Equivariant E-Theory for C*-Algebras

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Abstract. Let A and B be C^*-algebras which are equipped with continuous actions of a second countable, locally compact group G. We define a notion of equivariant asymptotic morphism, and use it to define equivariant E-theory groups $E_G(A, B)$ which generalize the E-theory groups of Connes and Higson. We develop the basic properties of equivariant E-theory, including a composition product and six-term exact sequences in both variables, and apply our theory to the problem of calculating K-theory for group C^*-algebras. Our main theorem gives a simple criterion for the assembly map of Baum and Connes to be an isomorphism. The result plays an important role in recent work of Higson and Kasparov on the Baum-Connes conjecture for groups which act isometrically and metrically properly on Hilbert space.

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Introduction

The notion of an *asymptotic morphism* between two C^*-algebras was introduced in a brief note of Connes and Higson [**11**]. An asymptotic morphism from A to B induces a homomorphism from the C^*-algebra K-theory of A to that of B. At the level of homotopy there is a composition law for asymptotic morphisms which is compatible with K-theory. The homotopy category of asymptotic morphisms so obtained is a powerful tool, closely related to Gennadi Kasparov's KK-theory [**22**], for calculating C^*-algebra K-theory groups.

The purpose of this article is to develop in some detail the theory of *equivariant* asymptotic morphisms, appropriate to C^*-algebras equipped with continuous actions of locally compact groups, and so construct tools very similar to those of Kasparov's equivariant KK-theory [**23**] for calculating the K-theory of group C^*-algebras. A central problem in C^*-algebra K-theory is the Baum-Connes conjecture [**4**], which proposes a formula for the K-theory of group C^*-algebras. A primary goal of the paper is to first formulate the conjecture in the language of asymptotic morphisms, and then describe a general method, due essentially to Kasparov, for attacking various cases of it. At present the method encompasses nearly all that is known about the Baum-Connes conjecture. The method can be implemented within either our theory or Kasparov's, but we note that the most recent progress on the conjecture [**19**] does not yet fit fully into the framework of equivariant KK-theory, so for the time being the theory of asymptotic morphisms appears to be an essential variation on Kasparov's work. We shall not describe how our approach to the Baum-Connes conjecture applies to the computation of K-theory for specific group C^*-algebras—for this we refer the reader to the paper [**19**]. Instead we concentrate on providing a reasonably conceptual framework for that and other investigations into the K-theory of group C^*-algebras.

Our starting point is a definition of asymptotic morphism which differs a little from the one introduced by Connes and Higson [**11**] and instead borrows from recent work of the same authors [**12**]. One can find in [**12**] a terse account of the material covered in the first several chapters of the present work: we hope that the more detailed treatment given here will be of use to some readers. The new definition of asymptotic morphism is presented in Chapter 1. Chapters 2–5 develop properties of the homotopy category of asymptotic morphisms and in Chapter 6 we introduce the equivariant E-theory groups $E_G(A, B)$. In Chapter 7 we summarize the homological properties of E_G-theory. In brief, these are as follows:

Composition Product. *The abelian groups $E_G(A, B)$ are the morphism sets in an additive category whose objects are the separable G-C^*-algebras. There is a functor into this category from the homotopy category of G-C^*-algebras and equivariant $*$-homomorphisms.*

Excision. *If $I \lhd A$ and $J \lhd B$ are G-C^*-ideals then the sequences*

$$E_G(A,J) \to E_G(A,B) \to E_G(A,B/J)$$
$$E_G(I,B) \leftarrow E_G(A,B) \leftarrow E_G(A/I,B)$$

are exact in the middle.

Stabilization. *If $\varphi\colon A \to B$ is a $*$-homomorphism of G-C^*-algebras and if, for some G-Hilbert space $\mathcal{H}$, the tensor product $\varphi \otimes 1\colon A \otimes \mathcal{K}(\mathcal{H}) \to B \otimes \mathcal{K}(\mathcal{H})$ is a G-homotopy equivalence (where $\mathcal{K}(\mathcal{H})$ denotes the C^*-algebra of compact operators on $\mathcal{H}$) then the morphism $[\varphi] \in E_G(A,B)$ is invertible.*

It is not hard to see that E_G is the *universal* theory with these properties. Of course, a further key property is that an (equivariant) asymptotic morphism from A to B determines an element of $E_G(A,B)$—the whole point of E_G-theory is to provide a framework for calculations involving asymptotic morphisms. Finally, for applications to C^*-algebra K-theory the following property, which concerns the full crossed product C^*-algebra $C^*(G,A)$, is crucial:

Descent. *There is a functor from the equivariant E-theory category to the non-equivariant E-theory category, mapping the class in $E_G(A,B)$ of an equivariant $*$-homomorphism from A to B to the class in $E(C^*(G,A), C^*(G,B))$ associated to the induced $*$-homomorphism from $C^*(G,A)$ to $C^*(G,B)$.*

The later chapters of our paper borrow from the second author's collaboration with Kasparov, to whom we are grateful for allowing us to include some of that joint work here. Following roughly the procedure in [**4**] we define the *Baum-Connes assembly map*

$$\mu\colon E_G(\mathcal{E}G,B) \to K_*(C^*(G,B)).$$

If G is compact then by adapting to E_G-theory a well-known argument of Green [**16**] and Julg [**21**] we prove that the assembly map is an isomorphism. In order to study assembly for non-compact groups we follow Kasparov's lead [**23**] and introduce a notion of *proper* G-C^*-algebra. A guiding principle is that the action of a non-compact group on a proper G-C^*-algebra is roughly the same as a compact group action, and with this in mind we seek to generalize the Green-Julg isomorphism to proper G-C^*-algebras. Unfortunately some technical obstacles arise, but we are at least able to prove the following result:

Generalized Green-Julg Theorem. *If G is a countable discrete group and if D is a proper G-C^*-algebra then the Baum-Connes assembly map*

$$\mu\colon E_G(\mathcal{E}G,D) \to K_*(C^*(G,D))$$

is an isomorphism.

Although we shall not go into it here, the theorem is also true for a variety of other classes of groups (for example, connected Lie groups and totally disconnected groups). However it is not clear to us that the statement is correct for general locally compact groups, particularly for groups of infinite dimension.

In any case, concentrating on discrete groups, we obtain the following important result which is central to the paper [**19**] and which might be regarded as the focus of the present article:

THEOREM. *If G is a countable discrete group and if the identity morphism $1 \in E_G(\mathbb{C}, \mathbb{C})$ factors through a proper G-C^*-algebra then, for* any *G-C^*-algebra B, the Baum-Connes assembly map*

$$\mu\colon E_G(\mathcal{E}G, B) \to K_*(C^*(G, B))$$

is an isomorphism.

The proof is very straightforward—if the identity on $\mathbb{C}$ factors through a proper G-C^*-algebra D then the assembly map for B identifies with a direct summand of the assembly map for the proper G-C^*-algebra $B \otimes D$, and by the generalized Green-Julg theorem the latter is an isomorphism.

Throughout the preceding discussion we have used the *full* crossed product C^*-algebra $C^*(G, B)$ rather than its reduced counterpart $C^*_{\mathrm{red}}(G, B)$. It is a definite shortcoming of E-theory that it is not as well adapted to the reduced crossed product as is Kasparov's KK-theory. On the other hand if a discrete group G is C^*-*exact,* in the sense that minimal tensor product with the reduced C^*-algebra $C^*_{\mathrm{red}}(G)$ preserves short exact sequences of C^*-algebras, then in the above discussion we can replace the full with the reduced crossed product. Conjecturally all discrete groups are C^*-exact, and many classes of groups are known to be so (most notably discrete subgroups of connected Lie groups), so from a practical perspective this shortcoming of E-theory is perhaps not so great. To reinforce this point, we note that in the key applications of the Baum-Connes theory to topology and geometry (via, for instance, the Novikov conjecture) it is sufficient to work with full crossed product C^*-algebras [**24**]. But the incompatibility of E-theory with reduced crossed products is nonetheless an awkward circumstance. It suggests that the machinery developed in this paper will not be the final and most suitable framework for the Baum-Connes theory—but this is a speculation which must be enlarged upon elsewhere.

While this paper was in the final stages of preparation we received from Klaus Thomsen an interesting paper [**35**] on essentially the same subject. There is less overlap between the two articles than one might expect: as we have noted above, the main emphasis of the present work is the applications of E-theory to the Baum-Connes conjecture, whereas Thomsen's article is concerned more with foundational questions concerning homology-type functors on the category of G-C^*-algebras. In fact the papers complement one another quite nicely, although we note that certain basic objects, such as the equivariant E-theory groups themselves, are defined differently in the two papers.

CHAPTER 1

Asymptotic Morphisms

Let B be a C^*-algebra. Denote by $\mathfrak{T}B$ the C^*-algebra of continuous, bounded functions from the locally compact space $T = [1, \infty)$ into B. Denote by $\mathfrak{T}_0 B$ the ideal in $\mathfrak{T}B$ comprised of continuous functions from T to B which vanish in norm at infinity.

1.1. Definition. Let A and B be C^*-algebras. The *asymptotic algebra* of B is the quotient C^*-algebra

$$\mathfrak{A}B = \mathfrak{T}B/\mathfrak{T}_0 B.$$

An *asymptotic morphism* from A to B is a $*$-homomorphism from A into the C^*-algebra $\mathfrak{A}B$.

1.2. Remark. In a moment, when we start to consider equivariant asymptotic morphisms, we shall modify the definition of $\mathfrak{A}B$ very slightly (see Definition 1.9 below).

One can extract from a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}B$ a family of functions

$$\{\varphi_t\}_{t\in[1,\infty)} : A \to B$$

by composing φ with any set-theoretic section from the quotient algebra $\mathfrak{A}B$ to $\mathfrak{T}B$, then composing with the $*$-homomorphisms from $\mathfrak{T}B$ to B given by evaluation at $t \in [1, \infty)$. The family $\{\varphi_t\}$ so obtained has the following properties:

(i) for every $a \in A$ the map $t \mapsto \varphi_t(a)$, from $[1, \infty)$ into B, is continuous and bounded; and

(ii) for every $a, a' \in A$ and $\lambda \in \mathbb{C}$,

$$\lim_{t\to\infty} \left\{ \begin{array}{c} \varphi_t(a)^* - \varphi_t(a^*) \\ \varphi_t(a) + \lambda\varphi_t(a') - \varphi_t(a + \lambda a') \\ \varphi_t(a)\varphi_t(a') - \varphi_t(aa') \end{array} \right\} = 0.$$

Conversely, a family of functions $\{\varphi_t\}_{t\in[1,\infty)} : A \to B$ satisfying these conditions determines an asymptotic morphism from A to B. Indeed if $a \in A$ then the function $t \mapsto \varphi_t(a)$ belongs to $\mathfrak{T}B$ and by associating to a the class of this function in the quotient $\mathfrak{A}B = \mathfrak{T}B/\mathfrak{T}_0 B$ we obtain a $*$-homomorphism from A into $\mathfrak{A}B$.

1.3. Definition. Let A and B be C^*-algebras. An *asymptotic family* mapping A to B is a family of functions

$$\{\varphi_t\}_{t\in[1,\infty)} : A \to B$$

satisfying the conditions (i) and (ii) above. Two asymptotic families $\{\varphi_t\}, \{\psi_t\} : A \to B$ are *equivalent* if $\lim_{t\to\infty}(\varphi_t(a) - \psi_t(a)) = 0$, for all $a \in A$.

The following result is clear from the above discussion:

1.4. Proposition. *There is a one-to-one correspondence between asymptotic morphisms from A to B and equivalence classes of asymptotic families $\{\varphi_t\}_{t\in[1,\infty)}: A \to B$.* $\square$

1.5. Remark. Our definition of asymptotic family is virtually the same as the original definition of asymptotic morphism ([**11**, Section 2] or [**10**]). We have added to the original the requirement that $\varphi_t(a)$ be a bounded function of t. In fact boundedness follows from the other parts of the definition of asymptotic family, but since the proof is not altogether simple (the one suggested in [**11**] is incomplete) it seems simpler to incorporate boundedness into our definition.

Every $*$-homomorphism from A to B determines an asymptotic morphism from A to B by means of the following device:

1.6. Definition. If B is a C^*-algebra then denote by $\alpha_B: B \to \mathfrak{A}B$ the $*$-homomorphism which associates to $b \in B$ the class in $\mathfrak{A}B$ of the constant function $t \longmapsto b \in \mathfrak{T}B$.

Thus if $\varphi: A \to B$ is a $*$-homomorphism then composing with α_B we obtain an asymptotic morphism from A to B. Of course, all we are doing here is constructing from φ the constant asymptotic family $\{\varphi_t = \varphi\}_{t\in[1,\infty)}: A \to B$. This idea is slightly generalized by the observation that a continuous family of $*$-homomorphisms $\{\varphi_t\}_{t\in[1,\infty)}: A \to B$ defines an asymptotic family, and hence an asymptotic morphism from A to B.

The most important feature of asymptotic morphisms is that they induce homomorphisms of C^*-algebra K-theory groups. From an asymptotic morphism $\varphi: A \to \mathfrak{A}B$ we obtain a homomorphism of abelian groups

$$\varphi_*: K_*(A) \to K_*(B),$$

in such a way that if φ is actually a $*$-homomorphism from A to B then φ_* is the usual induced map on K-theory groups. To see how this comes about, let φ be an asymptotic morphism from A to B and for simplicity consider a class in $K_0(A)$ represented by a projection $p \in A$. Let $\{\varphi_t\}_{t\in[1,\infty)} : A \to B$ be an asymptotic family corresponding to φ and consider the norm-continuous family of elements $f_t = \varphi_t(p)$ in B. It has the property that

$$\lim_{t\to\infty} \left\{ \begin{array}{c} f_t^2 - f_t \\ f_t^* - f_t \end{array} \right\} = 0.$$

By an easy application of the functional calculus for C^*-algebras, there is a norm-continuous family of actual projections $e_t \in B$ such that $\lim_{t\to\infty}(e_t - f_t) = 0$. The projections e_t define a common class $[e] \in K_0(B)$ and we define

$$K_0(A) \ni [p] \xrightarrow{\varphi_*} [e] \in K_0(B).$$

To give a fuller description of the induced map on K-theory, applicable to all classes in $K_0(A)$ as well as all classes in $K_1(A)$, we note that the ideal $\mathfrak{T}_0B \lhd \mathfrak{T}B$ is contractible; that is, it is homotopy equivalent in the C^*-algebra sense to the zero C^*-algebra (see the next chapter for a quick review of homotopy for C^*-algebras). It follows [**6,**Chapter 4] that the projection map $\mathfrak{T}B \to \mathfrak{A}B$ induces an isomorphism in K-theory, and we define $\varphi_*: K_*(A) \to K_*(B)$ to be the composition

$$K_*(A) \xrightarrow{\varphi} K_*(\mathfrak{A}B) \xleftarrow{\cong} K_*(\mathfrak{T}B) \xrightarrow[\text{at } t=1]{\text{Evaluation}} K_*(B).$$

Again, if $\varphi : A \to B$ is a $*$-homomorphism, viewed as an asymptotic morphism via α_B as above, then this construction gives the usual induced map on K-theory.

If φ is an asymptotic morphism from A to B, and if ψ is a $*$-homomorphism from A_1 to A then we can compose ψ with φ to obtain an asymptotic morphism from A_1 to B. Similarly, a $*$-homomorphism from B to B_1 induces a $*$-homomorphism from $\mathfrak{A}B$ to $\mathfrak{A}B_1$ (the functoriality of the asymptotic algebra will be considered more fully in the next chapter) and once again we can compose with φ to obtain an asymptotic morphism from A to B_1. It is easy to check that the induced map on K-theory just defined is compatible with these compositions, in the obvious sense.

The main achievement of the theory of asymptotic morphisms is the construction of a composition operation for a pair of asymptotic morphisms (not just one asymptotic morphism and one $*$-homomorphism, as we have just considered) which is compatible with the composition of induced maps on K-theory. This is not a simple matter; for instance the most obvious attempt to compose asymptotic families would be to form $\varphi_t \circ \psi_t$, but this fails to produce an asymptotic family. In fact the correct definition of composition is only well defined up to a suitable notion of homotopy for asymptotic morphisms. The construction will be given in the next chapter.

We now consider the definition of an *equivariant* asymptotic morphism between two G-C^*-algebras. Throughout the paper we shall denote by G a locally compact, second countable, Hausdorff topological group. In the later chapters we will limit ourselves to consideration of discrete groups, but for the next several chapters this restriction will not apply. In several places we could assume less of G—for instance we could drop the hypothesis of second countability—but for simplicity we do not do so.

1.7. Definition. A *G-C^*-algebra* is a C^*-algebra A equipped with a continuous action $G \times A \to A$ by $*$-automorphisms. An *equivariant asymptotic morphism* from one G-C^*-algebra A to another one B is an equivariant $*$-homomorphism from A to the asymptotic C^*-algebra $\mathfrak{A}B$.

It should be noted that while the action of G on B passes in a natural way to an action by $*$-automorphisms on the asymptotic algebra $\mathfrak{A}B$, this action is seldom continuous; G acts continuously on $\mathfrak{T}_0B$ but not on $\mathfrak{T}B$ or $\mathfrak{A}B$. Nevertheless, an asymptotic morphism from A to B necessarily maps A into the C^*-subalgebra comprised of elements $b \in \mathfrak{A}B$ which *are* G-continuous, in the sense that the map $g \mapsto g(b)$ is continuous from G to B. Indeed it is clear that any equivariant $*$-homomorphism between C^*-algebras equipped with actions of G by $*$-automorphisms must map G-continuous elements to G-continuous elements. This prompts us to alter the definition of $\mathfrak{A}B$ a little. Note first the following not altogether simple fact:

1.8. Lemma. *If a bounded continuous function $f\colon T \to B$ determines a G-continuous element of the asymptotic algebra $\mathfrak{A}B$ then f is a G-continuous element of $\mathfrak{T}B$.*

Proof. We must show that for all $\varepsilon > 0$ there exists a neighborhood U of the identity in G such that

$$\|g(f(t)) - f(t)\| < \varepsilon, \quad \text{for all } t \in T \text{ and } g \in U.$$

For this, it suffices to show that for every $m \in \mathbb{N}$ there is a neighborhood U of the identity in G and some $n \in \mathbb{N}$ such that

$$\sup_{t \geq n} \|g(f(t)) - f(t)\| \leq \frac{1}{m}, \qquad \text{for all } g \in U.$$

For each $m \in \mathbb{N}$ and $n \in \mathbb{N}$ define a closed subset W_{mn} of G by

$$W_{mn} = \{\, g \in G : \sup_{t \geq n} \|g(f(t)) - f(t)\| \leq 1/2m \,\}.$$

Our hypothesis on f amounts to the assertion that for every m, the union $\cup_{n=1}^{\infty} W_{mn}$ contains a neighborhood of the identity element of G. For every m, one of the sets W_{mn} ($n \in \mathbb{N}$) must therefore contain a non-empty open subset of G. This follows for instance from the Baire category theorem, although it is really the local compactness of G which is of significance. But if W_{mn_m} contains a non-empty open set then $\sup_{t \geq n_m} \|g(f(t)) - f(t)\| \leq 1/m$ for all $g \in W_{mn_m} W_{mn_m}^{-1}$, and this set contains a neighborhood of the identity of G. $\square$

1.9. Definition. Let B be a G-C^*-algebra. We henceforth denote by $\mathfrak{T}B$ the C^*-algebra of G-*continuous*, continuous and bounded functions from $T = [1, \infty)$ to B and by $\mathfrak{A}B$ the quotient of this C^*-algebra by the ideal of continuous functions from T to B which vanish at infinity.

In view of Lemma 1.8, the new asymptotic algebra $\mathfrak{A}B$ is just the C^*-subalgebra of G-continuous elements in the former asymptotic algebra of B. If G is a discrete group then the definition of $\mathfrak{A}B$ is not changed.

We conclude by setting down the obvious notion of equivariant asymptotic family, and its relation to the notion of asymptotic morphism.

1.10. Definition. An asymptotic family $\{\varphi_t\}_{t \in [1,\infty)} : A \to B$ is *equivariant* if

$$\lim_{t \to \infty} \|\varphi_t(g(a)) - g(\varphi_t(a))\| = 0,$$

for all $a \in A$ and $g \in G$.

1.11. Proposition. *Let A and B be G-C^*-algebras. There is a one-to-one correspondence between equivariant asymptotic morphisms from A to B and equivalence classes of equivariant asymptotic families $\{\varphi_t\}_{t \in [1,\infty)} : A \to B$.* $\square$

CHAPTER 2

The Homotopy Category of Asymptotic Morphisms

The purpose of this chapter is to construct a category $\mathfrak{A}$ whose objects are the G-C^*-algebras, and such that every equivariant asymptotic morphism determines a morphism in $\mathfrak{A}$. If A is a separable G-C^*-algebra then a morphism from A into a G-C^*-algebra B will be a homotopy class (defined below) of an equivariant asymptotic morphism from A to B. If A is not separable this need not be so.

The category $\mathfrak{A}$ will come equipped with a functor from the homotopy category of G-C^*-algebras, and we begin our construction of $\mathfrak{A}$ with a brief review of the notion of homotopy for C^*-algebras.

For the rest of this chapter the term 'C^*-algebra' will mean 'G-C^*-algebra' and the term '$*$-homomorphism' will mean 'equivariant $*$-homomorphism.'

2.1. Definition. If $I = [a, b]$ is a closed interval then let

$$IB = \{\, f\colon I \to B \mid f \text{ is continuous} \,\}.$$

Two $*$-homomorphisms $\varphi_0, \varphi_1\colon A \to B$ are *homotopic* if there is a closed interval I and a $*$-homomorphism $\varphi\colon A \to IB$ from which φ_0 and φ_1 can be recovered by composing with evaluation at the two endpoints of I.

It is easy to see that homotopy is an equivalence relation which is compatible with composition of $*$-homomorphisms. The *homotopy category of C^*-algebras* is the category whose objects are C^*-algebras and whose morphisms are homotopy classes of $*$-homomorphisms.

We are going to define a notion of homotopy for asymptotic morphisms, but before doing so it is convenient to note that the correspondence $B \mapsto \mathfrak{A}B$ is a functor on the category of C^*-algebras. Indeed a $*$-homomorphism $\varphi\colon B_1 \to B_2$ gives rise to a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{T}_0 B_1 \arrow[r] \arrow[d] & \mathfrak{T} B_1 \arrow[r] \arrow[d] & \mathfrak{A} B_1 \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \mathfrak{T}_0 B_2 \arrow[r] & \mathfrak{T} B_2 \arrow[r] & \mathfrak{A} B_2 \arrow[r] & 0,
\end{tikzcd}$$

in which the two leftmost vertical maps are given by composing a function $T \to B_1$ with φ, and the rightmost vertical map is induced from these two.

Denote by $\mathfrak{A}^n$ the n-fold composition of the functor $\mathfrak{A}$ with itself. It is convenient to denote by $\mathfrak{A}^0$ the identity functor.

2.2. Definition. Two $*$-homomorphisms $\varphi_0, \varphi_1\colon A \to \mathfrak{A}^n B$ are *n-homotopic* if there is a closed interval I and a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^n IB$ from which φ_0 and φ_1 can recovered upon composing with evaluation at the endpoints of I.

Two asymptotic morphisms from A to B are *homotopic* if they are 1-homotopic as $*$-homomorphisms from A to $\mathfrak{A}B$.

The special cases $n = 0$ and $n = 1$ are worth mentioning explicitly; when $n = 0$ we recover the notion of homotopy of $*$-homomorphisms and when $n = 1$ we obtain the notion of homotopy of asymptotic families as originally defined by Connes and Higson [**11**, Section 2].

A homotopy of asymptotic morphisms from A to B is not the same thing as a homotopy of $*$-homomorphisms from A to $\mathfrak{A}B$. For instance, a continuous family $\{\varphi_t\}_{t\in[1,\infty)}$ of $*$-homomorphisms from A to B determines an asymptotic morphism which is homotopic, as an asymptotic morphism, to the constant family $\{\varphi_1\}_{t\in[1,\infty)}$. But this asymptotic morphism will rarely be homotopic to φ_1 when considered as a $*$-homomorphism from A to $\mathfrak{A}B$, .

2.3. Proposition. *The relation of n-homotopy is an equivalence relation on the set of $*$-homomorphisms from A to $\mathfrak{A}^n B$.*

To prove the proposition we shall use the following calculations.

2.4. Lemma. *If J is an ideal in a C^*-algebra B then the sequence*

$$0 \to \mathfrak{A}^n J \to \mathfrak{A}^n B \to \mathfrak{A}^n (B/J) \to 0$$

is exact. In particular, a surjection of C^*-algebras induces a surjection of asymptotic algebras.

