structural analysis of the algebra $\mathcal{S}(G)$, and this in turn depends on a reasonably thorough understanding of the tempered dual of G. So we shall begin by giving a very telegraphic account of the tempered dual.

8.2. Definition. Let M be a standard Levi subgroup of G, as in Section 4. We denote by $E_2(M)$ denote the set of all representations σ in the discrete series of M, up to unitary equivalence.

The space $E_2(M)$ is a disjoint union of compact tori. Each representation in $E_2(M)$ extends to the parabolic subgroup P associated to M, and after unitary induction we obtain a tempered representation of the group G. The Weyl group $W(M)$ acts on M and hence on $E_2(M)$, and two representations of G obtained by induction from discrete series representations of M are unitarily equivalent if and only if they lie in the same orbit under the action of $W(M)$ on $E_2(M)$.

In the special case of $GL(N)$ all the representations obtained by induction from discrete series representations on M are irreducible, and we obtain in this way the full tempered dual of G. In other words if $\mathrm{Irr}^{\mathrm{t}}(G)$ denotes the tempered dual of $G = GL(N)$ then

$$\mathrm{Irr}^{\mathrm{t}}(G) \cong \cup_M E_2(M)/W(M),$$

where the disjoint union is over the Levi subgroups of G, up to conjugation in G.

The analysis of the Schwartz subalgebras $\mathcal{S}(G /\!\!/ K)$ follows a similar pattern. By associating to each point σ of $E_2(M)$ the Hilbert space H_σ of the induced representation we obtain a Hilbert space bundle $H(M)$ over $E_2(M)$ on each fiber of which there is a tempered representation of G. In fact all the representation spaces over a single component of $E_2(M)$ may in a natural way be chosen the same, so that the bundle we have constructed is componentwise trivial, as is the finite-dimensional bundle $H(M)_K$ comprised of the K-fixed vectors in each fiber of $H(M)$. Only finitely many components in $H(M)$ actually contain K-fixed vectors, so $H(M)_K$ is in effect a smooth vector bundle over a finite union of components of $E_2(M)$.

Let us continue to suppose that $G = GL(N)$. The action of $W(M)$ on $E_2(M)$ lifts to a projective unitary action on the bundle $H(M)$ since if $w \in W(M)$ then the irreducible representations H_σ and $H_{w\sigma}$ are unitarily equivalent, via a unitary which is unique up to a scalar (by Schur's Lemma). In fact these intertwining operators can be chosen so as to produce a smooth unitary action of $W(M)$ on $H(M)$ (not merely a projective unitary action). The action restricts to $H(M)_K$ and a very attractive theorem of Mischenko [Mi] asserts that the representation of G on $H(M)$ induces an isomorphism

$$\mathcal{S}(G /\!\!/ K) \cong \bigoplus_M [C^\infty(\mathrm{End}(H(M)_K))]^{W(M)}.$$

Here $C^\infty(\mathrm{End}(H(M)_K))$ denotes the smooth sections of the endomorphism bundle of $H(M)_K$ and of course $[C^\infty(\mathrm{End}(H(M)_K))]^{W(M)}$ denotes the $W(M)$-invariant subalgebra of $C^\infty(\mathrm{End}(H(M /\!\!/ K)))$. The (finite) direct sum is over conjugacy classes of Levi subgroups, and as we already mentioned each bundle $H(M)_K$ is defined over a compact smooth manifold.

8.3. Lemma. (See [BP], Lemma 5.) *Denote by $E_2(M)_K \subset E_2(M)$ the compact open subset of those σ for which the representation H_σ of G has nonzero K-fixed vectors. The Fréchet algebra $[C^\infty(\mathrm{End}(H(M)_K)]^{W(M)}$ is Morita equivalent to the commutative Fréchet algebra $[C^\infty(E_2(M)_K)]^{W(M)}$ via the equivalence bimodule comprised of the invariant smooth sections of the bundle $H(M)_K$.* $\square$

Thus thanks to Mischenko's result there is a Morita equivalence

$$\mathcal{S}(G /\!\!/ K) \underset{\mathrm{Morita}}{\sim} \bigoplus_M [C^\infty(E_2(M)_K)]^{W(M)}.$$

Having reduced $\mathcal{S}(G /\!\!/ K)$ by a Morita equivalence to a commutative nuclear Fréchet algebra it is now a simple matter to apply the available tools from cyclic theory [Wa,Co1] to deduce the following result:

8.4. Theorem. $HP_{\mathrm{ev/odd}}(\mathcal{S}(G /\!\!/ K)) \cong H^{\mathrm{ev/odd}}(E_2(M)_K/W(M))$.

On the right-hand side here is the ordinary cohomology (with complex coefficients) of the compact space $E_2(M)_K/W(M)$. Taking the direct limit over all compact open subgroups K we conclude that

$$HP_{\mathrm{ev/odd}}(\mathcal{S}(G)) \cong H^{\mathrm{ev/odd}}(\mathrm{Irr}^{\mathrm{t}}(G)),$$

where on the right-hand side is cohomology with complex coefficients and *compact* supports. See Theorem 7 in [BP].