Proof. This is proved by induction on n and a diagram chase, once it is shown that $\mathfrak{T}B$ and $\mathfrak{T}_0 B$ preserve short exact sequences. The only difficult point is to show that the map $\mathfrak{T}B \to \mathfrak{T}(B/J)$ is surjective. This follows by a partition of unity argument on T from the following assertion: for all $\varepsilon_1, \dots, \varepsilon_k > 0$, all compact sets $K_1, \dots, K_k \subset G$, every finite closed interval I, and every $f \in I(B/J)$ such that $\|f - g(f)\| < \varepsilon_j$, for all $g \in K_j$, there is a lift $\tilde{f} \in IB$ such that $\|\tilde{f} - g(\tilde{f})\| < 2\varepsilon_j$, for all $g \in K_j$. To prove the assertion, we use the fact there is an approximate unit $\{u_\lambda\}$ for J such that $\|g(u_\lambda) - u_\lambda\| \to 0$ as $\lambda \to \infty$, uniformly over compact sets in G. This is due to Kasparov [**23**, Lemma 1.4]; a proof will be given in Chapter 5. Granted the existence of $\{u_\lambda\}$, and given f as above, let $\tilde{f}_0$ be any lifting then define $\tilde{f} = (1 - u_\lambda)\tilde{f}_0$, for a sufficiently large λ. $\square$

2.5. Lemma. *Let $\varphi_1 \colon B_1 \to B$ and $\varphi_2 \colon B_2 \to B$, be $*$-homomorphisms, one of which is surjective, and let*

$$B_1 \underset{B}{\oplus} B_2 = \{\, b_1 \oplus b_2 \in B_1 \oplus B_2 \,:\, \varphi_1(b_1) = \varphi_2(b_2) \,\}.$$

The natural $*$-homomorphism

$$\mathfrak{A}^n (B_1 \underset{B}{\oplus} B_2) \to \mathfrak{A}^n B_1 \underset{\mathfrak{A}^n B}{\oplus} \mathfrak{A}^n B_2,$$

induced from the projections of $B_1 \underset{B}{\oplus} B_2$ onto B_1 and B_2, is an isomorphism.

Proof. According to the previous lemma, the functor $\mathfrak{A}$ transforms surjections to surjections, so by an induction argument it suffices to prove the present lemma for $n = 1$. Our surjectivity hypothesis ensures that the quotient map in the sequence

$$0 \to \mathfrak{T}_0 B_1 \underset{\mathfrak{T}_0 B}{\oplus} \mathfrak{T}_0 B_2 \to \mathfrak{T}B_1 \underset{\mathfrak{T}B}{\oplus} \mathfrak{T}B_2 \to \mathfrak{A}B_1 \underset{\mathfrak{A}B}{\oplus} \mathfrak{A}B_2 \to 0$$

is indeed surjective, and hence that the above is a short exact sequence. Consider now the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{T}_0 B_1 \underset{\mathfrak{T}_0 B}{\oplus} \mathfrak{T}_0 B_2 \arrow[r] \arrow[d, "\cong"'] & \mathfrak{T} B_1 \underset{\mathfrak{T} B}{\oplus} \mathfrak{T} B_2 \arrow[r] \arrow[d, "\cong"'] & \mathfrak{A} B_1 \underset{\mathfrak{A} B}{\oplus} \mathfrak{A} B_2 \arrow[r] \arrow[d, dotted, "\cong"'] & 0 \\
0 \arrow[r] & \mathfrak{T}_0(B_1 \underset{B}{\oplus} B_2) \arrow[r] & \mathfrak{T}(B_1 \underset{B}{\oplus} B_2) \arrow[r] & \mathfrak{A}(B_1 \underset{B}{\oplus} B_2) \arrow[r] & 0,
\end{tikzcd}$$

where the two leftmost vertical arrows send $f_1 \oplus f_2$ to the function $f(t) = f_1(t) \oplus f_2(t)$, and the rightmost vertical arrow is induced from the other two. The rightmost arrow is inverse to the $*$-homomorphism that we are asked to prove is an isomorphism. $\square$

Proof of Proposition 2.3. Reflexivity and symmetry of the n-homotopy relation are straightforward. We concentrate on transitivity.

Suppose that φ_0 is homotopic to φ_1 via an n-homotopy $\Phi_1 \colon A \to \mathfrak{A}^n I_1 B$ and that φ_1 is n-homotopic to φ_2 via $\Phi_2 \colon A \to \mathfrak{A}^n I_2 B$. We can assume that I_1 and I_2 are consecutive intervals on the real line, whose union is a third closed interval I. The $*$-homomorphisms Φ_1 and Φ_2 determine a $*$-homomorphism into the pullback C^*-algebra

$$\mathfrak{A}^n I_1 B \underset{\mathfrak{A}^n B}{\oplus} \mathfrak{A}^n I_2 B,$$

where $I_1 B$ is mapped to B by evaluation at the rightmost endpoint of I_1 and $I_2 B$ is mapped to B by evaluation at the leftmost endpoint. By Lemma 2.5, this C^*-algebra is isomorphic to $\mathfrak{A}^n(I_1 B \underset{B}{\oplus} I_2 B)$, and using the fact that $I_1 B \underset{B}{\oplus} I_2 B \cong IB$ we obtain a $*$-homomorphism from A into $\mathfrak{A}^n IB$ which implements an n-homotopy between φ_0 and φ_1. $\square$

2.6. Definition. Denote by $[\![A, B]\!]_n$ the n-homotopy classes of $*$-homomorphisms from A to $\mathfrak{A}^n B$.

Note that $[\![A, B]\!]_0$ is the set of homotopy classes of $*$-homomorphisms and $[\![A, B]\!]_1$ is the set of homotopy classes of asymptotic morphisms. Also, each $[\![A, B]\!]_n$ is a pointed set with the zero $*$-homomorphism $A \to \mathfrak{A}^n B$ as the basepoint.

We now assemble the sets $[\![A, B]\!]_n$, for $n \in \mathbb{N}$, into a single set $[\![A, B]\!]$.

Let $\alpha_B \colon B \to \mathfrak{A} B$ be the $*$-homomorphism described in Definition 1.6, which associates to each element in B the class of the corresponding constant function from T to B. Note that α defines a natural transformation from the identity functor on the category of C^*-algebras to the functor $\mathfrak{A}$. Given a $*$-homomorphism $\varphi \colon A \to \mathfrak{A}^n B$, form the composition

$$A \xrightarrow{\varphi} \mathfrak{A}^n B \xrightarrow{\mathfrak{A}^n(\alpha_B)} \mathfrak{A}^{n+1} B.$$

It follows from the functoriality of $\mathfrak{A}$ and the naturality of α_B that the $(n+1)$-homotopy class of the composition depends only on the n-homotopy class of φ. We obtain a map

$$[\![A, B]\!]_n \xrightarrow[\text{with } \mathfrak{A}^n(\alpha_B)]{\text{Composition}} [\![A, B]\!]_{n+1}.$$

2.7. Definition. Denote by $[\![A, B]\!]$ the direct limit of the system of pointed sets

$$[\![A, B]\!]_0 \to [\![A, B]\!]_1 \to [\![A, B]\!]_2 \to \cdots,$$

obtained from the linking maps $[\![A, B]\!]_n \to [\![A, B]\!]_{n+1}$ just described.

There is a second way of constructing a directed system, namely by associating to a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^n B$ the composition

$$A \xrightarrow{\varphi} \mathfrak{A}^n B \xrightarrow{\alpha_{\mathfrak{A}^n B}} \mathfrak{A}^{n+1} B.$$

In view of the commuting diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha_A"'] & \mathfrak{A}^n B \arrow[d, "\alpha_{\mathfrak{A}^n B}"] \\ \mathfrak{A}A \arrow[r, "\mathfrak{A}(\varphi)"'] & \mathfrak{A}^{n+1} B, \end{tikzcd}$$

this is the same as the composition

$$A \xrightarrow{\alpha_A} \mathfrak{A}A \xrightarrow{\mathfrak{A}(\varphi)} \mathfrak{A}^{n+1} B.$$

¿From this observation and the functoriality of $\mathfrak{A}$ it is easy to see that composition with $\alpha_{\mathfrak{A}^n B}$ maps n-homotopy classes to $(n+1)$-homotopy classes.

2.8. Proposition. *The maps*

$$[\![A, B]\!]_n \xrightarrow[\text{with } \mathfrak{A}^n(\alpha_B)]{\text{Composition}} [\![A, B]\!]_{n+1}$$

and

$$[\![A, B]\!]_n \xrightarrow[\text{with } \alpha_{\mathfrak{A}^n B}]{\text{Composition}} [\![A, B]\!]_{n+1}$$

are equal.

Before proving this, let us assemble some useful facts concerning n-homotopy.

2.9. Lemma.
(i) *If $\varphi_0, \varphi_1\colon A \to \mathfrak{A}^k B$ are k-homotopic then the $*$-homomorphisms*

$$\mathfrak{A}^j(\varphi_0), \mathfrak{A}^j(\varphi_1)\colon \mathfrak{A}^j A \to \mathfrak{A}^{j+k} B$$

are $(j+k)$-homotopic.
(ii) *If $\varphi_0, \varphi_1\colon A \to \mathfrak{A}^k D$ are k-homotopic $*$-homomorphisms, and if $D = \mathfrak{A}^j B$, then the $*$-homomorphisms $\varphi_0, \varphi_1\colon A \to \mathfrak{A}^{j+k} B$ are $(j+k)$-homotopic.*

Proof. Part (i) is immediate. Part (ii) is a consequence of the fact that if I is any closed interval then there is a $*$-homomorphism from $I\mathfrak{A}B$ into $\mathfrak{A}IB$ such that the diagram

$$\begin{tikzcd} I\mathfrak{A}B \arrow[r] \arrow[d, "\text{eval. at } t"'] & \mathfrak{A}IB \arrow[d, "\text{eval. at } t"] \\ \mathfrak{A}B \arrow[r, "="'] & \mathfrak{A}B \end{tikzcd}$$

commutes. To construct it, consider the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I\mathfrak{T}_0 B & \longrightarrow & I\mathfrak{T}B & \longrightarrow & I\mathfrak{A}B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \vdots & & \\ 0 & \longrightarrow & \mathfrak{T}_0 IB & \longrightarrow & \mathfrak{T}IB & \longrightarrow & \mathfrak{A}IB & \longrightarrow & 0 \end{array}$$

in which the two leftmost vertical arrows map a function $f\colon I \to \mathfrak{T}B$ to the function $\hat{f}\colon T \to IB$ defined by $\hat{f}(t)(s) = f(s)(t)$. The induced $*$-homomorphism on $I\mathfrak{A}B$ is the one we require. An iteration produces a similar $*$-homomorphism $I\mathfrak{A}^j B \to \mathfrak{A}^j IB$, so from a k-homotopy $\varphi\colon A \to \mathfrak{A}^k ID$, where $D = \mathfrak{A}^j B$, we obtain the $(j+k)$-homotopy

$$A \xrightarrow{\varphi} \mathfrak{A}^k I\mathfrak{A}^j B \to \mathfrak{A}^k \mathfrak{A}^j IB. \quad \square$$

2.10. Remark. In the next chapter we shall formalize the above 'commuting diagram' argument in order to limit its further repetition in the paper.

Let us also establish the following conventions concerning the description of elements in $\mathfrak{T}^2 D$ and $\mathfrak{A}^2 D$.

2.11. Notation. Let B be a C^*-algebra. An element $F \in \mathfrak{T}B$ is a function from $T = [1, \infty)$ into B, and we denote its value at t_1 by $F(t_1) \in B$. If B itself is a function algebra, say $B = \mathfrak{T}C$, then we may evaluate $F(t_1)$ at $t_2 \in T$; we shall denote the result by $F(t_1, t_2) \in C$. Finally, if $C = ID$ then we may evaluate $F(t_1, t_2)$ at $s \in I$; the result will be denoted by $F(t_1, t_2, s) \in D$. In this way $\mathfrak{T}^2 ID$ is identified identified with a subalgebra of the C^*-algebra of bounded continuous functions of three variables.

It is important to note that not *all* bounded continuous functions of three variables lie in $\mathfrak{T}^2 ID$. In addition to the G-continuity requirement discussed previously, an additional equicontinuity requirement must be satisfied; namely, for every $t_1 \in T$ and every $\varepsilon > 0$ there must be some $\delta > 0$ such that

$$|t_1' - t_1| < \delta \quad \Rightarrow \quad \|F(t_1, t_2, s) - F(t_1', t_2, s)\| < \varepsilon,$$

for all $t_2 \in T$ and $s \in I$.

The C^*-algebra $\mathfrak{A}^2 ID$ is the quotient of $\mathfrak{T}^2 ID$ corresponding to the C^*-seminorm

$$\|F\|_{\mathfrak{A}^2} = \limsup_{t_1 \to \infty} \limsup_{t_2 \to \infty} \sup_s \|F(t_1, t_2, s)\| \qquad (F \in \mathfrak{T}^2 ID).$$

The C^*-algebra $\mathfrak{A}^2 D$ may be described similarly by omitting the variable s in the above formula. The C^*-algebra $\mathfrak{A}D$ is the quotient of $\mathfrak{T}D$ corresponding to the C^*-seminorm

$$\|f\|_{\mathfrak{A}} = \limsup_{t \to \infty} \|f(t)\| \qquad (f \in \mathfrak{T}D).$$

Proof of Proposition 2.8. It suffices to show that the $*$-homomorphisms

$$\mathfrak{A}^n(\alpha_B),\ \alpha_{\mathfrak{A}^n B}\colon \mathfrak{A}^n B \to \mathfrak{A}^{n+1} B$$

are $(n+1)$-homotopic. By considering the chain of maps

$$\alpha_{\mathfrak{A}^n B},\ \mathfrak{A}(\alpha_{\mathfrak{A}^{n-1} B}),\ \mathfrak{A}^2(\alpha_{\mathfrak{A}^{n-2} B}),\ \ldots,\ \mathfrak{A}^n(\alpha_B),$$

and using Lemma 2.9 along with the transitivity of the $(n+1)$-homotopy relation, the proof is reduced to the assertion that for any C^*-algebra D the $*$-homomorphisms

$$\mathfrak{A}(\alpha_D), \alpha_{\mathfrak{A}D}\colon \mathfrak{A}D \to \mathfrak{A}^2 D$$

are 2-homotopic. This is what we shall prove.

The $*$-homomorphism $\alpha_{\mathfrak{A}D}\colon \mathfrak{A}D \to \mathfrak{A}^2 D$ is induced from the $*$-homomorphism of $\mathfrak{T}D$ into $\mathfrak{T}^2 D$ which maps a function $f \in \mathfrak{T}D$ to the two variable function $F(t_1, t_2) = f(t_2)$. Similarly the $*$-homomorphism $\mathfrak{A}(\alpha_D)\colon \mathfrak{A}D \to \mathfrak{A}^2 D$ is induced from the $*$-homomorphism which maps a function $f \in \mathfrak{T}D$ to the two variable function $F(t_1, t_2) = f(t_1)$. Let I be the unit interval, and define a 2-homotopy $\mathfrak{A}D \to \mathfrak{A}^2 ID$ by mapping $f \in \mathfrak{T}D$ to the function

$$F(t_1, t_2, s) = \begin{cases} f(t_1) & \text{if } t_1 > st_2 \\ f(st_2) & \text{if } t_1 \leq st_2. \end{cases}$$

One checks that F satisfies the appropriate uniformity conditions in Remark 2.11 and so determines an element of $\mathfrak{A}^2 ID$, and that $\|F\|_{\mathfrak{A}^2} \leq \|f\|_{\mathfrak{A}}$. Therefore the formula defines a $*$-homomorphism from $\mathfrak{A}D$ into $\mathfrak{A}^2 ID$. Evaluating at $s = 0$ we get $\mathfrak{A}(\alpha_D)$. Evaluating at $s = 1$, we see that $F(t_1, t_2, 1)$ and $f(t_2)$ agree for $t_2 \geq t_1$. Bearing in mind the formula for the norm $\| \|_{\mathfrak{A}^2}$ on $\mathfrak{T}^2 D$ given in Remark 2.11, we get $\alpha_{\mathfrak{A}D}$, as required. □

We are ready now to organize the sets $[\![A, B]\!]$ into the morphism sets of a category.

2.12. Proposition. *Given $*$-homomorphisms $\varphi\colon A \to \mathfrak{A}^j B$ and $\psi\colon B \to \mathfrak{A}^k C$, the construction*

$$A \xrightarrow{\varphi} \mathfrak{A}^j B \xrightarrow{\mathfrak{A}^j(\psi)} \mathfrak{A}^{j+k} C,$$

defines an associative composition law

$$[\![A, B]\!] \times [\![B, C]\!] \to [\![A, C]\!].$$

The identity $*$-homomorphisms $B \to \mathfrak{A}^0 B$ serve as left and right identity elements for this composition law.

Proof. Fix a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^j B$. The fact that composition with φ gives a well defined map from $[\![B, C]\!]$ to $[\![A, C]\!]$ is then a simple check of definitions. If we fix a $*$-homomorphism $\psi\colon B \to \mathfrak{A}^k(C)$ then the fact that composition with ψ gives a well defined map from $[\![A, B]\!]$ to $[\![A, C]\!]$ is a consequence of Proposition 2.8, which allows us to alter the procedure for constructing the direct limits $[\![A, B]\!] = \varinjlim [\![A, B]\!]_n$ and $[\![A, C]\!] = \varinjlim [\![A, C]\!]_n$. Associativity is immediate, since both ways of composing a triple of $*$-homomorphisms

$$\varphi\colon A \to \mathfrak{A}^j B, \quad \psi\colon B \to \mathfrak{A}^k(C), \quad \text{and} \quad \theta\colon C \to \mathfrak{A}^l D$$

produce the same $*$-homomorphism

$$A \xrightarrow{\varphi} \mathfrak{A}^j B \xrightarrow{\mathfrak{A}^j \psi} \mathfrak{A}^{j+k} C \xrightarrow{\mathfrak{A}^{j+k}\theta} \mathfrak{A}^{j+k+l} D.$$

It is clear that the identity $*$-homomorphisms serve as identity morphisms, as required. □

2.13. Definition. The *homotopy category of asymptotic morphisms* is the category $\mathfrak{A}$ whose objects are C^*-algebras and whose morphism sets are

$$\mathrm{Hom}_{\mathfrak{A}}(A, B) = [\![A, B]\!].$$

The law of composition of morphisms is given by Proposition 2.10.

2.14. Remark. Every $*$-homomorphism $\varphi\colon A \to B$ determines a morphism $[\varphi]\colon A \to B$ in the category $\mathfrak{A}$. The correspondence $\varphi \mapsto [\varphi]$ is a functor from the homotopy category of C^*-algebras into $\mathfrak{A}$.

2.15. Remark. An important special case of composition in $\mathfrak{A}$ occurs when one of the maps is a $*$-homomorphism. If $\varphi\colon A \to B$ is a $*$-homomorphism and if $\psi\colon B \to \mathfrak{A}C$ is an asymptotic morphism the composition of $[\varphi]$ with $[\psi]$ is the class of the composition $\psi \circ \varphi\colon A \to \mathfrak{A}C$. If $\varphi\colon A \to \mathfrak{A}B$ is an asymptotic morphism and $\psi\colon B \to C$ is a $*$-homomorphism the composition of $[\varphi]$ with $[\psi]$ is the class of the composition $\mathfrak{A}(\psi) \circ \varphi : A \to \mathfrak{A}C$ formed using the functoriality of the asymptotic algebra.

Note that in contrast to the theory of asymptotic morphisms developed in [**11**] we have not required our C^*-algebras to be separable. In fact it was rather important to consider separable and non-separable C^*-algebras at once, since $\mathfrak{A}B$ is non-separable (unless of course $B = 0$). What we gain from our approach is a simplified procedure for forming the composition of two asymptotic morphisms. The price we pay is that the homotopy sets $[\![A, B]\!]$ are, on the face of it, complicated direct limits. But if we restrict our attention to separable C^*-algebras we can recover the theory developed in [**11**].

2.16. Theorem. *If A is a separable C^*-algebra then the natural map*

$$[\![A, B]\!]_1 \to \varinjlim_n [\![A, B]\!]_n$$

is a bijection. Thus $[\![A, B]\!]$ is isomorphic to the set of homotopy equivalence classes of asymptotic morphisms from A to B.

The proof relies on a technical lemma. We shall state the lemma; prove the theorem; then prove the lemma.

2.17. Lemma.
(i) *Let D be a C^*-algebra and let $E \subset \mathfrak{A}^2 D$ be a separable C^*-subalgebra. There is a $*$-homomorphism $\psi\colon E \to \mathfrak{A}D$ such that the diagram:*

$$\begin{tikzcd} E \arrow[rr, "\text{inclusion}"] \arrow[dr, "\psi"'] & & \mathfrak{A}^2 D \\ & \mathfrak{A}D \arrow[ur, "\alpha_{\mathfrak{A}D}"'] & \end{tikzcd}$$

commutes up to 2-homotopy.
(ii) *Let $\varepsilon\colon D_1 \to D_2$ be a $*$-homomorphism of C^*-algebras; let E_1 be a separable C^*-subalgebra of $\mathfrak{A}^2 D_1$ and let E_2 be its image under ε in $\mathfrak{A}^2 D_2$. Suppose that C_2 is a separable C^*-subalgebra of $\mathfrak{A}D_2$ which is mapped into E_2 by the $*$-homomorphism $\alpha_{\mathfrak{A}D_2}$. There are $*$-homomorphisms $\psi_i\colon E_i \to \mathfrak{A}D_i$ $(i = 0, 1)$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & E_1 \arrow[r, "\psi_1"] \arrow[d, "\varepsilon"'] & \mathfrak{A}D_1 \arrow[d, "\varepsilon"] \\
C_2 \arrow[r, "\alpha_{\mathfrak{A}D_2}"'] \arrow[rr, bend right, "\text{inclusion}"'] & E_2 \arrow[r, "\psi_2"'] & \mathfrak{A}D_2
\end{tikzcd}$$

Proof of Theorem 2.16. It suffices to show that the map

$$[\![A, B]\!]_n \to [\![A, B]\!]_{n+1}$$

is bijective for any $n \geq 1$. Let $\varphi : A \to \mathfrak{A}^{n+1}B$ be a $*$-homomorphism. We shall write $D = \mathfrak{A}^{n-1}B$ and so consider φ as a homomorphism from A into $\mathfrak{A}^2D$. Since A is separable, the C^*-subalgebra $E = \varphi[A] \subset \mathfrak{A}^2D$ is separable. By Lemma 2.17 there is a $*$-homomorphism $\psi : E \to \mathfrak{A}D$ such that the composition

$$A \xrightarrow{\varphi} E \xrightarrow{\psi} \mathfrak{A}D \xrightarrow{\alpha_{\mathfrak{A}D}} \mathfrak{A}^2D$$

is 2-homotopic to $\varphi\colon A \to \mathfrak{A}^{n+1}B$. It follows from Lemma 2.9 that this composition, viewed as a $*$-homomorphism

$$A \xrightarrow{\psi\varphi} \mathfrak{A}^nB \xrightarrow{\alpha_{\mathfrak{A}^nB}} \mathfrak{A}^{n+1}B$$

is $(n+1)$-homotopic to φ. This shows that the map $[\![A, B]\!]_n \to [\![A, B]\!]_{n+1}$ is surjective.

To prove injectivity, suppose that $\varphi_0, \varphi_1\colon A \to \mathfrak{A}^nB$ are $*$-homomorphisms which become $(n+1)$-homotopic after composing with the $*$-homomorphism

$$\alpha_{\mathfrak{A}^nB}\colon \mathfrak{A}^nB \to \mathfrak{A}^{n+1}B.$$

Let $\varphi\colon A \to \mathfrak{A}^{n+1}IB$ be an $(n+1)$-homotopy between them and let $\varepsilon\colon IB \to B \oplus B$ be the $*$-homomorphism given by evaluation at the endpoints of I. Applying the second part of Lemma 2.17 we obtain the commuting diagram

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\varphi_0 \oplus \varphi_1"'] & \mathfrak{A}^{n+1}IB \arrow[r, dashed, "\psi_1"] \arrow[d, "\varepsilon"'] & \mathfrak{A}^nIB \arrow[d, "\varepsilon"] \\
\mathfrak{A}^n(B \oplus B) \arrow[r] \arrow[rr, bend right, "="'] & \mathfrak{A}^{n+1}(B \oplus B) \arrow[r, dashed, "\psi_2"'] & \mathfrak{A}^n(B \oplus B),
\end{tikzcd}$$

where the dashed arrows indicate $*$-homomorphisms defined only on suitable separable C^*-subalgebras. The composition along the top is an n-homotopy equivalence between φ_0 and φ_1. $\square$

Proof of Lemma 2.17. As in Remark 2.11, we shall view $\mathfrak{A}D$ as the C^*-algebra quotient of $\mathfrak{T}D$ associated to the C^*-seminorm

$$\|f\|_{\mathfrak{A}} = \limsup_{t \to \infty} \|f(t)\|,$$

and we shall view $\mathfrak{A}^2 D$ as the C^*-algebra quotient of $\mathfrak{T}^2 D$ associated to the C^*-seminorm

$$\|F\|_{\mathfrak{A}^2} = \limsup_{t_1\to\infty} \limsup_{t_2\to\infty} \|F(t_1,t_2)\|.$$

As noted in the proof of Proposition 2.8, the map $\alpha_{\mathfrak{A}D}\colon \mathfrak{A}D \to \mathfrak{A}^2 D$ associates to a function $f \in \mathfrak{T}D$ the function $F(t_1,t_2) = f(t_2)$ in $\mathfrak{T}^2 D$.

To prove part (i), given a separable C^*-subalgebra $E \subset \mathfrak{A}^2 D$ choose a separable C^*-subalgebra $\tilde{E} \subset \mathfrak{T}^2 D$ which maps onto E. We are going to prove the following:

2.18. Claim. *There exists a continuous function $r_0\colon [1,\infty) \to [1,\infty)$ such that $\lim_{t\to\infty} r_0(t) = \infty$ and if $r\colon [1,\infty) \to [1,\infty)$ is any continuous function for which $r(t) \le r_0(t)$, for all t, and for which also $\lim_{t\to\infty} r(t) = \infty$ then*

$$\limsup_{t\to\infty} \|F(r(t),t)\| \le \limsup_{t_1\to\infty} \limsup_{t_2\to\infty} \|F(t_1,t_2)\|,$$

for all $F \in \tilde{E}$.