All of the above isomorphisms decompose according to the components of the Bernstein variety:

8.5. Definition. Let Ω be a Bernstein component. We denote by $\mathrm{Irr}^{\mathrm{t}}(G)_\Omega$ the Ω-component of the tempered dual of G, so that the Bernstein decomposition of the tempered dual is

$$\mathrm{Irr}^{\mathrm{t}}(G) = \cup_\Omega \mathrm{Irr}^{\mathrm{t}}(G)_\Omega.$$

8.6. Definition. Let Ω be a Bernstein component. We define

$$\mathcal{S}(G/\!/K)_\Omega = \mathcal{S}(G/\!/K) \bigcap \mathcal{S}(G)_\Omega,$$

so that $\mathcal{S}(G)_\Omega = \cup_K \mathcal{S}(G/\!/K)_\Omega$ and $\mathcal{S}(G) = \oplus_\Omega \mathcal{S}(G)_\Omega$, the latter being an algebraic direct sum. We define

$$HP_*(\mathcal{S}(G)_\Omega) = \varinjlim_K HP_*(\mathcal{S}(G/\!/K)_\Omega).$$

Remark. The directed system $\{HP_*(\mathcal{S}(G/\!/K)_\Omega)\}_{K \subset G}$ actually stabilizes since for small enough K all the algebras $\mathcal{S}(G/\!/K)$ are Morita equivalent, in a manner compatible with the inclusions in the directed system.

8.7. Proposition. *If $G = GL(N)$ and if Ω is any component of the Bernstein variety then*

$$HP_*(\mathcal{S}(G)_\Omega) \cong H^*(\mathrm{Irr}^{\mathrm{t}}(G)_\Omega).$$

What is the structure of the space $\mathrm{Irr}^{\mathrm{t}}(G)$? For $G = GL(N)$ the answer may be read from the Langlands classification for G [Ku,Ze]. We refer the reader to the article [Pl2] for details, and here simply state the result:

8.8. Theorem. *Let Ω be a component in the Bernstein variety of $GL(N)$. There is a homeomorphism*

$$\mathrm{Irr}^{\mathrm{t}}(G)_\Omega \cong \widehat{\mathbb{T}^{d(\Omega)}}/W(\Omega).$$

Here $\widehat{\mathbb{T}^{d(\Omega)}}/W(\Omega)$ is the very same extended quotient that we considered in the last section in connection with our analysis of the periodic cyclic homology for the Bernstein components $\mathcal{H}(G)_\Omega$ of the Hecke algebra. Theorems 7.8 and 8.8 show that the periodic cyclic homology groups of $\mathcal{H}(G)_\Omega$ and and $\mathcal{S}(G)_\Omega$ are abstractly isomorphic. It is of course natural to guess a stronger assertion:

8.9. Conjecture. *For any reductive p-adic group G and any Bernstein component $\Omega \subset \Omega(G)$ the inclusion $\mathcal{H}(G)_\Omega \subset \mathcal{S}(G)_\Omega$ induces an isomorphism*

$$HP_*(\mathcal{H}(G)_\Omega) \xrightarrow{\cong} HP_*(\mathcal{S}(G)_\Omega).$$

Equivalently, the inclusion $\mathcal{H}(G) \subset \mathcal{S}(G)$ induces an isomorphism

$$HP_*(\mathcal{H}(G)) \xrightarrow{\cong} HP_*(\mathcal{S}(G)).$$

Remark. The periodic cyclic homology for $\mathcal{S}(G)_\Omega$ has been defined as a direct limit, whereas no such limit was involved in the definition of $HP_*(\mathcal{H}(G)_\Omega)$. So before presenting the conjecture we should really have checked that the natural map

$$\varinjlim_K HP_*(\mathcal{H}(G/\!/K)_\Omega) \to HP_*(\mathcal{H}(G)_\Omega)$$

is an isomorphism. In fact this is so, for the following reason. For each Ω there is a compact open subgroup K_Ω with the property that every smooth G-representation E for which $E = E_\Omega$ has K_Ω-fixed vectors. If e denotes the characteristic function of K_Ω then there is a Morita equivalence $\mathcal{H}(G)_\Omega \sim e\mathcal{H}(G)_\Omega e$ induced by the bimodule $e\mathcal{H}(G)_\Omega$. It follows that the directed system $\{HP_*(\mathcal{H}(G /\!\!/ K))_\Omega\}_{K \subset G}$ stabilizes (once $K \subset K_\Omega$) to $HP_*(\mathcal{H}(G)_\Omega)$. The same argument applies to the Schwartz algebra, which justifies the remark following Definition 8.6.

As we shall explain in the next section, Conjecture 8.9 is valid for $GL(N)$.

9. K-Theory and the Baum-Connes Conjecture

The purpose of this section is to place some of the ideas we have developed in the previous sections into an extremely general context which incorporates not just reductive p-adic groups but arbitrary locally compact groups. Doing so places some of the questions we have raised into contact with a diverse collection of problems, ranging from differential topology and Riemannian geometry to real harmonic analysis and operator theory, although in this paper we can no more than hint at the relations with these other areas. The reader is referred to the paper [BCH] and [Hi], as well as Connes' extraordinary book [Co2], for more information.

Throughout this section, let G be a second countable, Hausdorff locally compact group (for instance a p-adic group, a Lie group, or a countable discrete group).[13]

9.1. Definition. The *reduced C^*-algebra* of G, denoted $C_r^*(G)$, is the completion in the operator norm of the convolution algebra $L^1(G)$, viewed as an algebra of operators on the Hilbert space $L^2(G)$.

In the p-adic case $C_r^*(G)$ is a completion of both $\mathcal{H}(G)$ and $\mathcal{S}(G)$. In the discrete case it is a completion of the complex group algebra $\mathbb{C}[G]$.

According the general point of view developed by Alain Connes, and named by him *noncommutative geometry*, the C^*-algebra $C_r^*(G)$ should be studied as if the reduced dual[14] $\hat{G}_r$ of G was a locally compact and Hausdorff space, and as if $C_r^*(G)$ was (Morita equivalent to) the C^*-algebra of continuous complex-valued functions on $\hat{G}_r$ which vanish at infinity. If G is an *abelian* group then in fact $C_r^*(G)$ identifies by a Fourier isomorphism with the C^*-algebra of continuous complex-valued functions, vanishing at infinity, on the Pontrjagin dual $\hat{G} = \hat{G}_r$. If G is compact then $C_r^*(G)$ is Morita equivalent (in the sense of C^*-algebras) to the C^*-algebra of functions vanishing at infinity on the discrete space $\hat{G}_r$. For more general groups we cannot take Connes' point of view too literally: for instance for nonabelian free groups the standard topology on the reduced unitary dual has no nontrivial open sets, so that there are no nonconstant continuous functions at all, in the ordinary

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[13]Some points of the theory to be sketched below require an additional, tiny hypothesis on G. See item (1.5) in [BCH]. Since the hypothesis is satisfied for discrete groups, for totally disconnected groups, and for Lie groups — the most interesting cases — we shall not dwell on this point any further here.

[14]This is the subset of the unitary dual comprised of representations which are 'weakly contained' in the regular representation. If G is a reductive group then the reduced dual is the same thing as the tempered dual.

sense of the term. Here $C^*_r(G)$ provides a substitute for the ordinary notions of topological structure and continuous function.

The most interesting and best developed tool with which to analyze the 'non-commutative topological structure' carried by a C^*-algebra is K-theory. We refer the reader to one of several texts (for instance [Bl]), and of course to Connes' book [Co2], for a fuller discussion. Here let us just briefly recall that the K-theory groups of a C^*-algebra are the homotopy groups

$$K_j(A) = \pi_{j-1}(GL_\infty(A)) \qquad (j = 1, 2, \dots),$$

of the stable general linear group over A, and that the famous Bott Periodicity Theorem implies that

$$K_j(A) \cong K_{j-2}(A),$$

so that by periodicity the definition of K-theory may be extended to all $j \in \mathbb{Z}$. The definition of K-theory is so arranged that if A is a commutative C^*-algebra, and is hence of the form $A = C_0(X)$ for some locally compact Hausdorff space X, then the K-theory groups of A identify with the Atiyah-Hirzebruch K-theory groups of X:

$$K_*(A) \cong K^{-*}(X).$$

Thus K-theory fits very nicely with Connes' point of view which we summarized above

Is is natural ask, what is $K_*(C^*_r(G))$? According to Connes' philosophy, the groups $K_*(C^*_r(G))$ should be viewed as the Atiyah-Hirzebruch K-theory of the reduced unitary dual of G. If G is abelian then this is not just a point of philosophy but an actual theorem. On the other hand if G is, for example, a nonabelian free group then since the classical topological structure of the reduced dual is so poor, the ordinary topological invariants are meaningless. In this case C^*-algebra K-theory has no apparent counterpart in ordinary topology.