Granted this, choose any $r \le r_0$ for which $\lim_{t\to\infty} r(t) = \infty$ and define a $*$-homomorphism $\psi\colon \mathfrak{T}^2 D \to \mathfrak{T}D$ by associating to $F \in \mathfrak{T}^2 D$ the one-variable function $F(r(t),t)$. The estimate in Claim 2.18 says that ψ descends to a $*$-homomorphism from $E \subset \mathfrak{A}^2 D$ into $\mathfrak{A}D$. Part (i) of the lemma is now proved by using the homotopy

$$H(t_1,t_2,s) = \begin{cases} F(t_1,t_2) & \text{if } t_1 > sr(t_2) \\ F(sr(t_2),t_2) & \text{if } t_1 \le sr(t_2) \end{cases}$$

Let us continue for a moment to assume 2.18. To prove part (ii), choose separable covers $\tilde{C}_2$ of C_2 and $\tilde{E}_1$ of E_1, and let $\tilde{E}_2 \subset \mathfrak{T}^2 D_2$ be any separable C^*-subalgebra containing the images of $\tilde{C}_2$ and $\tilde{E}_1$. We now use the above claim to choose the same sufficiently slow growth function $r(t)$ to define $*$-homomorphisms $\psi_1\colon \mathfrak{T}^2 D_1 \to \mathfrak{T}D_1$ and $\psi_2\colon \mathfrak{T}_2 D_2 \to \mathfrak{T}D_2$, as above, which descend to $*$-homomorphisms from E_1 and E_2 into $\mathfrak{A}D_1$ and $\mathfrak{A}D_2$. Commutativity of the diagram in part (ii) of the lemma is now obvious, since in fact the diagram commutes at the level of covering algebras $\tilde{C}_2$, $\tilde{E}_1$ and $\tilde{E}_2$.

It remains then to prove the claim. Let $F_1, F_2, F_3, \ldots,$ be a sequence in $\tilde{E}$ whose image is dense, in the ordinary sup norm of $\mathfrak{T}^2 D$. By an approximation argument it suffices to find an r_0 such that the inequality in Claim 2.18 holds for each F_j: it will then automatically hold for every $F \in \tilde{E}$.

Choose an increasing sequence $1 < a_1 < a_2 < \ldots,$ converging to infinity, such that

$$\left\{ \begin{matrix} t_1 \ge a_n \\ n \ge j \ge 1 \end{matrix} \right\} \quad \Rightarrow \quad \limsup_{t_2\to\infty} \|F_j(t_1,t_2)\| \le \|F_j\|_{\mathfrak{A}^2} + \frac{1}{n}.$$

Choose an increasing sequence $1 < b_1 < b_2 < \ldots,$ also converging to infinity, such that

$$\left\{ \begin{matrix} a_{n+1} \ge t_1 \ge a_n \\ n \ge j \ge 1 \\ t_2 \ge b_n \end{matrix} \right\} \quad \Rightarrow \quad \|F_j(t_1,t_2)\| \le \|F_j\|_{\mathfrak{A}^2} + \frac{2}{n}.$$

(It follows from the definitions of $\|F_j\|_{\mathfrak{A}^2}$ and a_n that for each *individual* $t_1 \ge a_n$ there exists b_n such that $\|F_j(t_1,t_2)\| \le \|F_j\|_{\mathfrak{A}^2} + 2/n$, whenever $t_2 \ge b_n$. We can choose the *same* b_n for all t_1 in the range $a_{n+1} \ge t_1 \ge a_n$ because, as noted in Remark 2.11, the functions $F_j \in \mathfrak{T}^2 D$ are equicontinuous in the first variable.) Now

define $r_0(1) = 1$ and $r_0(b_n) = a_n$, and then extend r_0 to a function on $[1, \infty)$ by linear interpolation. If $r \leq r_0$ and $t \geq b_n$ then $\|F_j(r(t), t)\| \leq \|F_j\| + 2/n$, for all $j = 1, \ldots, n$. This proves the claim. $\square$

We close this chapter by recording a simple rigidity property which we shall use in Chapter 6 when we discuss the relation between K-theory and E-theory. We shall consider the C^*-algebra $C_0(\mathbb{R})$, although a limited number of other C^*-algebras exhibit the same behavior. However we note that the proposition below would be false if, for instance, we replaced $C_0(\mathbb{R})$ with $C_0(\mathbb{R}^2)$.

Denote by $[A, B]$ the set of homotopy classes of $*$-homomorphisms from the C^*-algebra A to B.

2.19. Proposition. *For any C^*-algebra B with trivial G-action the natural map*

$$[C_0(\mathbb{R}), B] \to [\![C_0(\mathbb{R}), B]\!]$$

is a bijection.

Proof. We will show that in each homotopy class of asymptotic morphisms there is a unique-up-to-homotopy $*$-homomorphism. In view of Theorem 2.16 this will suffice.

The function $a = 2i(x - i)^{-1} \in C_0(\mathbb{R})$ satisfies the relations

$$aa^* + a + a^* = 0$$
$$a^*a + a + a^* = 0$$

(in other words, $1 + a$ is a unitary). In fact $C_0(\mathbb{R})$ is the universal C^*-algebra generated by such an element. Indeed the C^*-algebra obtained by adjoining a unit to $C_0(\mathbb{R})$ is isomorphic, via the Cayley transform, to $C(S^1)$, in such a way that $1+a$ corresponds to the standard unitary generator u; but by spectral theory $C(S^1)$ is the universal C^*-algebra generated by a unitary. Now it follows from the functional calculus that in any C^*-algebra the set of elements b satisfying

$$\|bb^* + b + b^*\| < 1$$
$$\|b^*b + b + b^*\| < 1$$

retracts onto the set of elements for which these norms are zero. In fact there is a retraction r with the property that $\|b - r(b)\| \to 0$ as the above norms converge to zero. Thus if

$$\{\varphi_t\}_{t \in [1,\infty)} \colon C_0(\mathbb{R}) \to B$$

is any asymptotic family then by applying the retraction r to the elements $b_t = \varphi_t(a)$, for large t, and using the universality of $C_0(\mathbb{R})$, we see that $\{\varphi_t\}$ is in fact asymptotically equivalent to a family of $*$-homomorphisms $\{\psi_t\}$. This family is homotopic, in the sense of asymptotic morphisms, to the constant family $\{\psi_1\}$, and so every homotopy class of asymptotic morphisms does indeed contain a $*$-homomorphism. By applying the same construction to homotopies of asymptotic morphisms we see that the homotopy class of this $*$-homomorphism is unique. $\square$

CHAPTER 3

Functors on the Homotopy Category

Let F be a covariant functor from the category of G-C^*-algebras to itself. The goal of this chapter is to develop sufficient conditions under which F determines a functor from the homotopy category of asymptotic morphisms to itself. For the most part this a formal matter—we shall spend much of the chapter verifying that a large number of diagrams commute. In the next chapter we shall consider applications to tensor products and crossed products.

We shall continue to work with G-C^*-algebras and equivariant $*$-homomorphisms, and as in the last chapter we shall suppress explicit mention of the group G. But it is worth noting that the results of this chapter apply to functors from say G_1-C^*-algebras to G_2-C^*-algebras, where G_1 and G_2 are distinct groups. We shall see a number of instances of such functors in later chapters.

Let us fix a functor F, and let B be a C^*-algebra. If I is any closed interval and if $t \in I$ then 'evaluation' at t gives a $*$-homomorphism from $F(IB)$ into $F(B)$. Thus if $f \in F(IB)$ we can 'evaluate' f at any point of I and obtain an element of $F(B)$: in other words $f \in F(IB)$ determines a function $\hat{f}\colon I \to F(B)$.

3.1. Definition. We shall say that the functor F is *continuous* if, for every C^*-algebra B and every closed interval I, each of the functions $\hat{f}\colon I \to F(B)$ defined above is continuous.

If F is continuous then by associating to $f \in F(IB)$ the continuous function $\hat{f}\colon I \to F(B)$ we obtain a $*$-homomorphism

$$\iota\colon F(IB) \to IF(B).$$

In the same way, there are $*$-homomorphisms

$$\iota\colon F(\mathfrak{T}B) \to \mathfrak{T}F(B)$$

and

$$\iota_0\colon F(\mathfrak{T}_0B) \to \mathfrak{T}_0F(B),$$

obtained by 'evaluating' any f in $F(\mathfrak{T}B)$ or $F(\mathfrak{T}_0B)$ at each $t \in T$. We should like to define an induced $*$-homomorphism

$$\iota\colon F(\mathfrak{A}B) \to \mathfrak{A}F(B).$$

To do so we must make a further assumption concerning the functor F.

3.2. Definition. The functor F is *exact* if, for every short exact sequence

$$0 \to J \to B \to B/J \to 0,$$

the induced sequence of C^*-algebras

$$0 \to F(J) \to F(B) \to F(B/J) \to 0$$

is exact.

In practice continuity is a rather mild restriction on F, but exactness is more problematic. Proving that a functor is exact can sometimes involve significant calculations in functional analysis. Furthermore, it is an unfortunate fact that some functors of interest, particularly those associated with minimal tensor products, are not exact. Nevertheless we confine our attention to exact functors.

If F is a continuous and exact functor then the desired $*$-homomorphism $\iota\colon F(\mathfrak{A}B) \to \mathfrak{A}F(B)$ is defined by requiring that the diagram

$$\begin{tikzcd}
0 \arrow[r] & F(\mathfrak{T}_0 B) \arrow[r] \arrow[d, "\iota_0"'] & F(\mathfrak{T}B) \arrow[r] \arrow[d, "\iota"'] & F(\mathfrak{A}B) \arrow[r] \arrow[d, dotted, "\iota"'] & 0 \\
0 \arrow[r] & \mathfrak{T}_0 F(B) \arrow[r] & \mathfrak{T}F(B) \arrow[r] & \mathfrak{A}F(B) \arrow[r] & 0
\end{tikzcd}$$

be commutative.

Let us also define $*$-homomorphisms

$$\iota_n\colon F(\mathfrak{A}^n B) \to \mathfrak{A}^n F(B)$$

inductively, as follows:

$$F(\mathfrak{A}^n B) \xrightarrow{\iota} \mathfrak{A}F(\mathfrak{A}^{n-1}B) \xrightarrow{\mathfrak{A}(\iota_{n-1})} \mathfrak{A}^n F(B).$$

For later purposes we note that:

3.3. Lemma. *The diagram*

$$\begin{tikzcd}
F(\mathfrak{A}^{j+k}B) \arrow[rr, "\iota_{j+k}"] \arrow[dr, "\iota_j"'] & & \mathfrak{A}^{j+k}F(B) \\
& \mathfrak{A}^j F(\mathfrak{A}^k B) \arrow[ur, "\mathfrak{A}^j(\iota_k)"'] &
\end{tikzcd}$$

commutes. □

3.4. Definition. If $\varphi\colon A \to \mathfrak{A}B$ is an asymptotic morphism from A to B then construct from it an asymptotic morphism

$$\bar{F}(\varphi)\colon F(A) \to \mathfrak{A}F(B)$$

from $F(A)$ to $F(B)$ by forming the composition

$$F(A) \xrightarrow{F(\varphi)} F(\mathfrak{A}B) \xrightarrow{\iota} \mathfrak{A}F(B).$$

From a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^n B$ construct a $*$-homomorphism

$$\bar{F}(\varphi)\colon F(A) \to \mathfrak{A}^n F(B)$$

by forming the composition

$$F(A) \xrightarrow{F(\varphi)} F(\mathfrak{A}^n B) \xrightarrow{\iota_n} \mathfrak{A}^n F(B).$$

Our main result concerning the extension of functors to the homotopy category of asymptotic morphisms is the following theorem.

3.5. Theorem. *For each continuous and exact functor F there is an associated functor $\bar{F}$ from the homotopy category of asymptotic morphisms to itself which maps the class of a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^n B$ to the class of the above described $*$-homomorphism $\bar{F}(\varphi)\colon F(A) \to \mathfrak{A}^n F(B)$.*

We remark that on objects, $\bar{F}(B) = F(B)$.

Proof. We must check that the correspondence $\varphi \mapsto F(\varphi)$ is well defined at the level of homotopy classes; that it descends to a well defined map on the morphism sets $[\![A, B]\!] = \varinjlim [\![A, B]\!]_n$; and that it is compatible with composition of morphisms.

Applying the $\bar{F}$-construction to an n-homotopy $\varphi\colon A \to \mathfrak{A}^n IB$ we obtain $*$-homomorphism

$$\bar{F}(\varphi)\colon F(A) \to \mathfrak{A}^n F(IB).$$

By continuity there is a canonical $*$-homomorphism $F(IB) \to IF(B)$, so we obtain from $\bar{F}(\varphi)$ an n-homotopy

$$\bar{F}(\varphi)\colon F(A) \to \mathfrak{A}^n IF(B).$$

as required (we have taken a notational liberty and re-used the name $\bar{F}(\varphi)$ for this n-homotopy). It follows without difficulty that the correspondence $\varphi \mapsto \bar{F}(\varphi)$ defines a map

$$[\![A, B]\!]_n \to [\![F(A), F(B)]\!]_n.$$

Next we shall prove that this map is compatible with the direct limit construction used to obtain $[\![A, B]\!]$. It is readily checked that for any C^*-algebra D the diagram

$$\begin{array}{ccc} F(D) & \xrightarrow{F(\alpha_D)} & F(\mathfrak{T}D) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \iota} \\ F(D) & \xrightarrow[\alpha_{F(D)}]{} & \mathfrak{T}F(D) \end{array}$$

commutes. Passing to the quotient $\mathfrak{A}D = \mathfrak{T}D/\mathfrak{T}_0 D$ and substituting $D = \mathfrak{A}^n(B)$, it follows that the top square in the following diagram commutes:

$$\begin{array}{ccc} F(\mathfrak{A}^n B) & \xrightarrow{F(\alpha_{\mathfrak{A}^n B})} & F(\mathfrak{A}^{n+1} B) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \iota} \\ F(\mathfrak{A}^n B) & \xrightarrow{\alpha_{F(\mathfrak{A}^n B)}} & \mathfrak{A}F(\mathfrak{A}^n B) \\ {\scriptstyle \iota_n}\downarrow & & \downarrow{\scriptstyle \mathfrak{A}(\iota_n)} \\ \mathfrak{A}^n F(B) & \xrightarrow{\alpha_{\mathfrak{A}^n(FB)}} & \mathfrak{A}^{n+1} F(B). \end{array}$$

The bottom square commutes since α is a natural transformation. The right hand vertical composition is, by definition, ι_{n+1} and therefore, the commutativity of the diagram as a whole implies the commutativity of the diagram

$$\begin{array}{ccc} F(\mathfrak{A}^n B) & \xrightarrow{F(\alpha_{\mathfrak{A}^n B})} & F(\mathfrak{A}^{n+1} B) \\ {\scriptstyle \iota_n}\downarrow & & \downarrow{\scriptstyle \iota_{n+1}} \\ \mathfrak{A}^n F(B) & \xrightarrow{\alpha_{\mathfrak{A}^n(FB)}} & \mathfrak{A}^{n+1} F(B). \end{array}$$

It follows that the correspondence $\varphi \mapsto \bar{F}(\varphi)$ is compatible with direct limits and so defines a map

$$[\![A, B]\!] \to [\![F(A), F(B)]\!].$$

It remains then to prove that the correspondence is compatible with the composition law in the category $\mathfrak{A}$. Let $\varphi : A \to \mathfrak{A}^j B$ and $\psi : B \to \mathfrak{A}^k C$ represent elements of $[\![A, B]\!]$ and $[\![B, C]\!]$, respectively and consider the following diagram:

$$\begin{tikzcd}
F(A) \arrow[r, "F(\varphi)"] \arrow[d, "="'] & F(\mathfrak{A}^j B) \arrow[r, "F\mathfrak{A}^j(\psi)"] \arrow[d, "\iota_j"] & F(\mathfrak{A}^{j+k} C) \arrow[d, "\iota_{j+k}"] \\
F(A) \arrow[r, "\bar{F}(\varphi)"'] & \mathfrak{A}^j F(B) \arrow[r, "\mathfrak{A}^j \bar{F}(\psi)"'] & \mathfrak{A}^{j+k} F(C).
\end{tikzcd}$$

The composition of $\bar{F}(\varphi) \circ \bar{F}(\psi)$ is obtained by moving counterclockwise from the upper left to the lower right of the diagram, whereas $\bar{F}(\varphi \circ \psi)$ is obtained by moving clockwise around the diagram. We must show that the diagram commutes.

The first square commutes by definition of $\bar{F}(\varphi)$. The second square commutes by consideration of a final diagram

$$\begin{tikzcd}
F(\mathfrak{A}^j B) \arrow[r, "F\mathfrak{A}^j(\psi)"] \arrow[d, "\iota_j"'] & F(\mathfrak{A}^{j+k} C) \arrow[d, "\iota_j"] \\
\mathfrak{A}^j(FB) \arrow[r, "\mathfrak{A}^j F(\psi)"] \arrow[d, "="'] & \mathfrak{A}^j F(\mathfrak{A}^k C) \arrow[d, "\mathfrak{A}^j(\iota_k)"] \\
\mathfrak{A}^j(FB) \arrow[r, "\mathfrak{A}^j \bar{F}(\psi)"] & \mathfrak{A}^{j+k} F(C).
\end{tikzcd}$$

By Lemma 3.3 the perimeter of this diagram is the same as the second square in the former one. Commutativity of the top square expresses the easily verified fact that ι_j is a natural transformation, while commutativity of the bottom square follows from the definition of $\bar{F}(\psi)$ and the functoriality of $\mathfrak{A}^j$. $\square$

We conclude with one final piece of algebra—the proof is not difficult and is left to the reader.

3.6. Proposition. *Let F_1 and F_2 be continuous and exact functors and let β be a natural transformation from F_1 to F_2 (thus in particular, β is a collection of $*$-homomorphisms $\beta_B \colon F_1(B) \to F_2(B)$). If $\bar{\beta}_B \colon \bar{F}_1(B) \to \bar{F}_2(B)$ denotes the class in $\mathfrak{A}$ of the $*$-homomorphism β_B then $\bar{\beta}$ is natural transformation from $\bar{F}_1$ to $\bar{F}_2$.* $\square$

CHAPTER 4

Tensor Products and Descent

Let B and D be C^*-algebras. We shall denote by $D \otimes B$ the *maximal* C^*-algebra tensor product of D and B. This completion of the algebraic tensor product has the universal property that there is a one-to-one and onto correspondence between nondegenerate representations of $D \otimes B$ and commuting pairs of nondegenerate representations of D and B [**36**, Section 1.9]. If we fix D then the correspondence $B \mapsto D \otimes B$ is a functor from the category of C^*-algebras to itself. The following is well-known:

4.1. Lemma. *Let D be a C^*-algebra. The maximal tensor product $B \mapsto D \otimes B$ is an exact functor, in the sense of Definition 3.2.*

Proof. This is a simple consequence of the universal property of the maximal tensor product and the fact that a non-degenerate representation of an ideal in a C^*-algebra extends to a non-degenerate representation of the algebra itself. See for example [**36**, Section 1.9]. □

4.2. Lemma. *Let D be a C^*-algebra. The maximal tensor product functor $B \mapsto D \otimes B$ is continuous, in the sense of Definition 3.1.*

Proof. Denote by $C[I]$ the C^*-algebra of continuous functions on a closed interval I. Then $IB \cong C[I] \otimes B$. Continuity follows from the isomorphism $D \otimes IB \cong I(D \otimes B)$ provided by associativity and commutativity of the tensor product:

$$D \otimes IB \cong D \otimes (C[I] \otimes B) \cong C[I] \otimes (D \otimes B) \cong I(D \otimes B).$$

We shall leave the remaining details to the reader. □

¿From the previous chapter we obtain the following consequence:

4.3. Proposition. *Let D be a C^*-algebra, there is a functor from the homotopy category $\mathfrak{A}$ to itself which associates to the class of a $*$-homomorphism $\psi\colon A \to \mathfrak{A}^n B$ the composition*

$$D \otimes A \xrightarrow{1 \otimes \psi} D \otimes \mathfrak{A}^n A \xrightarrow{\iota_n} \mathfrak{A}^n(D \otimes A). \quad \square$$

There is of course a 'right-handed' version of this as well:

4.4. Proposition. *Let D be a C^*-algebra, there is a functor from $\mathfrak{A}$ to itself which associates to the class of a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^n B$ the composition*

$$A \otimes D \xrightarrow{\varphi \otimes 1} (\mathfrak{A}^n A) \otimes D \xrightarrow{\iota_n} \mathfrak{A}^n(A \otimes D). \quad \square$$

The following lemma expresses a crucial compatibility property of left and right tensor products.

4.5. Lemma. *If $\varphi\colon A_1 \to A_2$ and $\psi\colon B_1 \to B_2$ are morphisms in the homotopy category of asymptotic morphisms then the compositions*

$$A_1 \otimes B_1 \xrightarrow{\varphi\otimes 1} A_2 \otimes B_1 \xrightarrow{1\otimes\psi} A_2 \otimes B_2$$

and

$$A_1 \otimes B_1 \xrightarrow{1\otimes\psi} A_1 \otimes B_2 \xrightarrow{\varphi\otimes 1} A_2 \otimes B_2$$

are equal.

Proof. Let $\varphi\colon A_1 \to \mathfrak{A}^m A_2$ and $\psi\colon B_1 \to \mathfrak{A}^n B$ be $*$-homomorphisms representing the morphisms given in the lemma (we shall use the same symbol for the $*$-homomorphisms and their classes in the category $\mathfrak{A}$). The compositions in the lemma are represented by the compositions of the $*$-homomorphism

$$\varphi \otimes \psi\colon A_1 \otimes B_1 \to \mathfrak{A}^m A_2 \otimes \mathfrak{A}^n B_2$$

with two $*$-homomorphisms from $\mathfrak{A}^m A_2 \otimes \mathfrak{A}^n B_2$ to $\mathfrak{A}^{m+n}(A_2 \otimes B_2)$ obtained by taking the two possible routes around the diagram

$$\begin{array}{ccc}
\mathfrak{A}^m A_2 \otimes \mathfrak{A}^n B_2 & \xrightarrow{\iota_m} & \mathfrak{A}^m(A_2 \otimes \mathfrak{A}^n B_2) \\
{\scriptstyle \iota_n}\downarrow & & \downarrow{\scriptstyle \mathfrak{A}^m(\iota_n)} \\
\mathfrak{A}^n(\mathfrak{A}^m A_2 \otimes B_2) & \xrightarrow[\mathfrak{A}^n(\iota_m)]{} & \mathfrak{A}^{m+n}(A_2 \otimes B_2).
\end{array}$$

So it suffices to prove that the diagram commutes up to $(m+n)$-homotopy. By an induction argument, it suffices to consider the case $n = m = 1$, and therefore the diagram

$$\begin{array}{ccc}
\mathfrak{A}A_2 \otimes \mathfrak{A}B_2 & \xrightarrow{\iota} & \mathfrak{A}(A_2 \otimes \mathfrak{A}B_2) \\
{\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \mathfrak{A}(\iota)} \\
\mathfrak{A}(\mathfrak{A}A_2 \otimes B_2) & \xrightarrow[\mathfrak{A}(\iota)]{} & \mathfrak{A}^2(A_2 \otimes B_2).
\end{array}$$

Considering the same diagram with $\mathfrak{T}$ in place of $\mathfrak{A}$, the two compositions are represented by the $*$-homomorphisms

$$f \otimes h \mapsto f(t_1) \otimes h(t_2)$$

and

$$f \otimes h \mapsto f(t_2) \otimes h(t_1),$$

where the right hand sides in these displays represent functions of two variables, as in Remark 2.11. But now essentially the same argument as we gave in the proof of Proposition 2.8 shows that these two $*$-homomorphisms are 2-homotopic. □

The lemma allows us to define the tensor product of any two morphisms in the homotopy category $\mathfrak{A}$:

4.6. Theorem. *There is a functor $\otimes\colon \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ which is the maximal tensor product on objects; which associates to the pair of morphisms $(1, \psi)$ the morphism $1 \otimes \psi$, as in Proposition 4.3; and which associates to the pair of morphisms $(\varphi, 1)$ the morphism $\varphi \otimes 1$, as in Proposition 4.4.*

Proof. Define $\varphi \otimes \psi \colon A_1 \otimes B_1 \to A_2 \otimes B_2$ to be the composition

$$A_1 \otimes B_1 \xrightarrow{\varphi \otimes 1} A_2 \otimes B_1 \xrightarrow{1 \otimes \psi} A_2 \otimes B_2.$$

It follows from Lemma 4.5 that this is a functor with the required properties. □

Let us record for later use the following useful fact.

4.7. Proposition. *Let $\varphi \colon A \to B$ be a morphism in $\mathfrak{A}$. After $\mathbb{C} \otimes A$ and $\mathbb{C} \otimes B$ are identified with A and B, respectively, the morphism $1 \otimes \varphi \colon \mathbb{C} \otimes A \to \mathbb{C} \otimes B$ identifies with $\varphi \colon A \to B$.*

Proof. This is an immediate consequence of Proposition 3.6. □

The tensor product also has the usual properties of associativity, commutativity, and so on. We shall not bother to record these formally.

Let us now make a brief comment on the *minimal* C^*-algebra tensor product [**36**, Section 1.3]. This is a functorial tensor product on the category of C^*-algebras, but unfortunately it is not in general exact, and for this reason it does not in general pass to a tensor product on our homotopy category $\mathfrak{A}$. The best we can do is restrict attention to a smaller class of C^*-algebras:

4.8. Definition. Recall that a C^*-algebra D is *exact* [**36**, Section 2.5] if, for every short exact sequence

$$0 \to A \to B \to C \to 0,$$

the minimal tensor product sequence

$$0 \to A \otimes_{\min} D \to B \otimes_{\min} D \to C \otimes_{\min} D \to 0$$

is exact as well.

4.9. Theorem. *On the homotopy category of exact C^*-algebras and asymptotic morphisms there is a minimal tensor product bifunctor.* □

We shall not need this result, and so we leave it to the reader to formulate the theorem more precisely.

We turn now to crossed product C^*-algebras, beginning with a quick review of the definition.

Let B be a G-C^*-algebra. The linear space $C_c(G, B)$ of continuous and compactly supported functions from G to B is a $*$-algebra under the convolution and involution

$$(f * h)(x) = \int_G f(g) g(h(g^{-1}x))\, dg$$

and

$$f^*(g) = \Delta(g)^{-1} g(f(g^{-1})^*),$$

where dg is a left Haar measure and Δ is the modular function for G. A covariant representation π of B (meaning a $*$-representation of B as operators on a Hilbert space, together with a compatible unitary representation of G on the same space) determines a $*$-representation of $C_c(G, B)$ by the formula

$$\langle \pi(f)v, w \rangle = \int_G \langle \pi[f(g)]\pi(g)v, w \rangle\, dg.$$

The *full crossed product algebra* $C^*(G,B)$ associated to B is the C^*-algebra completion of $C_c(G,B)$ under the norm

$$\|f\| = \sup\{\, \|\pi(f)\| \,:\, \pi \text{ is a covariant representation of } B\,\}.$$

It has the universal property that every covariant representation of B extends from $C_c(G,B)$ to a representation of $C^*(G,B)$. An equivariant $*$-homomorphism $\varphi : A \to B$ induces a $*$-homomorphism

$$C^*(G,\varphi) : C^*(G,A) \to C^*(G,B),$$

so that the crossed product may be regarded as a functor from G-C^*-algebras to C^*-algebras (or to G-C^*-algebras if we consider $C^*(G,B)$ as equipped with the trivial action of G as is occasionally useful). To avoid clumsy notation, as often as possible we shall use the same symbol φ to denote an equivariant $*$-homomorphism $\varphi\colon A \to B$ and the induced $*$-homomorphism

$$\varphi = C^*(G,\varphi) : C^*(G,A) \to C^*(G,B).$$

4.10. Lemma. *The functor $B \mapsto C^*(G,B)$ is exact.* □

Proof. This follows from the universal property of the full crossed product. Given a short exact sequence

$$0 \to J \to B \to B/J \to 0,$$

the induced map $C^*(G,J) \to C^*(G,B)$ is injective by virtue of the fact that every covariant representation of J extends to a covariant representation of B. The quotient $C^*(G,B)/C^*(G,J)$ is isomorphic to $C^*(G,B/J)$ by virtue of the fact that it is a completion of $C_c(G,B/J)$ with the same universal property as the C^*-algebra $C^*(G,B/J)$. □

4.11. Lemma. *The functor $B \mapsto C^*(G,B)$ is continuous.*

Proof. There is an isomorphism

$$C^*(G,IB) \cong IC^*(G,B)$$

which is compatible with the evaluation maps to $C^*(G,B)$. This is a special case of the fact that if G acts trivially on A then there is a canonical isomorphism

$$C^*(G,A\otimes B) \cong A \otimes C^*(G,B),$$

and this in turn follows from the universal properties of the maximal tensor product and the full crossed product. □

Applying the machinery of the previous chapter we get the following result, concerning the procedure of *descent* from equivariant asymptotic morphisms to asymptotic morphisms of crossed product C^*-algebras.

4.12. Theorem. *There is a* descent functor, *from the homotopy category of G-C^*-algebras and equivariant asymptotic morphisms to the homotopy category of C^*-algebras and asymptotic morphisms, which associates to the class of an equivariant $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^n B$ the class of the composition*

$$C^*(G,A) \xrightarrow{\varphi} C^*(G,\mathfrak{A}^n B) \xrightarrow{\iota_n} \mathfrak{A}^n C^*(G,B). \quad \square$$

We conclude this chapter with a discussion of *reduced* crossed products, which are of some importance to the Baum-Connes conjecture for K-theory of group C^*-algebras [**4**].

We refer the reader elsewhere for a definition of the reduced crossed product $C^*_{\mathrm{red}}(G, B)$ (see [**28**, Chapter 7], for instance). As with the minimal C^*-algebra tensor product exactness is problematic, although there is as yet no known counterexample to the conjecture that every G is C^*-exact, in the following sense:

4.13. Definition. A group G is C^*-*exact* if, for every short exact sequence

$$0 \to J \to B \to B/J \to 0$$

of G-C^*-algebras, the induced sequence

$$0 \to C^*_{\mathrm{red}}(G, J) \to C^*_{\mathrm{red}}(G, B) \to C^*_{\mathrm{red}}(G, B/J) \to 0$$

is also exact.

The issue of C^*-exactness is actually closely related to the exactness of the minimal tensor product, as can be seen from the following elegant but unpublished result of Kirchberg and Wassermann.

4.14. Lemma. (Kirchberg and Wassermann) *A discrete group G is C^*-exact if and only if its reduced group C^*-algebra $C^*_{\mathrm{red}}(G)$ is an exact C^*-algebra.* □

The proof is not especially difficult, but we shall not go into it here. The lemma is of interest because it is often quite easy to show that $C^*_{\mathrm{red}}(G)$ is exact, for instance, by exhibiting it as a C^*-subalgebra of a nuclear C^*-algebra [**36**, Section 7]. A well-known unpublished argument of Connes shows in this way that every discrete subgroup of a connected Lie group is C^*-exact.

In further unpublished work, Kirchberg and Wassermann have also shown, among other things, that every connected Lie group is C^*-exact.

4.15. Lemma. *The functor $B \mapsto C^*_{\mathrm{red}}(G, B)$ is continuous.*

Proof. The proof is similar to that of Lemma 4.11, but uses the canonical isomorphism

$$C^*_{\mathrm{red}}(G, A \otimes_{\min} B) \cong A \otimes_{\min} C^*_{\mathrm{red}}(G, B),$$

valid for C^*-algebras A with trivial G-action. We apply this to $A = C[I]$. Since this C^*-algebra is nuclear the minimal tensor products agree with maximal tensor products. □

We obtain the following *reduced descent functor* from equivariant asymptotic morphisms to asymptotic morphisms of reduced crossed product algebras.

4.16. Theorem. *Let G be a C^*-exact group. There is a* reduced descent functor *from the homotopy category of G-C^*-algebras and equivariant asymptotic morphisms to the homotopy category of C^*-algebras and asymptotic morphisms which associates to the class of an equivariant $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^n B$ the class of the composition*

$$C^*_{\mathrm{red}}(G, A) \xrightarrow{\varphi} C^*_{\mathrm{red}}(G, \mathfrak{A}^n B) \xrightarrow{\iota_n} \mathfrak{A}^n C^*_{\mathrm{red}}(G, B). \quad \square$$

CHAPTER 5

C*-Algebra Extensions

The aim of this chapter is to associate to a short exact sequence

$$0 \to J \to B \to A \to 0$$

of separable G-C^*-algebras an equivariant asymptotic morphism from the *suspension* of B, defined below, to J, and to investigate its various properties. The construction appears in the original note of Connes and Higson [**11**, Section 5], and much of what we are going to do here simply involves carrying over their ideas to the equivariant setting.

An important aspect of the theory of asymptotic morphisms, which we will pursue fully in the next chapter, is the development of long exact sequences involving the morphism sets $[\![A, B]\!]$ in the homotopy category $\mathfrak{A}$. Here we shall make a start by investigating the relationship between asymptotic morphisms and some standard constructions in elementary homotopy theory, involving mapping cones and the like. Our results here are primarily adaptations to the present setting of familiar results from elsewhere. Apart from [**11**], basic references are the papers [**22**], [**15**] and [**31**] and the book [**6**].

For the next several paragraphs let us fix a short exact sequence

$$0 \to J \to B \xrightarrow{\pi} A \to 0$$

of *separable* C^*-algebras (we continue to suppress explicit mention of the group G except when necessary; thus we assume that the C^*-algebras are G-C^*-algebras and the $*$-homomorphisms are G-equivariant). We shall usually identify J with its image in B, which is a closed, two sided ideal.

A crucial technical tool in this chapter will be the theory of approximate units in C^*-algebras. For the next one or two pages, let us call a C^*-algebra with no action of G a *plain* C^*-algebra.

5.1. Definition. Let J be a plain C^*-algebra. An approximate unit for J is a norm-continuous family of elements $\{u_t\}$ of J, parametrized by $t \in T$, such that

(i) $0 \le u_t \le 1$; and
(ii) $\lim_{t\to\infty} \|u_t b - b\| = 0$, for all $b \in J$.

In C^*-algebra theory it is more usual to consider approximate units which are parametrized by $n \in \mathbb{N}$. From such an approximate unit $\{u_n\}$ we obtain a 'continuous' approximate unit $\{u_t\}$ by linear interpolation; for $s \in [0, 1]$ we define $u_{n+s} = (1 - s)u_n + su_{n+1}$.

We shall require the following more elaborate type of approximate unit.

5.2. Definition. Let B be a separable C^*-algebra and J be a C^*-subalgebra such that $[B, J] \subset J$ (for instance, J might be an ideal in B). An *approximate unit*

for the pair $J \subset B$ is a norm-continuous family of elements $\{u_t\}$ of J, parametrized by $t \in T$, such that

(i) $0 \leq u_t \leq 1$;
(ii) $\lim_{t\to\infty} \|u_t b - b\| = 0$, for all $b \in J$;
(iii) $\lim_{t\to\infty} \| [u_t, b] \| = 0$, for all $b \in B$; and
(iv) $\lim_{t\to\infty} \|g(u_t) - u_t\| = 0$, uniformly for g in compact subsets of G.

Approximate units with the listed properties are often called *quasicentral* [**1**, Section 1].

5.3. Lemma. *Let B be a separable C^*-algebra and J be a C^*-subalgebra such that $[B, J] \subset J$. There exists an approximate unit for the pair $J \subset B$.*

Proof. This is proved in Lemma 1.4 of [**23**], but for the sake of variety we indicate a second argument.

Suppose first that the group G is discrete. Realize B and G as operators on a Hilbert space (in a covariant fashion, of course) and let J' be the operator norm closure of $J + JG + JGB$. It is a separable plain C^*-algebra and if B' denotes the plain C^*-algebra generated by B and G then $[B', J'] \subset J'$. Now it is proved in [**1**, Section 1] that the convex hull of any approximate unit $\{v_t\}$ for J' contains an approximate unit $\{u_t\}$ for J' such that $\|[u_t, b']\| \to 0$, for all $b \in B'$. If we start with an approximate unit $\{v_t\}$ which is actually an approximate unit for the plain C^*-algebra $J \subset J'$ (and note that by definition of J', any approximate unit for J is an approximate unit for J') then we obtain an approximate unit $\{u_t\}$ lying in J which has the properties listed in Definition 5.2.

If G is not discrete then perform the above construction for a countable dense subgroup $G_0 \subset G$ (viewed as a discrete group), and after having obtained $\{u_t\}$ form the integrals

$$\int_G f(g)g(u_t)\, dg,$$

where f is a continuous and compactly supported function on G with total mass one. These averages constitute an approximate unit for J with all the required properties. □

We are almost ready to begin the construction of the asymptotic morphism associated to a short exact sequence of C^*-algebras. First, a familiar definition:

5.4. Definition. Let B be a C^*-algebra. The *suspension* of B is the C^*-algebra

$$\Sigma B = \{f\colon [0,1] \to B \mid f \text{ is continuous and } f(0) = 0 = f(1).\}.$$

It is occasionally convenient to let Σ itself denote the C^*-algebra of continuous functions on the unit interval which vanish at both endpoints, in which case $\Sigma B \cong \Sigma \otimes B$. The operation of suspension is a functor, and if $\varphi\colon A \to B$ is a $*$-homomorphism then we shall denote by $\Sigma\varphi\colon \Sigma A \to \Sigma B$ the induced $*$-homomorphism on tensor products. As in Chapter 3, a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^n B$ induces a $*$-homomorphism $\Sigma\varphi\colon \Sigma A \to \mathfrak{A}^n \Sigma B$.

5.5. Proposition. *Given a short exact sequence of separable C^*-algebras*

$$0 \to J \to B \xrightarrow{\pi} A \to 0$$

and an approximate unit $\{u_t\}$ for $J \subset B$ there is an asymptotic morphism $\sigma\colon A \to \mathfrak{A}J$ such that if $\{\sigma_t\}_{t\in[1,\infty)}\colon \Sigma A \to J$ is any associated asymptotic family (see Definition 1.3) and $s\colon A \to B$ is any set-theoretic section (not necessarily equivariant) of the quotient map π then $\sigma_t(f \otimes x)$ is asymptotic to $f(u_t)s(a)$, for all $f \in \Sigma$ and $a \in A$.

Note that the asymptotic equivalence $\sigma_t(f\otimes x) \sim f(u_t)s(a)$ uniquely determines $\sigma : \Sigma A \to \mathfrak{A}J$.

The proof is a simple calculation based on the following lemma:

5.6. Lemma. *Let B be a separable C^*-algebra and J be a C^*-subalgebra such that $[B, J] \subset J$. Let $\{u_t\}$ be an approximate unit for $J \subset B$. Let f be a continuous, complex valued function on the unit interval such that $f(0) = 0$.*

(i) *If $g \in G$ then $\lim_{t\to\infty} \|g(f(u_t)) - f(u_t)\| = 0$, the convergence being uniform on compact subsets of G.*
(ii) *If $b \in B$ then $\lim_{t\to\infty} \|[b, f(u_t)]\| = 0$.*
(iii) *The function $t \mapsto f(u_t)$ is G-continuous and so is an element of the C^*-algebra $\mathfrak{T}J$ of Definition 1.9.*
(iv) *If in addition $f(1) = 0$ then $\lim_{t\to\infty} \|f(u_t)b\| = 0$, for every $b \in J$.*

Proof. Let us consider the first two items. By the Weierstrass approximation theorem it suffices to prove them for the single function $f(x) = x$, and for this function they are immediate consequences of the definition of approximate unit. Item (iii) follows from item (i) and a compactness argument. As for item (iv), the set of f for which the limit is zero, for all $b \in J$, is an ideal in the C^*-algebra of continuous functions f which vanish at 0 and 1. So it suffices to prove (iv) for the function $f(x) = x(1 - x)$. But according to the definition of approximate unit, if $b \in J$ then

$$\lim_{t\to\infty} \|u_t(1 - u_t)b\| \le \lim_{t\to\infty} \|(1 - u_t)b\| = 0. \quad \square$$

Proof of Proposition 5.5. For later purposes (see in particular Proposition 5.8) we are going to prove something a little stronger. Let J_0 be a C^*-subalgebra of J and let $s\colon A \to B_0 \subset B$ be a section (not necessarily equivariant) of $\pi\colon B \to A$ which is a $*$-homomorphism, modulo J_0. Suppose that $[B_0, J_0] \subset J_0$. Then if $\{u_t\}$ is an approximate unit for the pair $J_0 \subset B_0$ there is an asymptotic morphism $\sigma\colon \Sigma A \to \mathfrak{A}J$ such that $\sigma_t(f\otimes x) \sim f(u_t)s(a)$. For the present proposition it suffices to take $J_0 = J$ and $B_0 = B$.

Denote by $\Sigma \odot A$ the algebraic tensor product of Σ and A. It follows from Lemma 5.6 that the formula

$$\sigma_t(f \otimes a) = f(u_t)s(a) \in \mathfrak{T}J,$$

when followed with the projection from $\mathfrak{T}J$ to $\mathfrak{A}J$, extends to an equivariant $*$-homomorphism from $\Sigma \odot A$ into $\mathfrak{A}J$. Now we invoke a characterization of the maximal tensor product which says that any $*$-homomorphism from $\Sigma\odot A$ into a C^*-algebra extends to a $*$-homomorphism of the maximal tensor product $\Sigma A = \Sigma \otimes A$ [**36**, Section 1.10]. We obtain then an asymptotic morphism $\sigma\colon \Sigma A \to \mathfrak{A}J$ as required. It follows from Lemma 5.6 that σ is independent of the choice of section $s\colon A \to B$, and the rest of the proposition follows easily. $\square$

5.7. Lemma. *The homotopy class of the asymptotic morphism $\sigma\colon A \to \mathfrak{A}J$ in Proposition 5.5 depends only on the short exact sequence*

$$0 \to J \to B \xrightarrow{\pi} A \to 0,$$

and not on the choice of approximate unit $\{u_t\}$.

Proof. Let $\{v_t\}$ be a second approximate unit for $J \subset B$ and denote by I the unit interval. The functions $w_t(s) = su_t + (1-s)v_t$ constitute an approximate unit for $IJ \subset IB$. If $\tilde{B} \subset IB$ denotes the C^*-algebra of continuous functions from I to B which are constant modulo J then the asymptotic morphism $A \to \mathfrak{A}IJ$ associated to the short exact sequence

$$0 \to IJ \to \tilde{B} \to A \to 0,$$

constructed using the approximate unit $\{w_t\}$, is a homotopy of asymptotic morphisms connecting those constructed from $\{u_t\}$ and $\{v_t\}$. $\square$

In the following proposition, if $\sigma\colon \Sigma A \to \mathfrak{A}J$ is the asymptotic morphism associated to the short exact sequence

$$0 \to J \to B \to A \to 0$$

then we use the same notation σ for its homotopy class $\sigma \in [\![\Sigma A, J]\!]$.

5.8. Proposition. *A commuting diagram of short exact sequences of separable C^*-algebras*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_0 & \longrightarrow & B_0 & \longrightarrow & A_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0
\end{array}$$

gives rise to a commuting diagram

$$\begin{array}{ccc}
\Sigma A_0 & \xrightarrow{\sigma_1} & J_0 \\
\downarrow & & \downarrow \\
\Sigma A_1 & \xrightarrow{\sigma_2} & J_1
\end{array}$$

in the category $\mathfrak{A}$.

Proof. There is a natural commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_0 & \longrightarrow & B_0 & \longrightarrow & A_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
0 & \longrightarrow & J_1 & \longrightarrow & B_1 \underset{A_1}{\oplus} A_0 & \longrightarrow & A_0 & \longrightarrow & 0 \\
 & & {=}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0.
\end{array}$$

Since we can use the same approximate unit for both of the bottom two rows when defining the associated asymptotic morphisms, it follows that the diagram in the homotopy category associated to them is commutative. Thus it suffices to consider

the top two rows. Reformulating things a little, it suffices to prove the proposition in the case of a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_0 & \longrightarrow & B_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{=} & & \\ 0 & \longrightarrow & J_1 & \longrightarrow & B_1 & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

We want then to show that the diagram

$$\begin{array}{ccc} \Sigma A & \xrightarrow{\sigma_1} & \mathfrak{A}J_0 \\ {=}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ \Sigma A & \xrightarrow{\sigma_2} & \mathfrak{A}J_1 \end{array}$$

commutes up to a homotopy of asymptotic morphisms. To construct the homotopy let $\{u_{0,t}\}$ and $\{u_{1,t}\}$ be the approximate units used to define σ_1 and σ_2 and let

$$w_t(s) = (1-s)u_{0,t} + su_{1,t} \in J_1,$$

where we use the $*$-homomorphism $J_0 \to J_1$ to map $u_{0,t}$ into J_1. View $\{w_t\}$ as a continuous family in IJ_1, but note that it need *not* be an approximate unit. Nevertheless we can follow the construction in the proof of Proposition 5.5 to obtain from it an asymptotic morphism. To do so we consider the short exact sequence

$$0 \to IJ_1 \to B \to A \to 0,$$

where B denotes the continuous functions from I to B_1 which are constant, modulo J_1. Define a section $s\colon A \to B$ by composing any section $s_0\colon A \to B_0$ with the $*$-homomorphism $\varphi : B_0 \to B_1$ and viewing the result as a constant B_1-valued function on I. The crucial property of $s\colon A \to B$ is that it is an equivariant $*$-homomorphism, modulo not only the ideal IJ_1 but also modulo the image of IJ_0 in IJ_1. Because of this the formula

$$\sigma_t(f \otimes a) = f(w_t)s(a)$$

defines an asymptotic morphism from ΣA to $\mathfrak{A}IJ_1$ by the remarks at the beginning of the proof of Proposition 5.5. It is the required homotopy. □

5.9. Proposition. *Let $\sigma\colon \Sigma A \to J$ be the morphism in $\mathfrak{A}$ associated to the short exact sequence*

$$0 \to J \to B \to A \to 0.$$

If D is a separable C^*-algebra and if

$$\sigma_D\colon \Sigma A \otimes D \to J \otimes D$$

denotes the morphism in $\mathfrak{A}$ associated to the short exact sequence

$$0 \to J \otimes D \to B \otimes D \to A \otimes D \to 0$$

then $\sigma_D = \sigma \otimes 1_D$.

Proof. Let $\{v_t\}$ be an approximate unit for D, and let $\{u_t\}$ be an approximate unit for $J \subset B$. Then $\{u_t \otimes v_t\}$ is an approximate unit for $J \otimes D \subset B \otimes D$. Choose any section $s\colon A \to B$, and choose any section of the quotient map $B \otimes D \to A \otimes D$ which maps the elementary tensor $a \otimes d$ to $s(a) \otimes d$. With these choices, define the asymptotic morphisms σ and σ_D, as in the proof of Proposition 5.5. The present proposition will be proved if we can show that $f(u_t \otimes v_t)(b \otimes d)$ is asymptotic to $f(u_t)b \otimes d$, for every $b \in B$ and $d \in D$. For then the asymptotic morphisms

$$\sigma_D,\ \sigma \otimes 1_D\colon \Sigma \otimes A \otimes D \to \mathfrak{A}(J \otimes D)$$

will agree on elementary tensors $a \otimes d$, and so will be equal. To prove this asymptotic equivalence it suffices, by the Weierstrass approximation theorem, to consider the function $f(x) = x$, and for this we get

$$(u_t \otimes v_t)(b \otimes d) = u_t b \otimes v_t d \sim u_t b \otimes d,$$

as required. □

The following definition and proposition show that our construction of the asymptotic morphism associated to a short exact sequence is non-trivial.

5.10. Definition. If A is a C^*-algebra then denote by CA the C^*-subalgebra of IA comprised of continuous functions from the unit interval into A which vanish at 1. This is the *cone* on A. Denote by $\pi_0 : CA \to A$ the $*$-homomorphism given by evaluation of a function at zero.

5.11. Proposition. *The asymptotic morphism $\sigma\colon \Sigma\mathbb{C} \to \mathfrak{A}\Sigma\mathbb{C}$ associated to the sequence*

$$0 \longrightarrow \Sigma\mathbb{C} \longrightarrow C\mathbb{C} \xrightarrow{\pi_0} \mathbb{C} \longrightarrow 0$$

is the identity morphism in the category $\mathfrak{A}$.

Proof. To construct the asymptotic morphism σ, we must choose an approximate unit $\{u_t\}$ for Σ and set-theoretic section $\mathbb{C} \to C\mathbb{C}$. Let $u_t \in \Sigma$ $(t \geq 2)$ be the piecewise linear function obtained by linear interpolation from the data

$$u_t(0) = 0, \quad u_t(t^{-1}) = 1, \quad u_t(1 - t^{-1}) = 1, \quad u_t(1) = 0.$$

Let $s\colon \mathbb{C} \to C\mathbb{C}$ be the linear map which sends 1 to a function $h \in C$ which is 1 on $[0, 1/3)$ and 0 on $[2/3, 1]$. The asymptotic morphism $\sigma\colon \Sigma \to \mathfrak{A}\Sigma$ is then determined by the asymptotic family $\sigma_t\colon f \mapsto f(u_t)h$. If x denotes the identity function on the unit interval then as long as $t \geq 3$, $\sigma_t(f) = f(tx)$, where we view f as defined on $[0, \infty)$ by extending it to be zero on $[1, \infty)$. The homotopy $f(sx + (1 - s)tx)$ connects σ_t to the constant asymptotic family $\{\varphi_t = \mathrm{id}\}_{t \in [1,\infty)}\colon \Sigma \to \Sigma$. □

We note in passing the following interesting fact. It plays an important role in the problem of characterizing E-theory (see the introduction), but we shall not use it in what follows. The content of the theorem is that, up to suspension, every asymptotic morphism is associated to some short exact sequence, which is another way of demonstrating the non-triviality of our construction.

5.12. Theorem. *Let A and B be separable C^*-algebras and let $\varphi\colon A \to \mathfrak{A}B$ be an asymptotic morphism. There is a short exact sequence*

$$0 \to \Sigma B \to E \to A \to 0$$

whose associated asymptotic morphism $\sigma\colon \Sigma A \to \mathfrak{A}\Sigma B$ is homotopic to $\Sigma\varphi$.

Sketch of Proof. Let $\{\varphi_t\}_{t\in[,1\infty)}$ be an asymptotic family corresponding to φ. Let $\mathfrak{C}B$ be the C^*-algebra of continuous and bounded functions from interval $(0,1]$ into B which vanish at 1, and let

$$E = \{\, a \oplus f \in A \oplus \mathfrak{C}B \,:\, f(s) \sim \varphi_{s^{-1}}(a) \,\}.$$

There is then a short exact sequence

$$0 \to \Sigma B \to E \to A \to 0,$$

in which the first map is the obvious inclusion $f \mapsto 0 \oplus f$ and the second is the projection $a \oplus f \mapsto a$. The proof that its associated asymptotic morphism is homotopic to $\Sigma\varphi$ is an extension of the proof of Proposition 5.11. □

We turn now to a discussion of mapping cones and their relation to short exact sequences and asymptotic morphisms. We begin with the following standard definition.