The development of a plausible conjectural formula for $K_*(C^*_r(G))$ is a long story, starting in manifold theory and passing through the representation theory of real semisimple groups and other topics. The final form of the conjecture involves the following key idea (see [BCH] for a more complete discussion of what follows):

9.2. Definition. A G-space X is *proper* if X and the quotient space X/G are metrizable topological spaces and if for each $x \in X$ there exists a compact subgroup J of G and G-map from a G-neighborhood of x to the homogeneous space G/J. A proper G-space X is *universal* if whenever Y is a proper G-space there exists a G-map $f : Y \to X$ and any two G-maps from Y to X are G-homotopic.

For every G there exists a universal proper G-space $\mathcal{E}G$, and it is evident from the definition that this space is unique up to G-homotopy. The task of locating a universal proper G-space for a given group G is made easier by the following simple *universality criteria* on a proper space X:

- The two projections $X \times X \to X$ should be G-homotopic.
- For every compact subgroup $J \subset G$ there should be a point in X which is fixed by J

The universality conditions are guaranteed by many geometric notions of non-positive curvature. In these cases the G-homotopy in the first item is constructed using the unique geodesic connecting any two points of X, while the fixed point in the second item is obtained as the barycenter of any J-orbit in X. Thus for example the following spaces are universal, assuming that G acts properly and isometrically: trees; symmetric spaces of noncompact type; Hadamard manifolds; and affine buildings. Figure 9 gives two attractive examples for the group $SL(2, \mathbb{Z})$ along with a G-homotopy equivalence between them.
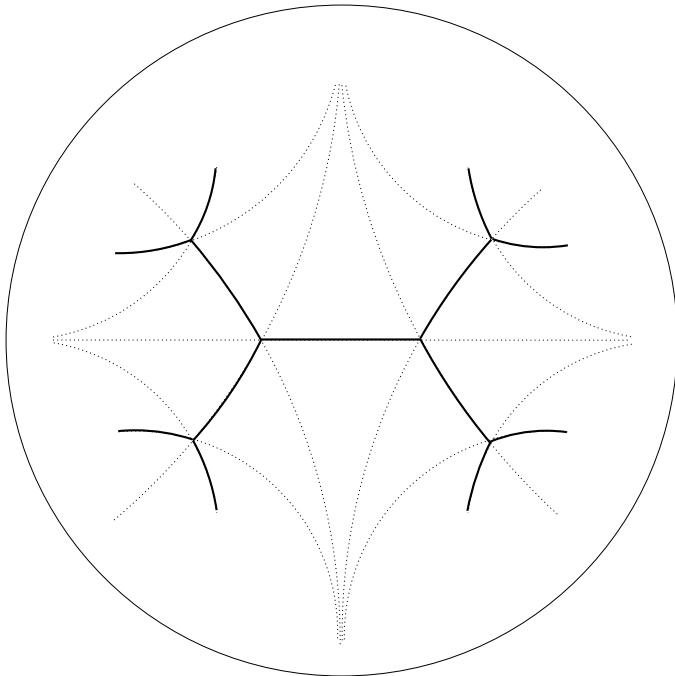

FIG. 9. Both the Poincaré disk and the embedded tree are universal proper spaces for the group $SL(2, \mathbb{Z})$. The disk retracts onto the tree using the nearest-point projection (along the dotted lines) in the Poincaré metric. This is an equivariant homotopy equivalence.

The conjectural formula for $K_*(C_r^*(G))$ involves the equivariant K-homology $K_*^G(\mathcal{E}G)$ of the space $\mathcal{E}G$. This is obtained using Kasparov's KK-theory; in the case where $\mathcal{E}G/G$ is compact (which is most relevant to us) the definition is

$$K_*^G(\mathcal{E}G) = KK_G^*(C_0(\mathcal{E}G), \mathbb{C}).$$

There is an *assembly map*

$$\mu : K_*^G(\mathcal{E}G) \to K_*(C_r^*(G))$$

which combines ideas from surgery theory (from whence it takes its name) and index theory.

9.3. The Baum-Connes Conjecture. *For every second countable, locally compact Hausdorff topological group the assembly map*

$$\mu : K_*^G(\mathcal{E}G) \to K_*(C_r^*(G))$$

is an isomorphism.

For an account of the current state of this conjecture see [Hi] (but note that the recent work of Lafforgue, discussed below, is not covered there).

How should we interpret the Baum-Connes Conjecture if G is a reductive p-adic group? The tempered dual of G is very close to Hausdorff in its natural topology, so we can expect $K_*(C_r^*(G))$ to model very closely the K-cohomology, with compact supports, of the tempered dual. Indeed we have already noted that if $G = GL(N)$ then the tempered dual really *is* a Hausdorff space; furthermore the description of the Schwartz algebra that we presented in Section 8 goes over to the reduced C^*-algebra, the only change being that the algebras $C^\infty(\text{End}(H(M)_K))$ of smooth sections are replaced with the corresponding C^*-algebras $C(\text{End}(H(M)_K))$ of continuous sections. It follows that $C_r^*(G)$ is Morita equivalent to the commutative C^*-algebra of continuous functions, vanishing at infinity, on the tempered dual of G [Pl1]. In this case, therefore, $K_*(C_r^*(G))$ is precisely $K^{-*}(\text{Irr}^t(G))$.

The K-homology of the affine building for G and the assembly map μ are more difficult to interpret directly. However there is a very simple relation between K-homology, the assembly map and the cyclic homology ideas discussed in the previous section:

9.4. Proposition. [BBH] *Let G be a reductive p-adic group and let X be its affine building. The Baum-Connes assembly map μ fits into a commutative diagram*

$$
\begin{array}{ccc}
K_*^G(X) & \xrightarrow{\ \mu\ } & K_*(C_r^*(G)) \\
{\scriptstyle\text{ch}}\downarrow & & \downarrow{\scriptstyle\text{ch}} \\
HP_*(\mathcal{H}(G)) & \longrightarrow & HP_*(\mathcal{S}(G))
\end{array}
$$

in which the vertical arrows are Chern characters and the bottom arrow is induced from the inclusion of $\mathcal{H}(G)$ into $\mathcal{S}(G)$. Upon tensoring with $\mathbb{C}$ the left vertical arrow becomes an isomorphism.

If $G = GL(N)$ then the right vertical arrow also becomes an isomorphism upon tensoring with $\mathbb{C}$. This can be seen by passing from $C_r^*(G))$ to a commutative algbra through a Morita equivalence, and then applying classical results in Atiyah-Hirzebruch K-theory. The same is probably also true for a general group G, and in view of this the Baum-Connes Conjecture amounts to essentially the same thing as Conjecture 8.9.

Some time ago Kasparov and Skandalis [KS] proved that for any reductive p-adic group G, the assembly map $\mu : K_*^G(X) \to K_*(C_r^*(G))$ is *injective*. But we noted in Sections 7 and 8 that if $G = GL(N)$ then when we decompose according to the Bernstein center the component groups $HP_*(\mathcal{H}(G)_\Omega)$ and $HP_*(\mathcal{S}(G)_\Omega)$ are

abstractly isomorphic, as finite-dimensional vector spaces. Since the natural map between them is injective, it must in fact be an isomorphism. This proves Conjecture 8.9 for $GL(N)$ and (with a little more work) also the Baum-Connes conjecture for $GL(N)$.

We close with the remarkable recent work of Lafforgue [La1,La2].

9.5. Theorem. *Let G be a reductive p-adic group. If X denotes the affine Bruhat-Tits building of G then the assembly map*

$$\mu : K_*^G(X) \to K_*(C_r^*(G))$$

is an isomorphism of abelian groups.

Lafforgue's proof is remarkable in that it is organized very directly around *geometric* ideas. While it is true that some of the fundamental harmonic analysis associated to the Schwartz algebra (as in [Si]) also plays a crucial role, the detailed smooth representation theory that we touched upon in Section 7 of this paper is entirely absent.

To go from Lafforgue's theorem to a proof of Conjecture 8.9 is probably only only a short journey. But it is far from clear that the conjectures in Sections 5 and 6 will be illuminated along the way. A proper account of these issues, hopefully at least partially based on geometrical aspects of the affine building, appears to be a challenging problem for the future.

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