5.13. Definition. Let $\theta : B \to A$ be a $*$-homomorphism between C^*-algebras. The *mapping cone* C_θ of θ is the C^*-algebra defined by

$$C_\theta = \{\, b \oplus f \in B \oplus CA \,|\, \theta(b) = f(0) \,\}.$$

Define $*$-homomorphisms

$$\alpha : C_\theta \to B \quad \text{and} \quad \beta : \Sigma A \to C_\theta$$

by $\alpha(b \oplus f) = b$ and $\beta(f) = 0 \oplus f$.

We are most interested in the special case of a *surjective* $*$-homomorphism $\pi\colon B \to A$ of separable C^*-algebras. Let J be kernel of π. Then J embeds as an ideal in the mapping cone C_π via the $*$-homomorphism $\tau\colon b \mapsto b \oplus 0$. Let us also define a $*$-homomorphism

$$\pi_1\colon CB \to C_\pi$$

by $\pi_1\colon f \mapsto f(0) \oplus \pi(f)$, where $\pi(f)$ denotes the composition of a function $f\colon I \to B$ with the $*$-homomorphism $\pi\colon B \to A$. The $*$-homomorphism π_1 is surjective, and there is a short exact sequence

$$0 \to \Sigma J \to CB \xrightarrow{\pi_1} C_\pi \to 0.$$

¿From it we obtain an asymptotic morphism

$$\sigma : \Sigma C_\pi \to \mathfrak{A}\Sigma J.$$

5.14. Proposition. *The inclusion $*$-homomorphism $\Sigma\tau\colon \Sigma J \to \Sigma C_\pi$ and the asymptotic morphism $\sigma : \Sigma C_\pi \to \mathfrak{A}\Sigma J$ determine mutually inverse morphisms in the category $\mathfrak{A}$.*

Proof. See [**15**, Theorem 13] or [**18**, Proposition 7.1]. Consider first the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma J & \longrightarrow & CJ & \xrightarrow{\pi_0} & J & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Sigma J & \longrightarrow & CB & \xrightarrow{\pi_1} & C_\pi & \longrightarrow & 0 \end{array}$$

in which the unlabeled $*$-homomorphisms are inclusions. By Proposition 5.8 it gives rise to a commuting diagram

$$\begin{array}{ccc} \Sigma J & \longrightarrow & \Sigma J \\ \downarrow & & \downarrow{\scriptstyle =} \\ \Sigma C_\pi & \xrightarrow[\sigma]{} & \Sigma J \end{array}$$

in the homotopy category of asymptotic morphisms. By Propositions 5.9 and 5.11 the top morphism is the identity. The diagram shows that the morphism in $\mathfrak{A}$ associated to the inclusion $\Sigma J \to \Sigma C_\pi$ is right inverse to σ. To prove that it is left inverse we consider the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma J & \longrightarrow & CB & \xrightarrow{\pi_1} & C_\pi & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \Sigma C_\pi & \longrightarrow & CC_\pi & \xrightarrow{\pi_0} & C_\pi & \longrightarrow & 0 \end{array}$$

where $\varphi\colon CB \to CC_\pi$ maps $f \in CB$ to the function $F\colon I \to C_\pi$ defined by $F(t) = f(t) \oplus f_t$, and $f_t(x) = \pi f(x+t)$ (in defining $f(x+t)$ we extend f to be zero on $[1,\infty)$). There is a corresponding commuting diagram

$$\begin{array}{ccc} \Sigma C_\pi & \xrightarrow{\sigma} & \Sigma J \\ {\scriptstyle =}\downarrow & & \downarrow \\ \Sigma C_\pi & \longrightarrow & \Sigma C_\pi \end{array}$$

in the homotopy category of asymptotic morphisms. Propositions 5.9 and 5.11 now show that σ is right inverse to the morphism $\Sigma J \to \Sigma C_\pi$. □

The following result is in the same spirit. We shall use it in the next chapter. Recall that τ denotes the inclusion of J into C_π, and that the $*$-homomorphism $\beta\colon \Sigma A \to C_\pi$ was introduced in Definition 5.13.

5.15. Lemma. *Let $\sigma : \Sigma A \to \mathfrak{A}J$ be the asymptotic morphism associated to the short exact sequence*

$$0 \to J \to B \xrightarrow{\pi} A \to 0.$$

In the category $\mathfrak{A}$ the composition

$$\Sigma^2 A \xrightarrow{\Sigma\sigma} \Sigma J \xrightarrow{\Sigma\tau} \Sigma C_\pi$$

is equal to $\Sigma\beta : \Sigma^2 A \to \Sigma C_\pi$.

Proof. Compare [**18**, Proposition 2.14]. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma J & \longrightarrow & \Sigma B & \xrightarrow{\Sigma\pi} & \Sigma A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Sigma\tau} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \Sigma C_\pi & \longrightarrow & CC_\pi & \xrightarrow{\pi_0} & C_\pi & \longrightarrow & 0, \end{array}$$

where the middle vertical map $\varphi\colon \Sigma B \to CC_\pi$ is same $*$-homomorphism that appeared in the proof of Proposition 5.14 but restricted to $\Sigma B \subset CB$. As in the proof of Proposition 5.14, the result now follows from the following commuting diagram in the category $\mathfrak{A}$:

$$\begin{array}{ccc} \Sigma^2 A & \xrightarrow{\Sigma\sigma} & \Sigma J \\ {\scriptstyle \Sigma\beta}\downarrow & & \downarrow{\scriptstyle \Sigma\tau} \\ \Sigma C_\pi & \xrightarrow[=]{} & \Sigma C_\pi. \end{array}$$

□

Our main application of mapping cones is to the development of excision results, such as the following:

5.16. Proposition. *Let $\theta\colon B \to A$ be a $*$-homomorphism and let D be a C^*-algebra. The sequence of pointed sets*

$$[\![D, C_\theta]\!] \xrightarrow{\alpha_*} [\![D, B]\!] \xrightarrow{\theta_*} [\![D, A]\!]$$

is exact. The maps α_* and θ_* in the sequence are induced by composition with the $*$-homomorphisms α and θ.

Before we begin the proof let us note the following fact. Let $\varphi\colon D \to \mathfrak{A}^n B$ be a $*$-homomorphism, and suppose that $\theta \circ \varphi : D \to \mathfrak{A}^n A$ is n-homotopic to zero. Then there is a $*$-homomorphism $\eta\colon D \to \mathfrak{A}^n CA$ from which $\theta \circ \varphi$ may be recovered by composition with evaluation at zero $\mathfrak{A}^n CA \to \mathfrak{A}^n A$. This is a consequence of Lemma 2.4.

Let us also note that

$$C_\theta = B \underset{A}{\oplus} CA,$$

where B maps to A via θ and CA maps to A via evaluation at zero. It follows from Lemma 2.5 that

$$\mathfrak{A}^n C_\theta \cong \mathfrak{A}^n B \underset{\mathfrak{A}^n A}{\oplus} \mathfrak{A}^n CA.$$

Proof of Proposition 5.16. It suffices to show that if $\varphi : D \to \mathfrak{A}^n B$ is a $*$-homomorphism, and if $\theta \circ \varphi : D \to \mathfrak{A}^n A$ is n-homotopic to zero, then there is a $*$-homomorphism $\psi\colon D \to \mathfrak{A}^n C_\theta$ such that $\alpha \circ \psi\colon D \to \mathfrak{A}^n B$ is n-homotopic to φ. In fact we shall construct ψ so that $\alpha \circ \psi$ is actually equal to φ.

Let $\eta\colon D \to \mathfrak{A}^n CA$ be a homotopy connecting $\theta \circ \varphi$ to zero. Together with φ, the $*$-homomorphism η determines a $*$-homomorphism

$$\psi = \varphi \oplus \eta\colon D \to \mathfrak{A}^n B \underset{\mathfrak{A}^n A}{\oplus} \mathfrak{A}^n CA.$$

As noted above, there is a natural isomorphism

$$\mathfrak{A}^n B \underset{\mathfrak{A}^n A}{\oplus} \mathfrak{A}^n CA \cong \mathfrak{A}^n (B \underset{A}{\oplus} CA),$$

while $B \underset{A}{\oplus} CA = C_\theta$. The $*$-homomorphism $\psi : D \to \mathfrak{A}^n(C_\theta)$ obtained through these identifications has the property that $\alpha \circ \psi = \varphi\colon D \to \mathfrak{A}B^n$, as required. $\square$

We should like to prove a version of Proposition 5.16 for the sequence

$$[\![A, D]\!] \xrightarrow{\theta^*} [\![B, D]\!] \xrightarrow{\alpha^*} [\![C_\theta, D]\!].$$

Unfortunately the best that can be done in this regard are the following two results. The first is a standard calculation, having nothing to do with asymptotic morphisms.

5.17. Proposition. *Let F be a contravariant functor from the homotopy category of C^*-algebras to pointed sets. If for every $\theta\colon B \to A$ the sequence of pointed sets*

$$F(C_\theta) \xrightarrow{\beta^*} F(\Sigma A) \xrightarrow{(\Sigma\theta)^*} F(\Sigma B)$$

is exact then the sequence

$$F(\Sigma A) \xrightarrow{(\Sigma\theta)^*} F(\Sigma B) \xrightarrow{(\Sigma\alpha)^*} F(\Sigma C_\theta)$$

is exact.

Proof. We apply the hypothesis to the mapping cone of the $*$-homomorphism $\alpha : C_\theta \to B$ of Definition 5.13. Note that α is surjective with kernel ΣA. In view of Definition 5.13 and the discussion following it there are $*$-homomorphisms

$$\beta\colon \Sigma B \to C_\alpha \quad \text{and} \quad \tau\colon \Sigma A \to C_\alpha.$$

The second is a homotopy equivalence [**31**, Theorem 3.8]. Applying our hypothesis on the functor F we see that the top row of the diagram

$$\begin{array}{ccccc} F(C_\alpha) & \xrightarrow{\beta^*} & F(\Sigma B) & \xrightarrow{(\Sigma\alpha)^*} & F(\Sigma C_\theta) \\ {\scriptstyle\tau^*}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle=} \\ F(\Sigma A) & \xrightarrow[(\Sigma\theta)^*]{} & F(\Sigma B) & \longrightarrow & F(\Sigma C_\theta) \end{array}$$

is exact. We wish to prove that the bottom row is exact. The left hand square does not commute, but if $\rho\colon \Sigma \to \Sigma$ is the inversion $f(x) \mapsto f(1-x)$ then the composition

$$\Sigma B \xrightarrow{\Sigma\theta} \Sigma A \xrightarrow{\rho} \Sigma A \xrightarrow{\tau} C_\alpha,$$

is homotopic to $\beta\colon \Sigma B \to C_\alpha$. Thus the square commutes 'modulo' the inversion isomorphism $\rho^*\colon F(\Sigma B) \to F(\Sigma B)$. This is enough to prove exactness in the bottom row of the diagram. $\square$

5.18. Proposition. *Let $\theta\colon B \to A$ be a $*$-homomorphism and let D be a C^*-algebra. If the vertical suspension maps in the commuting diagram*

$$\begin{array}{ccccc} & & [\![A, D]\!] & \xrightarrow{\theta^*} & [\![B, D]\!] \\ & & \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} \\ [\![C_\theta, \Sigma D]\!] & \xrightarrow{\beta^*} & [\![\Sigma A, \Sigma D]\!] & \xrightarrow{\Sigma\theta^*} & [\![\Sigma B, \Sigma D]\!]. \end{array}$$

are isomorphisms then the bottom row is exact. The maps β^* and θ^* are induced by composition with β and θ.

PROOF. It suffices to show that if $\varphi\colon A \to \mathfrak{A}^n D$ is a $*$-homomorphism and if the composition $\varphi \circ \theta\colon B \to \mathfrak{A}^n D$ is n-homotopic to zero then the suspension of φ, $\Sigma\varphi\colon \Sigma A \to \mathfrak{A}^n \Sigma D$, is n-homotopic to a composition

$$\Sigma A \xrightarrow{\beta} C_\theta \xrightarrow{\psi} \mathfrak{A}^n \Sigma D,$$

for some ψ. Let $\Sigma_1 D$ denote the C^*-algebra of continuous functions from $[-1,1]$ into D which vanish at both endpoints. There is a natural inclusion $\Sigma D \to \Sigma_1 D$, and since it is a homotopy equivalence, to prove the proposition it suffices to construct a $*$-homomorphism

$$\psi\colon C_\theta \to \mathfrak{A}^n \Sigma_1 D$$

such that the composition $\beta \circ \psi : \Sigma A \to \mathfrak{A}^n \Sigma_1 D$ is homotopic to the suspension of φ, followed with the inclusion of $\mathfrak{A}^n \Sigma D$ into $\mathfrak{A}^n \Sigma_1 D$.

Let $C_1 D$ denote the continuous functions from $[-1,0]$ to D which vanish at -1 and let

$$\eta\colon B \to \mathfrak{A}^n C_1 D$$

be a homotopy connecting zero to the composition $\varphi \circ \theta$. Form the cone on φ,

$$C\varphi\colon CA \to \mathfrak{A}^n CD,$$

which, as described in Chapter 3, is the composition $CA \to C\mathfrak{A}^n D \to \mathfrak{A}^n CD$ and has the property that when followed by the map $\mathfrak{A}^n CD \to \mathfrak{A}^n D$ of evaluation at t it yields the composition of evaluation at t and φ, $CA \to A \to \mathfrak{A}^n D$.

The $*$-homomorphisms η and $C\varphi$ determine a $*$-homomorphism

$$C_\theta \cong B \underset{A}{\oplus} CA \xrightarrow{\eta \oplus C\varphi} \mathfrak{A}^n C_1 D \underset{\mathfrak{A}^n D}{\oplus} \mathfrak{A}^n CD \cong \mathfrak{A}^n (C_1 D \underset{D}{\oplus} CD).$$

But $C_1 D \underset{D}{\oplus} CD = \Sigma_1 D$, and we obtain a $*$-homomorphism $\psi\colon C_\theta \to \mathfrak{A}^n \Sigma_1 D$ with the required property. $\square$

CHAPTER 6

E-Theory

In this chapter we define the equivariant E-theory groups $E_G(A, B)$ for C^*-algebras A and B. We develop some of the basic properties of these groups: namely stability, excision, and Bott periodicity. In the case of a trivial group G we also identify $E_G(\mathbb{C}, B)$ with the K-theory group $K_0(B)$.

6.1. Definition. A *G-Hilbert space* is a separable Hilbert space equipped with a continuous unitary action of G. The *standard G-Hilbert space* is the countable direct sum of Hilbert spaces

$$\mathcal{H} = L^2(G) \oplus L^2(G) \oplus \cdots .$$

equipped with the left regular representation of G.

Henceforth we shall say 'Hilbert space' in place of 'G-Hilbert space'. Unitary isomorphisms and isometries between Hilbert spaces will be assumed to be equivariant.

The standard Hilbert space $\mathcal{H}$ has the following useful and well-known properties:

6.2. Lemma. *If $\mathcal{H}'$ is any G-Hilbert space then $\mathcal{H} \cong \mathcal{H} \otimes \mathcal{H}'$.*

Proof. See for instance [**26**, Theorem 2.4]. □

6.3. Lemma. *Any isometry $V\colon \mathcal{H} \to \mathcal{H}$ is path connected to the identity, in the $*$-strong topology.*

Proof. Write $\mathcal{H}$ as a tensor product

$$\mathcal{H} \cong L^2(G) \otimes \mathcal{H}_0,$$

where $\mathcal{H}_0$ is a separable Hilbert space equipped with the trivial G-action. Let $\{W_t\}$ $(0 < t \leq 1)$ be a $*$-strongly continuous path of isometries on $\mathcal{H}_0$ such that $W_1 = I$ and $\lim_{t\to 0} W_t W_t^* = 0$, in the strong topology. If V is an isometry on $\mathcal{H}$ then define a $*$-strongly continuous path of isometries connecting $V_1 = V$ to $V_0 = 1$ by

$$V_t = (1 \otimes W_t)V(1 \otimes W_t^*) + (1 - 1 \otimes W_t W_t^*). \quad \square$$

6.4. Definition. Denote by $\mathcal{K}(\mathcal{H})$ the C^*-algebra of compact operators on $\mathcal{H}$, equipped with the continuous action of G induced from the unitary action of G on $\mathcal{H}$.

¿From Lemma 6.3 we obtain:

6.5. Lemma. *Any two $*$-homomorphisms from $\mathcal{K}(\mathcal{H})$ into $\mathcal{K}(\mathcal{H})$ which are induced from isometries of $\mathcal{H}$ into itself are homotopic.* $\square$

Since $\mathcal{H} \oplus \mathcal{H} \cong \mathcal{H}$ there is a canonical, up to homotopy, map

$$\Delta\colon \mathcal{K}(\mathcal{H}) \oplus \mathcal{K}(\mathcal{H}) \hookrightarrow \mathcal{K}(\mathcal{H} \oplus \mathcal{H}) \xrightarrow{\cong} \mathcal{K}(\mathcal{H}),$$

where the first map is the inclusion as the diagonal and the second is induced from a unitary isomorphism $\mathcal{H} \oplus \mathcal{H} \cong \mathcal{H}$.

6.6. Lemma. *Let A and B be C^*-algebras. The set $[\![A, B \otimes \mathcal{K}(\mathcal{H})]\!]$ of homotopy classes of asymptotic morphisms from A into $B \otimes \mathcal{K}(\mathcal{H})$ becomes a commutative semigroup under the direct sum operation which associates to a pair of $*$-homomorphisms $\varphi_0, \varphi_1\colon A \to \mathfrak{A}^n(B \otimes \mathcal{K}(\mathcal{H}))$ the $*$-homomorphism*

$$A \xrightarrow{\varphi_0 \oplus \varphi_1} \mathfrak{A}^n(B \otimes \mathcal{K}(\mathcal{H})) \oplus \mathfrak{A}^n(B \otimes \mathcal{K}(\mathcal{H})) \xrightarrow{\Delta} \mathfrak{A}^n(B \otimes \mathcal{K}(\mathcal{H})).$$

The zero element of this semigroup is represented by the zero asymptotic morphism.

Proof. The proof that the direct sum operation is well-defined is left to the reader. It follows from Lemma 6.5 that the zero asymptotic morphism is a zero for addition. Commutativity of addition follows from the observation that the maps $\mathcal{K}(\mathcal{H}) \oplus \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H} \oplus \mathcal{H})$ given by

$$(T, S) \longmapsto \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \quad \text{and} \quad (T, S) \longmapsto \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$$

are homotopic as $*$-homomorphisms, by a standard rotation homotopy. $\square$

6.7. Lemma. *Let A and B be C^*-algebras. The set $[\![A, \Sigma B \otimes \mathcal{K}(\mathcal{H})]\!]$ is an abelian group.*

Proof. We must show that the semigroup $[\![A, \Sigma B \otimes \mathcal{K}(\mathcal{H})]\!]$ contains additive inverses. Denote by $\rho\colon \Sigma \to \Sigma$ the inversion $*$-homomorphism $f(x) \mapsto f(1-x)$. Then the additive inverse of $\varphi : A \to \mathfrak{A}^n(\Sigma B \otimes \mathcal{K}(\mathcal{H}))$ is given by composition with ρ. The proof of this is a standard argument [**31**, Theorem 3.1] and is left to the reader. $\square$

6.8. Definition. Let A and B be G-C^*-algebras. The equivariant E-theory group $E_G(A, B)$ is defined by

$$E_G(A, B) = [\![\Sigma A \otimes \mathcal{K}(\mathcal{H}), \Sigma B \otimes \mathcal{K}(\mathcal{H})]\!].$$

¿From Chapter 3 we obtain the following result:

6.9. Theorem. *The groups $E_G(A, B) = [\![\Sigma A \otimes \mathcal{K}(\mathcal{H}), \Sigma B \otimes \mathcal{K}(\mathcal{H})]\!]$ are the morphism sets in an additive category whose objects are the G-C^*-algebras, and whose associative composition law*

$$E_G(A, B) \otimes E_G(B, C) \to E_G(A, C)$$

is composition in the homotopy category of asymptotic morphisms. There is a functor from the homotopy category of C^*-algebras into the E-theory category which associates to a $*$-homomorphism $\varphi\colon A \to B$ the class of the $*$-homomorphism

$$1 \otimes \varphi \otimes 1\colon \Sigma A \otimes \mathcal{K}(\mathcal{H}) \to \Sigma B \otimes \mathcal{K}(\mathcal{H}).$$

Proof. We need only verify that composition is bilinear. This is left to the reader. □

By incorporating $\mathcal{K}(\mathcal{H})$ into the definition of $E_G(A, B)$ we have made the following matrix-stability property of E-theory almost a tautology:

6.10. Proposition. *Let $\mathcal{H}_0$ be a Hilbert space (as usual with a continuous unitary action of G) and let $\varphi\colon A \to B$ be a $*$-homomorphism such that the tensor product $\varphi \otimes 1\colon A \otimes \mathcal{K}(\mathcal{H}_0) \to B \otimes \mathcal{K}(\mathcal{H}_0)$ is equivariantly homotopy equivalent to a $*$-isomorphism. Then φ determines an invertible element of $E_G(A, B)$.*

Proof. Under the identifications $\mathcal{H} \otimes \mathcal{H}_0 \cong \mathcal{H}$ and $\mathcal{K}(\mathcal{H} \otimes \mathcal{H}_0) \cong \mathcal{K}(\mathcal{H})$ the assumptions imply that $1 \otimes \varphi \otimes 1$ is equivariantly homotopic to an isomorphism. It follows immediately from the definitions that equivariantly homotopic $*$-homomorphisms determine the same element of an E_G-theory group and that $*$-isomorphisms determine invertible elements. □

Although we shall not need it here, Proposition 6.10 implies stronger forms of stability, related to the theory of Morita equivalence. Let us quickly sketch two results in this direction.

6.11. Proposition. *Let A be a C^*-algebra and let p be a G-invariant projection in the multiplier algebra of A such that ApA is dense in A. Then the inclusion $pAp \hookrightarrow A$ determines an invertible element of $E_G(pAp, A)$.*

Proof. This follows from the fact that the inclusion

$$pAp \otimes \mathcal{K}(\mathcal{H}) \hookrightarrow A \otimes \mathcal{K}(\mathcal{H})$$

is homotopic to a $*$-isomorphism. In the non-equivariant case this is a consequence the results of Brown, Green, Rieffel [**9**] and Brown [**7**]. In the equivariant case the proof is similar, but uses the results of Curto, Muhly and Williams [**14**] and Mingo, Phillips [**26**, Corollary 2.6] instead. □

Here is one more formulation of stability:

6.12. Theorem. *An equivariant Morita equivalence between C^*-algebras A and B determines an invertible element $\alpha \in E_G(A, B)$.*

Proof. An equivariant Morita equivalence [**14**] may be viewed as a diagram

$$A \hookrightarrow C \hookleftarrow B,$$

in which both inclusions are of the type considered in the previous proposition. □

We now investigate the exactness properties of E_G-theory.

6.13. Definition. A functor F from C^*-algebras to abelian groups is *half-exact* if, for every short exact sequence of separable C^*-algebras

$$0 \to J \to B \to A \to 0$$

the sequence of abelian groups

$$F(J) \to F(B) \to F(A)$$

is exact at $F(B)$.

We have stated the definition for covariant functors but it has an obvious contravariant counterpart, as does the following well-known result.

6.14. Proposition. *Let F be a half-exact homotopy functor from the category of C^*-algebras to abelian groups. For every short exact sequence of separable C^*-algebras*

$$0 \to J \to B \to A \to 0$$

there is a long exact sequence of abelian groups

$$\cdots \to F(\Sigma B) \to F(\Sigma A) \xrightarrow{\partial_*} F(J) \to F(B) \to F(A),$$

in which the connecting homomorphism $\partial_*\colon F(\Sigma A) \to F(J)$ fits into a commuting triangle (which determines it uniquely)

$$\begin{tikzcd} F(\Sigma A) \arrow[rr, "\partial_*"] \arrow[dr, "\beta_*"'] & & F(J) \arrow[dl, "\tau_*"] \\ & F(C_\pi) & \end{tikzcd} .$$

□

For a proof, see for instance [**6**, Theorem 21.4.1].

6.15. Theorem. *Let D be a separable C^*-algebra, let*

$$0 \to J \to B \to A \to 0$$

be a short exact sequence of separable C^*-algebras, and let $\sigma \in E_G(\Sigma A, J)$ be the E-theory class of the associated asymptotic morphism, as in Proposition 5.5. There is a long exact sequences of E_G-theory groups:

$$\cdots \to E_G(D, \Sigma A) \xrightarrow{\partial_*} E_G(D, J) \to E_G(D, B) \to E_G(D, A),$$

in which the map ∂_* is given by the E_G-theory product with $\sigma \in E_G(\Sigma A, J)$.

Proof. It follows from Proposition 5.16 that the functor $F(A) = E_G(D, A)$ is half exact. The result follows from Theorem 6.14, with exception of the identification of the connecting homomorphism ∂_* as E_G-theory product with σ. This follows from Lemma 5.15 which shows that the diagram

$$\begin{tikzcd} E_G(D, \Sigma A) \arrow[rr, "\sigma_*"] \arrow[dr, "\beta_*"'] & & E_G(D, J) \arrow[dl, "\tau_*"] \\ & E_G(C, C_\pi) & \end{tikzcd}$$

commutes. □

Before discussing long exact sequences in the first variable of $E_G(A, B)$, we need to gather together some results concerning suspension in E-theory. To begin with, the tensor product in the category $\mathfrak{A}$ determines a tensor product operation

$$E_G(A, B) \to E_G(A \otimes D, B \otimes D)$$

which is compatible with composition. In particular there is a *suspension functor*

$$\Sigma\colon E_G(A,B) \to E_G(\Sigma A, \Sigma B).$$

6.16. Proposition. *There is an asymptotic morphism*

$$\sigma\colon \Sigma^2 \to \mathfrak{A}(\mathcal{K}(\mathcal{H}_0)),$$

(*where $\mathcal{H}_0$ denotes a Hilbert space with a trivial G-action*) *such that for every C^*-algebra B the asymptotic morphism*

$$\sigma \otimes 1\colon \Sigma^2 \otimes B \to \mathfrak{A}(\mathcal{K}(\mathcal{H}_0) \otimes B)$$

determines an invertible class $\sigma_B \in E_G(\Sigma^2 B, B)$. The class $\sigma_B \in E_G(\Sigma^2 B, B)$ has the property that for every $\alpha \in E_G(B_1, B_2)$ the diagram

$$\begin{array}{ccc} \Sigma^2 B_1 & \xrightarrow{\sigma_{B_1}} & B_1 \\ {\scriptstyle \Sigma^2\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \Sigma^2 B_2 & \xrightarrow{\sigma_{B_2}} & B_2 \end{array}$$

commutes.

Proof. We use an elegant version of the Bott periodicity theorem, due to Cuntz [**13**, Section 4], which asserts that every stable, half-exact homotopy functor on the category of C^*-algebras satisfies Bott periodicity. The theorem gives a concrete description of the Bott isomorphism, of which we make use as well.

Define a stable, half-exact homotopy functor F on category of separable C^*-algebras without G-action by $F(C) = E_G(A, B \otimes C)$. The conclusion of Cuntz's theorem is that there is a natural isomorphism $E_G(A, \Sigma^2 B) \cong E_G(A, B)$. Furthermore, there is a C^*-algebra D and a short exact sequence

$$0 \to \mathcal{K}(\mathcal{H}_0) \to D \to \Sigma \to 0.$$

such that the periodicity isomorphism is the connecting homomorphism

$$F(\Sigma^2 B) \to F(\mathcal{K}(\mathcal{H}_0) \otimes B) \cong F(B)$$

associated to the short exact sequence

$$0 \to \mathcal{K}(\mathcal{H}_0) \otimes B \to D \otimes B \to \Sigma \otimes B \to 0.$$

By Proposition 5.9 the asymptotic morphism associated to this short exact sequence is $\sigma \otimes 1_B$, where σ denotes the asymptotic morphism associated to the previous short exact sequence.

That $\sigma \otimes 1 \in E_G(\Sigma^2 \otimes B, \mathcal{K}(\mathcal{H}_0) \otimes B)$ is invertible follows from the discussion of tensor products in Chapter 4 and the fact that $\sigma \in E_G(\Sigma^2, \mathcal{K}(\mathcal{H}_0))$ itself is invertible. To deduce this latter fact take $A = \mathbb{C} = B$ in the above discussion to deduce that product with σ gives an isomorphism $E_G(\mathbb{C}, \Sigma^2) \to E_G(\mathbb{C}, \mathbb{C})$. Thus, there exists $\rho \in E_G(\mathbb{C}, \Sigma^2)$ such that $\sigma\rho = 1_{\mathbb{C}} \in E_G(\mathbb{C}, \mathbb{C})$. On the other hand, taking $A = \Sigma^2$ and $B = \mathbb{C}$ we see that product with σ gives an isomorphism $E_G(\Sigma^2, \Sigma^2) \to E_G(\Sigma^2, \mathbb{C})$. This isomorphism maps both 1_{Σ^2} and $\rho\sigma$ to σ, hence $\rho\sigma = 1_{\Sigma^2}$.

We turn now to the commutativity of the diagram in the statement of the proposition. Let $\alpha \in E_G(B_1, B_2)$. To simplify notation write $D_k = \Sigma \otimes B_k \otimes \mathcal{K}(\mathcal{H})$,

for $k = 1, 2$. Then, α is an asymptotic morphism $D_1 \to \mathfrak{A}D_2$. The element $\sigma_{B_k} \in E_G(\Sigma^2 \otimes B_k, B_k)$ is given by the explicit asymptotic morphism

$$\sigma \otimes 1_{B_k} : \Sigma^2 \otimes D_k \to \mathcal{K}(\mathcal{H}_0) \otimes D_k.$$

The commutativity of

$$\begin{array}{ccc} \Sigma^2 \otimes D_1 & \xrightarrow{\sigma_{B_1}} & \mathcal{K}(\mathcal{H}_0) \otimes D_1 \\ {\scriptstyle 1_{\Sigma^2} \otimes \alpha} \downarrow & & \downarrow {\scriptstyle 1_{\mathcal{K}(\mathcal{H}_0)} \otimes \alpha} \\ \Sigma^2 \otimes D_2 & \xrightarrow{\sigma_{B_2}} & \mathcal{K}(\mathcal{H}_0) \otimes D_2 \end{array}$$

is simply the identity for composition of asymptotic morphisms

$$1_{\mathcal{K}(\mathcal{H}_0)} \otimes \alpha \circ \sigma \otimes 1_{D_1} = \sigma \otimes 1_{D_2} \circ 1_{\Sigma^2} \otimes \alpha.$$

which follows from Lemma 4.5. □

6.17. Proposition. *For all separable G-C^*-algebras A and B the suspension map $\Sigma : E_G(A, B) \to E_G(\Sigma A, \Sigma B)$ is an isomorphism.*

Proof. It suffices to show that

$$\Sigma^2 : E_G(A, B) \to E_G(\Sigma^2 A, \Sigma^2 B)$$

is an isomorphism for all such A and B. It follows from Proposition 6.16, in the notation of that proposition, that

$$\Sigma^2 \alpha = \sigma_B^{-1} \alpha \sigma_A,$$

for all $\alpha \in E_G(A, B)$. Thus, Σ^2 is the product on the left and right by invertible elements, and is therefore an isomorphism. □

6.18. Theorem. *Let D be a separable C^*-algebra, let*

$$0 \to J \to B \to A \to 0$$

be a short exact sequence of separable C^*-algebras, and let $\sigma \in E_G(\Sigma A, J)$ be the E-theory class of the associated asymptotic morphism, as in Proposition 5.5. There is a long exact sequences of E_G-theory groups:

$$\ldots \leftarrow E_G(\Sigma A, D) \xleftarrow{\partial^*} E_G(J, D) \leftarrow E_G(B, D) \leftarrow E_G(A, D),$$

in which the map ∂^* is given by the E_G-theory product with $\sigma \in E_G(\Sigma A, J)$.

Proof. By the contravariant version of Proposition 6.14, and the method of proof of Proposition 6.15, it suffices to show that the sequence

$$E_G(J, D) \leftarrow E_G(B, D) \leftarrow E_G(A, D)$$

is exact at $E_G(B, D)$. To do this define a contravariant homotopy functor by

$$F(C) = E_G(C, \Sigma D).$$

To simplify notation write $D' = \Sigma D \otimes \mathcal{K}$. Consider the following diagram;

$$\begin{tikzcd}
F(C_\pi) \arrow[r, "\beta^*"] & F(\Sigma A) \arrow[r, "(\Sigma\pi)^*"] & F(\Sigma B) \\
{[\![\Sigma C_\pi \otimes \mathcal{K}, \Sigma D']\!]} \arrow[u, "="] \arrow[r] & {[\![\Sigma^2 A \otimes \mathcal{K}, \Sigma D']\!]} \arrow[u, "="] \arrow[r] & {[\![\Sigma^2 B \otimes \mathcal{K}, \Sigma D']\!]} \arrow[u, "="] \\
 & {[\![\Sigma A \otimes \mathcal{K}, D']\!]} \arrow[u, "\cong"] \arrow[r] & {[\![\Sigma B \otimes \mathcal{K}, D']\!]} \arrow[u, "\cong"]
\end{tikzcd}$$

By Proposition 6.17 the vertical arrows are isomorphisms, and by Proposition 5.18 the middle row is exact. Therefore, by Proposition 5.17 the rows in the diagram

$$\begin{tikzcd}
F(\Sigma A) \arrow[r, "(\Sigma\pi)^*"] & F(\Sigma B) \arrow[r, "(\Sigma\alpha)^*"] & F(\Sigma C_\pi) \\
E_G(\Sigma A, \Sigma D) \arrow[u, "="] \arrow[r] & E_G(\Sigma B, \Sigma D) \arrow[u, "="] \arrow[r] & E_G(\Sigma C_\pi, \Sigma D) \arrow[u, "="] \\
E_G(A, D) \arrow[u, "\cong"', "\Sigma"] \arrow[r, "\pi^*"] & E_G(B, D) \arrow[u, "\cong"', "\Sigma"] \arrow[r, "\alpha^*"] & E_G(C_\pi, D) \arrow[u, "\cong"', "\Sigma"]
\end{tikzcd}$$

are exact, the lower vertical arrows being isomorphisms by Proposition 6.17. Identifying $E_G(C_\pi, D) \cong E_G(J, D)$ via the isomorphism τ^* completes the proof. $\square$

As an immediate corollary of Proposition 6.16 we obtain a version of Bott periodicity for E_G-theory.

6.19. Bott Periodicity Theorem. *The E_G-theory satisfies Bott periodicity in each of its variables. Precisely, there are isomorphisms*

$$E_G(A, B) \cong E_G(A, \Sigma^2 B) \quad \textit{and} \quad E_G(A, B) \cong E_G(\Sigma^2 A, B)$$

natural in the separable G-C^*-algebras A and B. $\square$

We come to our main results, beginning with the existence of cyclic 6-term exact sequences in each of the variables of E_G-theory. The following theorem follows immediately from Theorems 6.15, 6.18 and 6.19.

6.20. Theorem. *Let D be a separable C^*-algebra and let*

$$0 \to J \to B \to A \to 0$$

be a short exact sequence. There are exact sequences

$$\begin{tikzcd}
E_G(D, \Sigma J) \arrow[r] & E_G(D, \Sigma B) \arrow[r] & E_G(D, \Sigma A) \arrow[d] \\
E_G(D, A) \arrow[u] & E_G(D, B) \arrow[l] & E_G(D, J) \arrow[l]
\end{tikzcd}$$

and

$$\begin{array}{ccccc} E_G(\Sigma J, D) & \longleftarrow & E_G(\Sigma B, D) & \longleftarrow & E_G(\Sigma A, D) \\ \downarrow & & & & \uparrow \\ E_G(A, D) & \longrightarrow & E_G(B, D) & \longrightarrow & E_G(J, D). \end{array}$$

The boundary maps are given by Bott periodicity and product with the E_G-theory class $\sigma \in E_G(\Sigma A, J)$ associated to the short exact sequence. □

Proposition 6.17 also allows us to define a tensor product bifunctor on the E-theory category:

6.21. Theorem. *There is a tensor product bifunctor*

$$E_G(A, B) \otimes E_G(C, D) \to E_G(A \otimes C, B \otimes D)$$

which is compatible with composition in E-theory, and which, on classes determined by $*$-homomorphisms, agrees with the maximal tensor product of $*$-homomorphisms. If $1 \in E_G(\mathbb{C}, \mathbb{C})$ denotes the identity then $1 \otimes \varphi \in E_G(\mathbb{C} \otimes A, \mathbb{C} \otimes B)$ is equal to $\varphi \in E_G(A, B)$, once $\mathbb{C} \otimes A$ and $\mathbb{C} \otimes B$ are identified with A and B.

Proof. The tensor product on the category $\mathfrak{A}$ defined in Chapter 4 gives a bifunctor

$$E_G(A, B) \otimes E_G(C, D) \to E_G(\Sigma A \otimes C, \Sigma B \otimes D).$$

The result follows from this together with the suspension isomorphism of Proposition 6.17. □

We turn now to the descent functor.

6.22. Theorem. *There is a descent functor*

$$E_G(A, B) \to E(C^*(G, A), C^*(G, B))$$

which is compatible with composition in E-theory and which associates to the class of a $*$-homomorphism the class of the induced $*$-homomorphism on crossed products.

Proof. This follows from Theorem 4.12, together with the well-known isomorphism

$$C^*(G, D \otimes \mathcal{K}(\mathcal{H})) \cong C^*(G, D) \otimes \mathcal{K}(\mathcal{H}),$$

which is in turn a consequence of the fact that the action of G on $\mathcal{K}(\mathcal{H})$ is inner.□

For the remainder of this chapter we consider the non-equivariant version of E-theory and hence drop the group G from our notation. We identify the E-theory group $E(\mathbb{C}, B)$ with the C^*-algebra K-theory group $K_0(B)$.

Our proof is based on the following is a result of Rosenberg [**31**, Theorem 4.1].

6.23. Proposition. *Fix a generator of the K-theory group $K_1(\Sigma)$. The map which assigns to a $*$-homomorphism $\Sigma \to \Sigma B \otimes \mathcal{K}(\mathcal{H})$ the image under the induced map on K-theory of this generator in $K_1(\Sigma B \otimes \mathcal{K}(\mathcal{H})) \cong K_0(B)$ induces an isomorphism*

$$[\Sigma, \Sigma B \otimes \mathcal{K}(\mathcal{H})] \cong K_0(B). \quad \square$$

6.24. Theorem. *There is a natural isomorphism $E(\mathbb{C}, B) \cong K_0(B)$.*

For the proof of Theorem 6.24 we require one final result. The following simple lemma will also be proved later (see Corollary 9.8) in a more general setting.

6.25. Lemma. *Let A and B be C^*-algebras, A separable. Then*

$$E(A,B) \cong [\![\Sigma A, \Sigma B \otimes \mathcal{K}]\!]$$

Proof. Let p be a rank-one projection in $\mathcal{K}$. A map

$$[\![\Sigma A \otimes \mathcal{K}, \Sigma B \otimes \mathcal{K}]\!] \to [\![\Sigma A, \Sigma B \otimes \mathcal{K}]\!]$$

is constructed by composition with the $*$-homomorphism $1 \otimes p \colon \Sigma A \to \Sigma A \otimes \mathcal{K}$. Its inverse map

$$[\![\Sigma A, \Sigma B \otimes \mathcal{K}]\!] \to [\![\Sigma A \otimes \mathcal{K}, \Sigma B \otimes \mathcal{K}]\!]$$

is constructed by tensoring with $\mathcal{K}$. $\square$

Proof of Theorem 6.24. According to Proposition 2.19 the natural map

$$[\Sigma, \Sigma B \otimes \mathcal{K}(\mathcal{H})] \to [\![\Sigma, \Sigma B \otimes \mathcal{K}(\mathcal{H})]\!],$$

taking homotopy classes of $*$-homomorphisms to homotopy classes of asymptotic morphisms, is an isomorphism. The theorem is therefore proved by the chain of isomorphisms

$$E(\mathbb{C}, B) \cong [\![\Sigma, \Sigma B \otimes \mathcal{K}(\mathcal{H})]\!] \cong [\Sigma, \Sigma B \otimes \mathcal{K}(\mathcal{H})] \cong K_0(B). \quad \square$$

CHAPTER 7

Cohomological Properties

In this chapter we shall apply some standard homological machinery to the bifunctor $E_G(A, B)$ to obtain $\varprojlim^1$ and universal coefficient exact sequences. For the most part the proofs of these results exactly parallel corresponding proofs in Kasparov's KK-theory [**6**], so we shall need to do little more than state the theorems. We shall in any case have no need for these results in what follows. We conclude with a discussion of the relation between E-theory and K-homology theory, and again we shall refer the reader to papers on Kasparov's KK-theory for complete proofs.

We begin with perhaps the only point in the discussion which requires a detailed treatment. We continue to work in the category of G-C^*-algebras.

Let $A = \oplus_{n=1}^{\infty} A_n$ be the direct sum (in the terminology of C^*-algebra theory) of a sequence of separable C^*-algebras $\{A_1, A_2, \dots\}$. Thus A is comprised of sequences $\{a_n\}$, with $a_n \in A_n$, for which $\lim_{n\to\infty} \|a_n\| = 0$. There are natural inclusions $A_n \to A$, and so if B is any C^*-algebra there are natural projections from $E_G(A, B)$ to $E_G(A_n, B)$, for every n. They combine to form a product map

$$\pi\colon E_G(A, B) \to \prod_n E_G(A_n, B).$$

7.1. Proposition. *Let $A = \oplus_{n=1}^{\infty} A_n$ be the direct sum of a sequence of separable C^*-algebras. The product map $\pi\colon E_G(A, B) \to \prod E_G(A_n, B)$ is an isomorphism of abelian groups.*

Proof. We shall prove the proposition by defining an inverse map

$$\prod_n E_G(A_n, B) \to E_G(A, B).$$

Let e_n be the orthogonal projection onto the nth standard basis vector of the Hilbert space $\ell^2\mathbb{N}$. Given asymptotic morphisms

$$\varphi_n\colon \Sigma A_n \otimes \mathcal{K}(\mathcal{H}) \to \mathfrak{A}(\Sigma B \otimes \mathcal{K}(\mathcal{H})),$$

for $n \in \mathbb{N}$, we wish to define

$$\varphi\colon \Sigma A \otimes \mathcal{K}(\mathcal{H}) \to \mathfrak{A}(\Sigma B \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\ell^2\mathbb{N}))$$

by the formula

$$\varphi(\oplus a_n) = \sum_{n=1}^{\infty} \varphi_n(a_n) \otimes e_n.$$

To make sense of the formula, consider first the restriction of φ to the finite direct sum $A^N = \oplus_{n=1}^N A_n \subset A$. The inclusion

$$\underbrace{\Sigma B \otimes \mathcal{K}(\mathcal{H}) \oplus \ldots \oplus \Sigma B \otimes \mathcal{K}(\mathcal{H})}_{N \text{ times}} \to \Sigma B \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\ell^2 \mathbb{N}),$$

defined by $d_1 \oplus \cdots \oplus d_N \mapsto \sum_{n=1}^N d_n \otimes e_n$ induces a $*$-homomorphism

$$\underbrace{\mathfrak{A}(\Sigma B \otimes \mathcal{K}(\mathcal{H})) \oplus \ldots \oplus \mathfrak{A}(\Sigma B \otimes \mathcal{K}(\mathcal{H}))}_{N \text{ times}} \to \mathfrak{A}\left(\Sigma B \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\ell^2 \mathbb{N})\right)$$

By composing the $*$-homomorphism

$$\varphi_1 \oplus \cdots \oplus \varphi_N \colon \Sigma A^N \to \Sigma B \otimes \mathcal{K}(\mathcal{H}) \oplus \ldots \oplus \Sigma B \otimes \mathcal{K}(\mathcal{H})$$

with this map we obtain a $*$-homomorphism

$$\varphi^N \colon \Sigma A^N \otimes \mathcal{K}(\mathcal{H}) \to \mathfrak{A}(\Sigma B \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\ell^2 \mathbb{N}))$$

as required. This is how we interpret our formula for finite direct sums. For infinite sums, note that the φ^N are compatible with the inclusions $A^N \subset A^{N+1}$, and so define a $*$-homomorphism φ on the *algebraic* direct sum $A^\infty \subset A$. But any $*$-homomorphism defined on this dense $*$-subalgebra extends by continuity to A.

One checks that the above construction is well defined on homotopy classes by performing the same construction on homotopies. We therefore obtain a map from $\prod_n E_G(A_n, B)$ into $E_G(A, B)$, as required. It may be checked that the composition

$$\prod_n E_G(A_n, B) \to E_G(A, B) \to \prod_n E_G(A_n, B)$$

is the identity by restricting to each $E_G(A_n, B)$. To check that the composition

$$E_G(A, B) \to \prod_n E_G(A_n, B) \to E_G(A, B)$$

is the identity, observe first that the composition maps a $*$-homomorphism

$$\varphi \colon \Sigma A \otimes \mathcal{K}(\mathcal{H}) \to \mathfrak{A}(\Sigma B \otimes \mathcal{K}(\mathcal{H}))$$

to the $*$-homomorphism from $\Sigma A \otimes \mathcal{K}(\mathcal{H})$ into $\mathfrak{A}(\Sigma B \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\ell^2 \mathbb{N}))$ defined by

$$\psi(\oplus a_n) = \sum \varphi(a_n) \otimes e_n.$$

For clarity in what follows, let us write $\Sigma A \otimes \mathcal{K}(\mathcal{H}) = A'$ and $\Sigma B \otimes \mathcal{K}(\mathcal{H}) = B'$, so that ψ is a $*$-homomorphism from A' into $\mathfrak{A}(B' \otimes \mathcal{K}(\ell^2 \mathbb{N}))$. To see that φ is homotopic to ψ, regard ψ as the composition

$$A' \to A' \otimes \mathcal{K}(\ell^2(\mathbb{N})) \xrightarrow{\varphi} \mathfrak{A}(B' \otimes \mathcal{K}(\ell^2 \mathbb{N})),$$

where the first map is the $*$-homomorphism which maps $\oplus a_n$ to $\oplus(a_n \otimes e_n)$. This $*$-homomorphism is homotopic to the inclusion $\oplus a_n \mapsto \oplus(a_n \otimes e_1)$, and the stabilization property of E_G-theory asserts that the induced map on E_G-groups is an isomorphism. This completes the proof. $\square$

A standard argument now shows that the functor $A \longmapsto E_G(A, B)$ has Milnor $\varprojlim{}^1$-sequences. For a proof, along with further discussion, see [**6**, Section 21].

7.2. Proposition. *If A is a direct limit of a system $A_1 \to A_2 \to A_3 \to \cdots$ of separable C^*-algebras then there is a functorial short exact sequence*

$$0 \to \varprojlim{}^1 E_G(A_n, \Sigma B) \to E_G(A, B) \xrightarrow{\lambda} \varprojlim E_G(A_n, B) \to 0,$$

where λ is induced by the $*$-homomorphisms $A_n \to A$. □

We turn now to the universal coefficient theorem, and here we shall be even more brief. The formulation and proof of the universal coefficient theorem proceeds exactly as the proof for Kasparov's KK-theory given in [**32**]. For simplicity we shall state the result for commutative C^*-algebras only, but both the result and the proof in [**32**] work for all C^*-algebras in the so called 'bootstrap' category described there.

In the following theorem, we assume that G is the trivial group and drop it from the notation. Further, we write $E_n(A, B) = E(\Sigma^n A, B)$ in agreement with the usual conventions of C^*-algebra K-theory.

7.3. Universal Coefficient Theorem. *Let A be a commutative separable C^*-algebra. For any separable C^*-algebra B (in fact for any B) there is a short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} E_*(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \to 0,$$

natural in each of the variables A and B. □

In the statement of the theorem $\mathrm{Ext}^1_{\mathbb{Z}}$ is the usual derived functor from homological algebra. The map γ is induced by the E-theory product and the identification $E_*(\mathbb{C}, A) \cong K_*(A)$.

We conclude by continuing to consider the case of a trivial group G, and specializing to the groups $E(A, \mathbb{C})$. In fact we shall further assume that A is abelian.

7.4. Definition. If X is a compact metrizable space then let $E_{-n}(X) = E(\Sigma^n C(X), \mathbb{C})$. If X^+ is a pointed compact metrizable space let $\tilde{E}_{-n}(X^+) = E(\Sigma^n C_0(X), \mathbb{C})$, where X denotes X^+ with the base point removed. We extend these definitions to $n \in \mathbb{Z}$ by Bott periodicity (so that $E_n(X^+) \cong E_{n+2}(X^+)$). If (X, X_1) is a compact metrizable pair we define $E_n(X, X_1) = \tilde{E}_n(X/X_1)$.

7.5. Theorem. *The functors $\{E_n\}$ constitute a generalized homology theory on the category of compact metrizable pairs. Similarly, the functors $\tilde{E}_n$ are a reduced generalized homology theory on the category of pointed compact metrizable spaces.* □

Let us now restrict attention to the category of finite CW-pairs.

The generalized homology theory, K-homology, which is dual [**37**] to K-theory may be defined by Spanier-Whitehead duality [**33,34**]. We adopt this approach, setting $K_n(X) = K^{-n}(DX)$, where DX is the "dual" of X defined as a strong deformation retract of the complement of an embedding of X into an odd-dimensional sphere.

7.6. Theorem. *On the category of pairs of finite CW-complexes the generalized homology theories $E_*(X)$ and $K_*(X)$ are naturally isomorphic.*

Proof. We follow the argument of Atiyah [**2**, Section 3] and Brown, Douglas and Fillmore [**8**, Theorem 7.7]. Using the definition of $E_n(X)$ and the isomorphisms $K^0(X \times Y) \cong E(\mathbb{C}, C(X \times Y))$ and $K^0(Y) \cong E(\mathbb{C}, C(Y))$, the E-theory product gives a pairing

$$E_n(X) \otimes K^m(X \times Y) \to K^{m-n}(Y).$$

Following Atiyah's argument, we specialize to the case $Y = DX$. Evaluation on a fundamental class in $K^0(X \times DX)$ gives a natural transformation of generalized homology theories

$$E_n(X) \to K^{-n}(DX) \cong K_n(X).$$

The map is an isomorphism for the one-point space, so standard arguments show that it is an isomorphism for all finite CW-complexes. Thus E-homology and K-homology are naturally isomorphic. $\square$

We close with a remark relating E-theory to Steenrod homology [**25**]. Recall that a pointed compact metrizable space Z is the *strong wedge* of a sequence Z_k of subspaces if each pair of Z_k intersect precisely in the base point of Z and if the diameters of the Z_k tend to zero. If Z^0 and Z_k^0 denote the spaces obtained by removing base points then from Proposition 7.1 we conclude that

$$E_*(Z^0) \cong \prod E_*(Z_k^0),$$

or equivalently

$$\tilde{E}_*(Z) \cong \prod \tilde{E}_*(Z_k).$$

Thus, E-theory satisfies the so called strong wedge axiom of a generalized Steenrod homology theory on the category of pointed compact metric spaces [**25**, Axiom 9].

CHAPTER 8

Proper Algebras

We now commence our study of the Baum-Connes theory [**3,4**], the aim of which is to calculate the K-theory of crossed product C^*-algebras. Following Kasparov [**23**], we are going to study a class of G-C^*-algebras which extends the class of locally compact proper G-spaces. On the one hand, these algebras, and their associated crossed products, are particularly easy to study from the point of view of K-theory. On the other, they appear to play an important role in the K-theory of general crossed product C^*-algebras.

We begin by formulating the definition of proper G-space which seems best adapted to C^*-algebra K-theory. It is the same as the one used in [**4**].

8.1. Definition. A topological space X equipped with a continuous action of a topological group G is a *proper G-space* if:

(i) X is paracompact and Hausdorff;
(ii) X/G is paracompact and Hausdorff; and
(iii) for every $x \in X$ there is a G-neighborhood U of x; a compact subgroup K of G, and a continuous G-map $U \to G/K$.

Most of the time we shall be concerned with locally compact spaces X. If G is discrete and X is locally compact then properness in the sense of Definition 8.1 is the same as the requirement that the structural map $G \times X \to X \times X$ be proper. If G is non-discrete then our definition incorporates a slice property for the action (compare [**27**]).

8.2. Definition. A G-C^*-algebra D is *proper* if there exists

(i) a second countable, locally compact, proper G-space X; and
(ii) an equivariant $*$-homomorphism from $C_0(X)$ into the *center* of the multiplier algebra of D; such that $C_0(X)D$ is dense in D.

If D is a proper C^*-algebra, and X is the locally compact space which appears in Definition 8.2, then we shall say that D is *proper over X*. There is usually some freedom in the choice of X. Indeed if $f\colon X \to Y$ is a continuous and equivariant map of second countable, proper G-spaces then any C^*-algebra D which is proper over X becomes, via f, a C^*-algebra which is proper over Y. This is because f induces a $*$-homomorphism $f^*\colon C_0(Y) \to C_b(X)$, while the given structure map from $C_0(X)$ into the center of the multiplier algebra of D extends to $C_b(X)$. One checks easily that $f^*C_0(Y)D$ is dense in D. Note that to carry out this construction we do not need to require that the map $f\colon X \to Y$ be proper.

8.3. Example. If X is a second countable, locally compact proper G-space then $C_0(X)$ is clearly a proper G-C^*-algebra over X. Furthermore, if D is proper over X and B is any G-C^*-algebra then $D \otimes B$ is proper over X as well. In particular, $C_0(X) \otimes D$ is proper over X for any G-C^*-algebra D.

8.4. Example. If G is compact then every G-C^*-algebra is proper (over the one-point space).

Roughly speaking, our aim in the coming chapters is to examine the extent to which the K-theory for crossed products of proper algebras mirrors the K-theory for crossed products of C^*-algebras by compact groups.

To conclude this chapter we define a useful notion of *support* for elements of proper C^*-algebras, derived from the example $D = C_0(X)$.

8.5. Definition. Let D be proper over X. The *support* of an element $d \in D$ is the complement of the largest open subset $W \subset X$ for which $fd = 0$, for all $f \in C_0(W)$. If U is an open subset of X then we shall denote by $D(U)$ the C^*-algebra generated by the elements in D whose support lies within U.

8.6. Example. If $D = C_0(X)$ then the support of $f \in C_0(X)$ is the usual support of the function f and $D(U) = C_0(U)$.

If U is a G-invariant open set in X then $D(U)$ is a proper G-C^*-algebra in its own right. One can think of it as being proper over U, or proper over $\overline{U}$, or even proper over X.

Let us record the following simple observation:

8.7. Lemma. *Let D be proper over X. If U is any open subset of X then $D(U)$ is an ideal in $D = D(X)$. If U is G-invariant then the short exact sequence*

$$0 \longrightarrow D(U) \longrightarrow D(X) \longrightarrow D(X)/D(U) \longrightarrow 0$$

is a short exact sequence of proper G-C^*-algebras, where the quotient $D(X)/D(U)$ is proper over $X \setminus U$. $\square$

CHAPTER 9

Stabilization

Let D be a proper G-C^*-algebra. The purpose of this chapter is to define and examine the properties of an important *stabilization homomorphism*

$$\kappa\colon D \to D \otimes \mathcal{K}(L^2(G)).$$

The definition requires the following ancillary notion.

9.1. Definition. A *cut-off* function for a locally compact, second countable, proper G-space X is a non-negative, bounded, continuous function θ on X for which

(i) the intersection of Support(θ) with any G-compact set in X is compact; and
(ii) $\int_G \theta(g^{-1}x)^2\, dg = 1$, for every $x \in X$.

Note that by virtue of condition (i), the integral in condition (ii) is compactly supported, and so convergent.

It is a straightforward exercise to see that cut-off functions always exist.

9.2. Definition. If $\mathcal{H}$ is a G-Hilbert space then denote by $L^2(G, \mathcal{H})$ the completion of the space of compactly supported, continuous functions $G \to \mathcal{H}$, in the norm associated to the inner product

$$\langle \xi_1, \xi_2 \rangle = \int_G \langle \xi_1(g), \xi_2(g) \rangle\, dg.$$

The group G acts by unitary operators U_g on $L^2(G, \mathcal{H})$ according to the formula

$$(U_g \xi)(g_1) = \pi[g]\xi(g^{-1}g_1), \qquad g \in G,$$

where π denotes the given representation of G on $\mathcal{H}$.

The G-Hilbert space $L^2(G, \mathcal{H})$ is unitarily equivalent to the tensor product $L^2(G) \otimes \mathcal{H}$ of the left regular representation and the given representation of G on $\mathcal{H}$, in such a way that if ξ is a continuous and compactly supported scalar function on G, and if $v \in \mathcal{H}$, then $\xi \otimes v$ corresponds to the function $g \mapsto \xi(g)v$.

Suppose now that a G-C^*-algebra B is represented faithfully and covariantly on a G-Hilbert space $\mathcal{H}_B$. Each continuous and compactly supported function $k\colon G \times G \to B$ determines an operator K on $L^2(G, \mathcal{H}_B)$ by the formula

$$K\xi(g_1) = \int_G \pi(k(g_1, g_2))\xi(g_2)\, dg_2.$$

Under the unitary equivalence $L^2(G, \mathcal{H}_B) \cong L^2(G) \otimes \mathcal{H}_B$ the C^*-algebra closure of the set of all such operators is isomorphic to $\mathcal{K}(L^2(G)) \otimes B$. Observe that if $g \in G$ then the operator $U_g K U_g^*$ is represented by the kernel

$$k^g(g_1, g_2) = g[k(g^{-1}g_1, g^{-1}g_2)].$$

We are now ready to define the stabilization homomorphism.

9.3. Definition. Let D be a proper G-C^*-algebra over X, and suppose that D is represented faithfully and covariantly on a G-Hilbert space $\mathcal{H}_D$. Let θ be a cut-off function on X. Define the stabilization homomorphism

$$\kappa\colon D \to \mathcal{K}(L^2(G)) \otimes D$$

(which depends on the choice of θ) by associating to each $d \in D$ the kernel

$$k_d(g_1, g_2) = g_1(\theta) g_2(\theta) d.$$

To make sense of the definition, note that if d is compactly supported, in the sense of Definition 8.5, then the kernel $k_d(g_1, g_2)$ is continuous and compactly supported, and so defines an element of $\mathcal{K}(L^2(G)) \otimes D$. The formula in Definition 9.3 thus determines an equivariant $*$-homomorphism from the $*$-algebra $D_c(X)$ of compactly supported elements into $\mathcal{K}(L^2(G)) \otimes D$. Since $D_c(X)$ is an increasing union of C^*-subalgebras of D and since C^*-algebra $*$-homomorphisms are automatically contractive, our $*$-homomorphism extends by continuity to D.

9.4. Example. If G is compact and if we regard D as proper over a point, then the stabilization homomorphism $\kappa\colon D \to \mathcal{K}(L^2(G)) \otimes D$ is defined by $\kappa(d) = p \otimes d$, where $p \in \mathcal{K}(L^2(G))$ is the orthogonal projection onto the constant functions in $L^2(G)$.

Although the stabilization homomorphism depends on the choice of cut-off function θ, it is readily checked that since any two choices of θ are path-connected, any two stabilization homomorphisms are equivariantly homotopic.

¿From here on it will be convenient to reverse the order of the factors in the tensor product $\mathcal{K}(L^2(G)) \otimes D$ and write the stabilization homomorphism as

$$\kappa\colon D \to D \otimes \mathcal{K}(L^2(G)).$$

9.5. Proposition. *Let D be a proper G-C^*-algebra and let $\kappa\colon D \to D \otimes \mathcal{K}(L^2(G))$ be a stabilization homomorphism. If $\mathcal{H}$ is the standard G-Hilbert space of Definition 6.1 then the tensor product homomorphism*

$$D \otimes \mathcal{K}(\mathcal{H}) \xrightarrow{\kappa \otimes 1} D \otimes \mathcal{K}(L^2(G)) \otimes \mathcal{K}(\mathcal{H}),$$

is homotopic, through equivariant $*$-homomorphisms, to a $*$-isomorphism. In fact, once $\mathcal{K}(L^2(G)) \otimes \mathcal{K}(\mathcal{H})$ is identified with $\mathcal{K}(\mathcal{H})$ via a unitary equivalence $L^2(G) \otimes \mathcal{H} \cong \mathcal{H}$, the above tensor product homomorphism $\kappa \otimes 1$ becomes equivariantly homotopic to the identity on $D \otimes \mathcal{K}(\mathcal{H})$.

Proof. Form the Hilbert space $\mathbb{C} \oplus L^2(G)$, where G acts trivially on $\mathbb{C}$, and denote by $e \in \mathcal{K}(\mathbb{C} \oplus L^2(G))$ the orthogonal projection onto $\mathbb{C}$. We will show that the composition

$$D \xrightarrow{\kappa} D \otimes \mathcal{K}(L^2(G)) \to D \otimes \mathcal{K}(\mathbb{C} \oplus L^2(G))$$

(the second map is induced from the inclusion $L^2(G) \hookrightarrow \mathbb{C} \oplus L^2(G)$) is homotopic to the $*$-homomorphism $d \mapsto d \otimes e$. This will suffice since both the maps

$$\mathcal{K}(L^2(G)) \otimes \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathbb{C} \oplus L^2(G)) \otimes \mathcal{K}(\mathcal{H})$$

and

$$\mathbb{C} \otimes \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathbb{C} \oplus L^2(G)) \otimes \mathcal{K}(\mathcal{H})$$

are equivariantly homotopic to $*$-isomorphisms.

Represent D on a G-Hilbert space $\mathcal{H}_D$, as we did in our definition of the stabilization map, and for each compactly supported $d \in D$ define an operator

$$\eta(d)\colon L^2(G, \mathcal{H}_D) \to \mathcal{H}_D$$

by $\eta(d)\xi = \int_G \pi[g(\theta)d]\xi(g)\,dg$, where θ is the cut-off function used to define the stabilization homomorphism κ. We observe that

$$\begin{aligned} \eta(d_1)^*\eta(d_2) &= \kappa(d_2^* d_1) \qquad & \pi(d_1)\eta(d_2) &= \eta(d_1 d_2) \\ \eta(d_1)\eta(d_2)^* &= \pi(d_1 d_2^*) \qquad & \eta(d_1)\kappa(d_2) &= \eta(d_1 d_2), \end{aligned}$$

in consequence of which, for each $s \in [0,1]$ the formula

$$d \mapsto \begin{pmatrix} s^2\pi(d) & s(1-s^2)^{1/2}\eta(d) \\ s(1-s^2)^{1/2}\eta(d^*)^* & (1-s^2)\kappa(d) \end{pmatrix}$$

defines a $*$-homomorphism from D into the bounded operators on the Hilbert space $\mathcal{H}_D \oplus L^2(G, \mathcal{H}_D) \cong \mathcal{H}_D \otimes (\mathbb{C} \oplus L^2(G))$. Its range lies within $D \otimes \mathcal{K}(\mathbb{C} \oplus L^2(G))$ and we have obtained the required homotopy. □

9.6. Corollary. *Let D be a proper G-C^*-algebra. Every stabilization homomorphism $\kappa\colon D \to D \otimes \mathcal{K}(L^2(G))$ determines the same invertible morphism in $E_G(D, D \otimes \mathcal{K}(L^2(G))$.*

Proof. This follows immediately from Proposition 9.5 and the stabilization property of E_G-theory (Proposition 6.10). □

9.7. Corollary. *Let D be a proper G-C^*-algebra. Under the canonical isomorphism*

$$E_G(D, D \otimes \mathcal{K}(L^2(G))) \cong E_G(D, D)$$

the class in $E_G(D, D \otimes \mathcal{K}(L^2(G)))$ determined by any stabilization homomorphism corresponds to the identity $1 \in E_G(D, D)$.

Proof. The isomorphism comes about by identifying $\mathcal{K}(L^2(G)) \otimes \mathcal{K}(\mathcal{H})$ with $\mathcal{K}(\mathcal{H})$ via a unitary equivalence $L^2(G) \otimes \mathcal{H} \cong \mathcal{H}$, so once again the proof is an immediate consequence of Proposition 9.5. □

It is occasionally useful to observe that stabilization gives a means to simplify the definition of E_G-theory:

9.8. Corollary. *If D is a proper G-C^*-algebra and B is any G-C^*-algebra, and if $\mathcal{H}$ denotes the standard G-Hilbert space of Definition 6.1 then*

$$E_G(D, B) \cong [\![\Sigma D, \Sigma B \otimes \mathcal{K}(\mathcal{H})]\!],$$

i.e., every class in $E_G(D, B)$ is represented by a $*$-homomorphism

$$\varphi\colon \Sigma D \to \mathfrak{A}^n(\Sigma B \otimes \mathcal{K}(\mathcal{H})).$$

Proof. Since $L^2(G) \subset \mathcal{H}$, the stabilization homomorphism gives us a $*$-homomorphism

$$\kappa\colon D \to D \otimes \mathcal{K}(\mathcal{H}).$$

Composition with κ defines a homomorphism

$$E_G(D, B) \to [\![\Sigma D, \Sigma B \otimes \mathcal{K}(\mathcal{H})]\!].$$

A homomorphism

$$[\![\Sigma D, \Sigma B \otimes \mathcal{K}(\mathcal{H})]\!] \to E_G(D, B)$$

is defined using the tensor product construction in the asymptotic category to tensor with the identity on $\mathcal{K}(\mathcal{H})$, and identifying $\mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H}) \cong \mathcal{K}(\mathcal{H})$. It follows in a straightforward manner from Proposition 9.5 that these homomorphisms are inverses of each other. $\square$

If G is compact then these corollaries apply to *any* G-C^*-algebra D.

We conclude by noting a small variation on our definition of the stabilization homomorphism. Let H be a compact subgroup of G and suppose that the cutoff function θ used to define the $*$-homomorphism κ is H-invariant. If we identify $L^2(G/H)$ with the subspace of $L^2(G)$ comprised of right H-invariant functions on G then in fact κ maps D into $D \otimes \mathcal{K}(L^2(G/H))$. Suppose for example that G is discrete and that D is proper over the discrete coset space G/H. If we choose θ to be the characteristic function of the single point $eH \in G/H$ then the $*$-homomorphism $\kappa\colon D \to D \otimes \mathcal{K}(\ell^2(G/H))$ is defined by

$$\kappa(d) = \sum_{g \in G/H} p_g d \otimes e_g,$$

where p_g denotes the characteristic function of $gH \in G/H$ and e_g denotes the rank-one projection corresponding to the basis element $[gH] \in \ell^2(G/H)$. We shall use this formula in Chapter 12.

CHAPTER 10

Assembly

Paul Baum and Alain Connes have defined a C^*-algebraic *assembly map* which relates equivariant K-homology to the K-theory of (reduced) group C^*-algebras [**3**,**4**]. The main reason for studying proper G-C^*-algebras is that they appear to play an important role in the analysis of this assembly map.

We begin our discussion of assembly by defining E_G-theory for spaces which are not necessarily locally compact.

10.1. Definition. Let Y be a proper G-space and let B be a G-C^*-algebra. We define

$$E_G(Y, B) = \varinjlim E_G(C_0(X), B),$$

where the direct limit is over all G-compact, locally compact and second countable subsets $X \subset Y$ (a set is G-compact if it is the G-saturation of a compact set).

Observe that if X_1 and X_2 are G-compact, locally compact and second countable subsets of Y, and if $X_1 \subset X_2$, then there is a restriction homomorphism $C_0(X_2) \to C_0(X_1)$ and hence an induced homomorphism in E-theory,

$$E_G(C_0(X_1), B) \to E_G(C_0(X_2), B).$$

This explains the direct limit in Definition 10.1.

It is important to note that if Y is locally compact but not G-compact then $E_G(Y, B)$ is not the same as $E_G(C_0(Y), B)$. For instance the latter is functorial for proper G-maps $Y \to Y'$ whereas the former is functorial for arbitrary continuous G-maps between proper G-spaces. In particular, the group $E_G(Y, B)$ depends only on the G-homotopy type of the space Y.

10.2. Definition. Let X be a G-compact, locally compact, second countable, proper G-space (from now on we shall just say 'G-compact proper G-space'). Let θ be a cut-off function for X: since X is G-compact, θ is a compactly supported continuous function from X into $[0, \infty)$ such that $\int_G \theta^2(g^{-1}x)\, dg = 1$, for all $x \in X$. The function $p\colon G \to C_0(X)$ given by

$$p(g) = g(\theta)\, \theta,$$

is a projection in the crossed product algebra $C^*(G, X)$, which we shall call the *basic projection* in $C^*(G, X)$. Note that for commutative C^*-algebras $C_0(X)$ we write $C^*(G, X)$ in place of $C^*(G, C_0(X))$.

The basic projection $p \in C^*(G, X)$ depends on the choice of cut-off function θ, but since any two such choices are clearly path connected, the basic projection is well-defined at the level of homotopy.

10.3. Definition. Let X be a G-compact proper G-space and let B be any G-C^*-algebra. The *Baum-Connes assembly map*

$$\mu\colon E_G(C_0(X), B) \to E(\mathbb{C}, C^*(G, B))$$

is the composition

$$E_G(C_0(X), B) \xrightarrow{\text{descent}} E(C^*(G, X), C^*(G, B)) \xrightarrow[p \in E(\mathbb{C}, C^*(G,X))]{\text{composition with}} E(\mathbb{C}, C^*(G, B)).$$

If Y is a general proper G-space—not necessarily G-compact—then the assembly map for Y is the homomorphism $\mu\colon E_G(Y, B) \to E(\mathbb{C}, C^*(G, B))$ obtained as the direct limit of the assembly maps for the G-compact subsets of Y.

10.4. Remark. It is of some importance to consider also the *reduced* assembly map

$$\mu_{\text{red}}\colon E_G(Y, B) \to E(\mathbb{C}, C^*_{\text{red}}(G, B)),$$

which is obtained from the homomorphism in Definition 10.3 by composing with the E-theory map induced from the natural $*$-homomorphism from $C^*(G, B)$ onto the reduced crossed product algebra $C^*_{\text{red}}(G, B)$. In this paper we shall concentrate on μ rather than μ_{red} because, as we noted in Chapter 4, E-theory is more compatible with full than with reduced crossed products. However if G is C^*-exact in the sense of Definition 4.13 then everything we have to say will carry over without alteration to the reduced crossed product case.

The assembly map has the following naturality property: if $\varphi \in E_G(A, B)$ then there is a commuting diagram

$$\begin{array}{ccc} E_G(Y, A) & \xrightarrow{\mu} & E(\mathbb{C}, C^*(G, A)) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ E_G(Y, B) & \xrightarrow[\mu]{} & E(\mathbb{C}, C^*(G, B)), \end{array}$$

where the vertical maps are given by E-theoretic composition with φ (and on the right we use the descent functor of Chapter 4). If G is C^*-exact then there is a similar commutative diagram for the reduced assembly map.

We shall be interested in the Baum-Connes assembly map for a particular choice of proper G-space.

10.5. Definition. A proper G-space Y is *universal* [**4**] if:

(i) for every proper G-space Z there is a continuous and equivariant map from Z into Y; and
(ii) any two such maps are equivariantly homotopic to one another.

It is shown in [**4**, Appendix 1] that universal proper G-spaces always exist; it is clear from the definition that they are unique up to equivariant homotopy.

10.6. Definition. We shall denote by $\mathcal{E}G = Y$ a universal proper G-space.

It follows from the uniqueness of $\mathcal{E}G$, up to equivariant homotopy, that the group $E_G(\mathcal{E}G, B)$ is independent of the model for $\mathcal{E}G$, up to canonical isomorphism. Nevertheless it is occasionally useful to pick a particular model—for instance one which is G-compact, if this can be arranged, or one which is a simplicial complex.

For the sake of completeness, here is the statement of the well-known conjecture of Baum and Connes [**4**]:

10.7. Baum-Connes Conjecture. *Let G be a locally compact, second countable, Hausdorff topological group. The assembly map*

$$\mu_{\mathrm{red}}\colon E_G(\mathcal{E}G, \mathbb{C}) \to E(\mathbb{C}, C^*_{\mathrm{red}}(G))$$

is an isomorphism of abelian groups.

10.8. Remark. The Baum-Connes conjecture is formulated in [**4**] in just the above way, but using the language of Kasparov's KK-theory. See the short note [**20**] for a proof that the two formulations are equivalent.

Fix a countable discrete group G. Our goal in the coming chapters is to give a criterion, involving proper G-C^*-algebras, which is sufficient to imply that the assembly map

$$\mu\colon E_G(\mathcal{E}G, B) \to E(\mathbb{C}, C^*(G, B))$$

is an isomorphism, for every G-C^*-algebra B (Theorem 14.1). Thus our work will not bear directly on the Baum-Connes conjecture, which concerns μ_{red} not μ. Nevertheless there is a connection. For instance our criterion plays an important role in recent work of Higson and Kasparov [**19**], which proves the Baum-Connes conjecture for amenable discrete groups, among others. Of course for amenable groups $\mu = \mu_{\mathrm{red}}$ (compare [**28**, Section 7.7]). In fact for all the groups considered in [**19**]—namely those which act isometrically and properly (in the sense of [**17**] and [**5**]) on Euclidean space—it is possible to show that the reduction map

$$E(\mathbb{C}, C^*(G, B)) \xrightarrow{\cong} E(\mathbb{C}, C^*_{\mathrm{red}}(G, B))$$

is an isomorphism. Thus the reduced and non-reduced assembly maps are the same.

Another reason to consider μ as well as (or instead of) μ_{red} is that injectivity of μ suffices for most of the applications of the Baum-Connes theory to topology and geometry. For example injectivity of μ for a discrete group G implies the Novikov higher signature conjecture for G (compare [**24**]). In connection with this, apart from giving a criterion for isomorphism of the assembly map we will also give a broader criterion for (split) injectivity of μ (Theorem 14.2).

Finally, although we shall not discuss it further, in the following chapters we can work equally well with the reduced assembly map μ_{red}, as long as the groups G we consider are assumed to be C^*-exact in the sense of Definition 4.19. Thus for C^*-exact groups the main results of this monograph apply equally well to the assembly map and its reduced counterpart.

CHAPTER 11

The Green-Julg Theorem

We begin our investigation of the assembly map by considering the rather easy case of compact groups.

If G is a compact group then we may set $\mathcal{E}G = \text{point}$. The assembly map thus becomes a homomorphism

$$\mu\colon E_G(\mathbb{C}, B) \to E(\mathbb{C}, C^*(G, B))$$

(and we might note that $\mu = \mu_{\text{red}}$ since G is amenable). The basic projection of Definition 10.2 is the projection in $C^*(G)$ corresponding to the trivial representation of G: if Haar measure is normalized so that the total mass of G is 1 then p is just the constant function 1 on G.

In a slightly different context Green [**16**] and Julg [**21**] have proved:

11.1. Theorem. *If G is a compact group and B is any G-C^*-algebra then the assembly map*

$$\mu\colon E_G(\mathbb{C}, B) \to E(\mathbb{C}, C^*(G, B))$$

is an isomorphism.

The proof of the theorem relies on a well-known description of $C^*(G, B)$ as a fixed-point algebra [**30**, Proposition 4.3]:

$$C^*(G, B) \cong \{B \otimes \mathcal{K}(L^2(G))\}^G.$$

We shall need to know some of the details of the isomorphism, but granted its existence for a moment, observe that it supplies for us a $*$-homomorphism

$$\psi\colon C^*(G, B) \to B \otimes \mathcal{K}(L^2(G)).$$

Furthermore if we give the crossed product C^*-algebra the trivial G-action then this $*$-homomorphism ψ is *equivariant.* Consider now the following diagram,

$$\begin{array}{ccc} E_G(\mathbb{C}, C^*(G, B)) & \xrightarrow{\psi_*} & E_G(\mathbb{C}, B \otimes \mathcal{K}(L^2(G))) \\ \uparrow & & \cong\uparrow \kappa \\ E(\mathbb{C}, C^*(G, B)) & \xrightarrow{\nu} & E_G(\mathbb{C}, B) \end{array}$$

where: the left hand vertical arrow is defined by virtue of the fact that $C^*(G, B)$ may be viewed as a G-algebra with trivial G-action; the right hand vertical map is induced from the stabilization homomorphism of Chapter 9; and the bottom map ν is chosen so as to make the square commutative. We will show that the homomorphism ν inverts the assembly map.

Let us return now to the identification of $C^*(G,B)$ with the fixed point algebra $\{B\otimes\mathcal{K}(L^2(G))\}^G$. Pick a faithful, covariant and non-degenerate representation π of B on a separable Hilbert space $\mathcal{H}_B$ and define a covariant representation

$$\sigma\colon B,G\to\mathcal{B}(L^2(G,\mathcal{H}_B))$$

by

$$\begin{cases}\sigma(b)\xi\,(g_1)=\pi[g_1(b)]\,\xi(g_1)\\ \sigma(g)\xi\,(g_1)=\xi(g_1g).\end{cases}$$

Note that we have switched here from our customary left regular representation to a right regular representation, but since G is compact no modular function need be introduced. The associated representation

$$\sigma\colon C^*(G,B)\to\mathcal{B}(L^2(G,\mathcal{H}_B))$$

maps a continuous, compactly supported function $f\colon G\to B$ to the operator

$$\begin{aligned}\sigma(f)\xi\,(g_1)&=\int_G\pi[g_1(f(g_2))]\xi(g_1g_2)\,dg_2\\&=\int_G\pi[g_1(f(g_1^{-1}g_2))]\xi(g_2)\,dg_2\end{aligned}$$

Thus $\sigma(f)$ is represented by the continuous B-valued kernel

$$k(g_1,g_2)=g_1(f(g_1^{-1}g_2)).$$

It follows from these formulas and from the remarks made in Chapter 9 that σ maps $C^*(G,B)$ into $B\otimes\mathcal{K}(L^2(G))$ and that furthermore σ maps $C^*(G,B)$ into the G-fixed part of $B\otimes\mathcal{K}(L^2(G))$. It follows from the general theory of crossed products that σ is injective (it is injective on $C_{\mathrm{red}}(G,B)$ and $C_{\mathrm{red}}(G,B)=C^*(G,B)$ for compact groups). On the other hand any continuous, G-invariant kernel $k(g_1,g_2)$ is of the form $g_1(f(g_1^{-1}g_2))$, for some continuous function $f\colon G\to B$, and so σ surjects onto $\{B\otimes\mathcal{K}(L^2(G))\}^G$. In summary:

11.2. Proposition. *Let G be a compact group and let B be any G-C^*-algebra. There is an isomorphism*

$$C^*(G,B)\cong\{B\otimes\mathcal{K}(L^2(G))\}^G$$

under which a continuous functions $f\colon G\to B$, regarded as an element of $C^*(G,B)$, corresponds to the kernel $k(g_1,g_2)=g_1(f(g_1^{-1}g_2))$, regarded as an element of $B\otimes\mathcal{K}(L^2(G))$. □

Let us observe that the isomorphism in the proposition is functorial, in the sense that a G-equivariant $*$-homomorphism $\varphi\colon A\to B$ gives rise to a commuting diagram

$$\begin{array}{ccc}C^*(G,A) & \xrightarrow{C^*(G,\varphi)} & C^*(G,B)\\ \psi\big\downarrow & & \big\downarrow\psi\\ A\otimes\mathcal{K}(L^2(G)) & \xrightarrow{\varphi\otimes 1} & B\otimes\mathcal{K}(L^2(G)).\end{array}$$

Similarly, a $*$-homomorphism $\varphi\colon A \to \mathfrak{A}^n B$ gives rise to a commuting diagram

$$\begin{array}{ccc} C^*(G,A) & \xrightarrow{C^*(G,\varphi)} & \mathfrak{A}^n(C^*(G,B)) \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ A\otimes\mathcal{K}(L^2(G)) & \xrightarrow{\varphi\otimes 1} & \mathfrak{A}^n(B\otimes\mathcal{K}(L^2(G))). \end{array}$$

This follows from the remarks in Chapter 4 on tensor products and crossed products, along with Proposition 3.6.

We are now ready to prove the Green-Julg theorem. For clarity we shall consider injectivity and surjectivity separately.

Proof of Theorem 11.1—Injectivity. Let us start with a $*$-homomorphism

$$\varphi\colon \Sigma \to \mathfrak{A}^n(\Sigma B\otimes\mathcal{K}(\mathcal{H})),$$

representing a class $[\varphi]\in E_G(\mathbb{C},B)$ (by Corollary 9.8 all classes may be represented in this way). Consider next the commuting diagram

$$\begin{array}{ccccc} \Sigma & \xrightarrow{p} & C^*(G,\Sigma) & \xrightarrow{C^*(G,\varphi)} & \mathfrak{A}^n(C^*(G,\Sigma B\otimes\mathcal{K}(\mathcal{H}))) \\ & & {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ & & \Sigma\mathcal{K}(L^2(G)) & \xrightarrow{\varphi\otimes 1} & \mathfrak{A}^n(\Sigma B\otimes\mathcal{K}(\mathcal{H})\otimes\mathcal{K}(L^2(G))), \end{array}$$

in which the $*$-homomorphism $p\colon\Sigma\to C^*(G,\Sigma)$ maps an element $f\in\Sigma$ to $f\otimes p$ in $C^*(G,\Sigma)\cong\Sigma\otimes C^*(G)$. Applying to the class $[\varphi]\in E_G(\mathbb{C},B)$ first the assembly map μ; then its supposed inverse ν described earlier in this chapter; then the stabilization map κ; we get the class in $E_G(\mathbb{C},B\otimes\mathcal{K}(L^2(G)))$ represented by the composition of the top and rightmost vertical maps in the diagram. To calculate this composition, note that the composition in the leftmost part of the diagram, namely

$$\begin{array}{ccc} \Sigma & \xrightarrow{p} & C^*(G,\Sigma) \\ & & {\scriptstyle\psi}\big\downarrow \\ & & \Sigma\mathcal{K}(L^2(G)) \end{array}$$

is the stabilization $*$-homomorphism $\kappa\colon\Sigma\to\Sigma\mathcal{K}(L^2(G))$. Consider now the commuting diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \mathfrak{A}^n(\Sigma B\otimes\mathcal{K}(\mathcal{H})) \\ {\scriptstyle\kappa}\big\downarrow & & \big\downarrow{\scriptstyle\kappa} \\ \Sigma\mathcal{K}(L^2(G)) & \xrightarrow{\varphi\otimes 1} & \mathfrak{A}^n(\Sigma B\otimes\mathcal{K}(\mathcal{H})\otimes\mathcal{K}(L^2(G))). \end{array}$$

The composition around the top is $\kappa[\varphi]\in E_G(\mathbb{C},B\otimes\mathcal{K}(L^2(G)))$, while the composition around the bottom is the class $\kappa\nu\mu[\varphi]$ we are trying to calculate. By commutativity,

$$\kappa\nu\mu[\varphi]=\kappa[\varphi]\in E_G(\mathbb{C},B\otimes\mathcal{K}(L^2(G))),$$

and hence $\nu\mu[\varphi]=[\varphi]$, which proves that ν is left inverse to μ. $\square$

Proof of Theorem 11.1—Surjectivity. Consider a $*$-homomorphism

$$\varphi\colon \Sigma \to \mathfrak{A}^n(\Sigma C^*(G,B)\otimes \mathcal{K}(\mathcal{H}_0))$$

representing a class in $E(\mathbb{C}, C^*(G,B))$. Here $\mathcal{H}_0$ is a Hilbert space with no action of G (or, what is the same, the trivial action of G). To apply our supposed inverse ν to the assembly map, we regard φ as being equivariant for the trivial action of G on all C^*-algebras, then form the composition

$$\Sigma \xrightarrow{\varphi} \mathfrak{A}^n(\Sigma C^*(G,B)\otimes \mathcal{K}(\mathcal{H}_0)) \xrightarrow{1\otimes\psi\otimes 1} \mathfrak{A}^n(\Sigma B\otimes \mathcal{K}(L^2(G))\otimes \mathcal{K}(\mathcal{H}_0)),$$

which is an equivariant $*$-homomorphism defining a class in $E_G(\mathbb{C}, B)$. We are now going to streamline notation by dropping Σ and $\mathcal{K}(\mathcal{H}_0)$ from our formulas: the reader can readily check that since G acts trivially on them they could be easily reinserted into what follows. So we are considering a class in $E_G(\mathbb{C}, B)$ represented by the composition

$$\mathbb{C} \xrightarrow{\varphi} \mathfrak{A}^n(C^*(G,B)) \xrightarrow{\psi} \mathfrak{A}^n(B\otimes \mathcal{K}(L^2(G))).$$

Applying assembly to it we get

$$\mathbb{C} \xrightarrow{p} C^*(G) \xrightarrow{C^*(G,\varphi)} C^*(G, C^*(G,B)) \xrightarrow{C^*(G,\psi)} \mathfrak{A}^n(C^*(G, B\otimes \mathcal{K}(L^2(G)))).$$

Consider now the diagram

$$\begin{array}{ccccc}
C^*(G) & \xrightarrow{C^*(G,\varphi)} & \mathfrak{A}^n(C^*(G,C^*(G,B))) & \xrightarrow{C^*(G,\psi)} & \mathfrak{A}^n(C^*(G,B\otimes\mathcal{K}(L^2(G)))) \\
{\scriptstyle p}\uparrow & & {\scriptstyle p\otimes 1}\uparrow & & {\scriptstyle C^*(G,\kappa)}\uparrow \\
\mathbb{C} & \xrightarrow{\varphi} & \mathfrak{A}^n(C^*(G,B)) & \xrightarrow{=} & \mathfrak{A}^n(C^*(G,B)),
\end{array}$$

in which the middle vertical arrow is defined using the canonical isomorphism

$$C^*(G, C^*(G,B)) \cong C^*(G)\otimes C^*(G,B)$$

(valid since the action of G on $C^*(G,B)$ is trivial). The first square commutes, but the second does *not*. However it does commute up to n-homotopy. To see this, write

$$C^*(G, B\otimes\mathcal{K}(L^2(G))) \cong \{B\otimes\mathcal{K}(L^2(G))\otimes\mathcal{K}(L^2(G))\}^G$$

as in Proposition 11.2. The flip automorphism of $B\otimes\mathcal{K}(L^2(G))\otimes\mathcal{K}(L^2(G))$, which interchanges the two copies of $\mathcal{K}(L^2(G))$, is equivariantly homotopic to the identity and it is precisely this automorphism which accounts for the lack of commutativity in the right hand square. Since the diagram commutes up to homotopy, and since composition along the diagram represents the composition of ν, followed by μ, followed by κ, we see that $\mu\nu[\varphi] = [\varphi]$. $\square$

CHAPTER 12

Induction and Compression

The statement of the Green-Julg theorem extends very naturally to proper G-C^*-algebras:

12.1. Statement. *Let G be any second countable, locally compact group and let D be a proper G-C^*-algebra. The assembly map*

$$\mu\colon E_G(\mathcal{E}G, D) \to E(\mathbb{C}, C^*(G, D))$$

is an isomorphism.

Unfortunately we are not able to prove this statement, and there is even a possibility that it is incorrect for certain groups. However we are able to prove the statement for discrete groups, and it is the purpose of this chapter to lay the groundwork for our argument. Thus *in this chapter, G will denote a countable discrete group.* Some of our constructions work equally well for general groups, but some are quite particular to the discrete case.

Our general strategy is to reduce the proof of Statement 12.1 to the case of finite groups, already covered in the last chapter, using the following device:

12.2. Definition. Let H be a finite subgroup of G and let B be an H-C^*-algebra. We define the *induced C^*-algebra* $\operatorname{Ind}_H^G B$ by the formula

$$\operatorname{Ind}_H^G B = \{\, f \in C_0(G, B) \mid f(gh) = h^{-1}(f(g)), \quad \forall g \in G,\ \forall h \in H \,\}.$$

The group G acts on $\operatorname{Ind}_H^G B$ by left translation: $(gf)(g_1) = f(g^{-1}g_1)$, and $\operatorname{Ind}_H^G B$ is a proper G-C^*-algebra over the space G/H, if we regard a function on G/H as a function on G which is constant on the left H-cosets, and define its action on $\operatorname{Ind}_H^G B$ by pointwise multiplication.

According to Definition 8.1 every proper space is a union of open subsets, each of which is proper over some G/H. In the light of this, the following calculation shows that all proper G-C^*-algebras are built up out of induced C^*-algebras:

12.3. Lemma. *Let H be a finite subgroup of G. If D is a G-C^*-algebra which is proper over the space G/H then there is an H-C^*-algebra B such that $\operatorname{Ind}_H^G B \cong D$, as G-C^*-algebras.*

Proof. Let $p \in C_0(G/H)$ be the characteristic function of the identity coset and let $B = pD$. This is a C^*-subalgebra of D which is invariant under the restriction to H of the given G-action on D. Define a $*$-homomorphism $\operatorname{Ind}_H^G B \to D$ by

$$f \mapsto \sum_{g \in G/H} g(f(g)),$$

where the sum is over representatives of the left H-cosets in G (note that each summand is independent of the choice of representative). To make sense of the infinite sum, observe that the displayed formula defines a $*$-homomorphism on the subalgebra of $\operatorname{Ind}_H^G B$ comprised of finitely supported functions. This subalgebra is an increasing union of C^*-algebras. Therefore the $*$-homomorphism we have defined is automatically contractive and extends by continuity to $\operatorname{Ind}_H^G B$. The $*$-homomorphism on $\operatorname{Ind}_H^G B$ so obtained is inverse to the map from D to $\operatorname{Ind}_H^G B$ which takes $d \in D$ to the function $f(g) = pg^{-1}(d)$. $\square$

If D is a proper G-C^*-algebra then we can of course restrict the action of G to any finite subgroup $H \subset G$, so as to obtain an H-C^*-algebra. If D is proper over G/H then the proof of Lemma 12.3 suggests that a more drastic restriction operation will be of importance:

12.4. Definition. Let H be a finite subgroup of G and let D be proper over G/H. The *compression* of D to H is the C^*-subalgebra $B = pD$, where $p \in C_0(G/H)$ is the characteristic function of the identity coset in G/H.

Suppose that B is an H-C^*-algebra and that $D = \operatorname{Ind}_H^G B$. Then the compression of D to H is isomorphic to B, via the map which takes $f \in D$ to $f(e)$. The inverse map assigns to $b \in B$ the function

$$f_b(g) = \begin{cases} g^{-1}(b) & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases}$$

It follows easily from the proof of Lemma 12.3 that the operations of induction and compression are inverse to one another, but we shall not need to elaborate on this.

We come now to an important induction operation on morphisms:

12.5. Definition.

Let A and B be H-C^*-algebras and let $\varphi\colon A \to B$ be an H-equivariant $*$-homomorphism. Denote by

$$\operatorname{Ind}_H^G \varphi\colon \operatorname{Ind}_H^G A \to \operatorname{Ind}_H^G B$$

the G-equivariant $*$-homomorphism which maps a function $f \in \operatorname{Ind}_H^G A$ to the function $\varphi \circ f \in \operatorname{Ind}_H^G B$. Similarly, if $\varphi\colon A \to \mathfrak{A}^n B$ is an H-equivariant $*$-homomorphism then denote by

$$\operatorname{Ind}_H^G \varphi\colon \operatorname{Ind}_H^G A \to \mathfrak{A}^n \operatorname{Ind}_H^G B$$

the G-equivariant $*$-homomorphism obtained by following the above construction with the natural map

$$\operatorname{Ind}_H^G \mathfrak{A}^n B \to \mathfrak{A}^n \operatorname{Ind}_H^G B,$$

as in Chapter 3.

Before going on let us note a simple fact:

12.6. Lemma. *If B is an H-C^*-algebra and C is a G-C^*-algebra then there is an isomorphism of G-C^*-algebras*

$$(\operatorname{Ind}_H^G B) \otimes C \cong \operatorname{Ind}_H^G (B \otimes C),$$

under which the elementary tensor $f \otimes c \in (\operatorname{Ind}_H^G B) \otimes C$ *corresponds to the function* $g \mapsto f(g) \otimes g^{-1}(c)$ *in* $\operatorname{Ind}_H^G(B \otimes C)$. $\square$

Let H be a finite subgroup of G. Combining Definition 12.5 with Lemma 12.6, and using the fact that the universal Hilbert space $\mathcal{H}$ for the group H be be regarded as the restriction to H of the universal Hilbert space for G, we obtain from an H-equivariant $*$-homomorphism

$$\varphi\colon \Sigma A \otimes \mathcal{K}(\mathcal{H}) \to \mathfrak{A}^n(\Sigma B \otimes \mathcal{K}(\mathcal{H}))$$

an induced G-equivariant $*$-homomorphism

$$\operatorname{Ind}_H^G \varphi\colon \Sigma \operatorname{Ind}_H^G A \otimes \mathcal{K}(\mathcal{H}) \to \mathfrak{A}^n(\Sigma \operatorname{Ind}_H^G B \otimes \mathcal{K}(\mathcal{H})).$$

12.7. Definition. Let A and B be H-C^*-algebras. Denote by

$$\operatorname{Ind}_H^G\colon E_H(A, B) \to E_G(\operatorname{Ind}_H^G A, \operatorname{Ind}_H^G B)$$

the homomorphism obtained from the above construction.

The induction homomorphism of Definition 12.7 is in fact a functor from the H-equivariant E-theory category to the G-equivariant E-theory category; in other words it is compatible with E-theory products. This follows from Chapter 3.

We shall find it convenient to work with an induction homomorphism which is a small modification of the one just described. To describe it we need another definition:

12.8. Definition. Let A be a G-C^*-algebra which is proper over a G-space Y. If X is a closed subset of Y then denote by $A[X]$ the quotient of $A = A(Y)$ by the ideal $A(Y \setminus X)$ (see Definition 8.5 for the meaning of $A(Y)$ and $A(Y \setminus X)$). If X is an H-invariant and compact set in Y, and if Z is a G-compact subset of Y which contains X then define a G-equivariant $*$-homomorphism

$$A[Z] \longrightarrow \operatorname{Ind}_H^G A[X]$$

by mapping $a \in A[Z]$ to the function $f(g) = g^{-1}(a)$ in $\operatorname{Ind}_H^G A[X]$. By $g^{-1}(a)$ we mean here the element of $A[X]$ obtained by first transforming $a \in A[Z]$ by g^{-1}, then mapping to the quotient $A[X]$ of $A[Z]$.

Let A be a G-C^*-algebra which is proper over Y. By composing with the induction homomorphism of Definition 12.7 with the map on E_G-theory induced from the $*$-homomorphism in Definition 12.8, we obtain for each G-compact Z an induction homomorphism

$$\operatorname{Ind}_H^G\colon \varinjlim_{X \subset Z} E_H(A[X], B) \longrightarrow E_G(A[Z], \operatorname{Ind}_H^G B),$$

where the direct limit is over the H-compact subsets of Z. For the most part we shall be concerned with the special case where $A = C_0(Z)$, from which we obtain an induction homomorphism

$$\operatorname{Ind}_H^G\colon E_H(Z, B) \to E_G(Z, \operatorname{Ind}_H^G B).$$

We remind the reader that $E_H(Z, B)$ is by definition a direct limit over the H-compact sets $X \subset Z$ (see Definition 10.1). An element of $E_H(Z, B)$ is represented by a $*$-homomorphism

$$\varphi\colon \Sigma C(X) \to \mathfrak{A}^n(\Sigma B \otimes \mathcal{K}(\mathcal{H})),$$

and if we regard B as included in $\operatorname{Ind}_H^G B$ in the way we indicated earlier then our E-theoretic induction homomorphism maps the class of φ to the class of the $*$-homomorphism

$$\Phi\colon \Sigma C_0(Z) \to \mathfrak{A}^n(\Sigma \operatorname{Ind}_H^G B \otimes \mathcal{K}(\mathcal{H}))$$

defined by

$$\Phi(f \otimes h) = \sum_{g \in G/H} g(\varphi(f \otimes g^{-1}(h)),$$

where $g^{-1}(h)$ is regarded as a function on $X \subset Z$ by restriction. We note that if h is compactly supported then the sum is in fact finite. See the proof of Lemma 12.11 below for further remarks on how to interpret the formula.

Taking a direct limit over G-compact subsets $Z \subset \mathcal{E}G$ we obtain an induction homomorphism

$$\operatorname{Ind}_H^G\colon E_H(\mathcal{E}G, B) \to E_G(\mathcal{E}G, \operatorname{Ind}_H^G B).$$

The goal of this chapter is to prove the following result:

12.9. Proposition. *If H is a finite subgroup of a discrete group G and if B is an H-C^*-algebra then the induction homomorphism*

$$\operatorname{Ind}_H^G\colon E_H(\mathcal{E}G, B) \to E_G(\mathcal{E}G, \operatorname{Ind}_H^G B)$$

is an isomorphism.

The statement makes sense for arbitrary G, and conjecturally the assertion is correct for all G, but the argument we are going to give depends very much on G being discrete. It would be both interesting and useful to have a proof of the general case.

We shall prove the theorem by introducing a homomorphism which is closely related to induction, but which is much easier to analyze. We note that in the following definition it is crucial that the group G be discrete.

12.10. Definition. Let H be a finite subgroup of a discrete group G. Let A be a G-C^*-algebra which is proper over G/H and let D be any G-C^*-algebra (not necessarily proper). Let $\operatorname{Comp}_H^G A$ be the compression of A described in Definition 12.4. Define the *compression homomorphism*

$$\operatorname{Comp}_H^G\colon E_G(A, D) \to E_H(\operatorname{Comp}_H^G A, D)$$

by restricting a G-equivariant $*$-homomorphism

$$\varphi\colon \Sigma A \otimes \mathcal{K}(\mathcal{H}) \to \mathfrak{A}^n(\Sigma D \otimes \mathcal{K}(\mathcal{H}))$$

to the H-invariant C^*-subalgebra $\operatorname{Comp}_H^G A \subset A$ so as to obtain an H-equivariant $*$-homomorphism

$$\varphi\colon \Sigma \operatorname{Comp}_H^G A \to \mathfrak{A}^n(\Sigma D \otimes \mathcal{K}(\mathcal{H})).$$

12.11. Lemma. *Let H be a finite subgroup of a discrete group G and let A be a G-C^*-algebra which is proper over G/H. For every G-C^*-algebra D (not necessarily proper) the compression map*

$$\operatorname{Comp}_H^G\colon E_G(A, D) \to E_H(\operatorname{Comp}_H^G A, D)$$

is an isomorphism.

Proof. Let $B = \mathrm{Comp}_H^G A$. We shall define an 'inflation' map

$$I_H^G \colon E_H(B, D) \to E_G(A, D),$$

and show that it is inverse to the compression homomorphism. For the sake of brevity we shall set $A_1 = \Sigma A \otimes \mathcal{K}(\mathcal{H})$, $B_1 = \Sigma B \otimes \mathcal{K}(\mathcal{H})$ and $D_1 = \Sigma D \otimes \mathcal{K}(\mathcal{H})$. Note that B_1 is the compression of A_1. An element of $E_H(B, D)$ is represented by an H-equivariant $*$-homomorphism

$$\varphi \colon B_1 \to \mathfrak{A}^n D_1.$$

We define the *inflation* of φ, which is a G-equivariant $*$-homomorphism

$$\Phi \colon A_1 \to \mathfrak{A}^n(D_1 \otimes \mathcal{K}(\ell^2 G/H))$$

by

$$\Phi(a) = \sum_{g \in G/H} g(\varphi(g^{-1}(a))) \otimes e_g.$$

Here the sum is over representatives of the left cosets in G/H and e_g denotes the rank-one projection corresponding to the basis element $[gH] \in \ell^2 G/H$. If $a \in A_1$ and $g \in G$ then we regard $g^{-1}(a)$ as an element of B_1 (so as to apply φ) by compression—that is by multiplication with the projection $p \in C_0(G/H)$. We interpret the sum in the formula much as we interpreted a similar sum in the proof of Proposition 7.1, by assuming first that a is finitely supported; then extending by continuity to arbitrary $a \in A_1$. Note that Φ defines a class in $E_G(A, D)$, as required.

It is straightforward to check that inflation, followed by compression, gives the identity on $E_H(B, D)$. To calculate the other composition, suppose given a $*$-homomorphism

$$\Psi \colon A_1 \to \mathfrak{A}^n D_1.$$

Applying compression, followed by inflation, we obtain the $*$-homomorphism

$$\Phi \colon A_1 \to \mathfrak{A}^n(D_1 \otimes \mathcal{K}(\ell^2(G/H)))$$

defined by

$$\Phi(a) = \sum_{g \in G/H} g(\Psi(g^{-1}(a))) \otimes e_g.$$

But this is precisely the composition

$$A_1 \xrightarrow{\kappa} A_1 \otimes \mathcal{K}(\ell^2 G/H) \xrightarrow{\Psi \otimes 1} \mathfrak{A}^n(D_1 \otimes \mathcal{K}(\ell^2 G/H)),$$

where κ is the stabilization homomorphism considered in Chapter 9. According to what we showed in Chapter 9, the $*$-homomorphisms Φ and Ψ determine the same E-theory class. $\square$

The main step in our proof of Proposition 12.9 is the following calculation of the induction map for spaces of the form $Z = G/J$, where J is a finite subgroup of G.

12.12. Lemma. *Let H and J be finite subgroups of a discrete group G and let B be an H-C^*-algebra. The induction map*

$$\mathrm{Ind}_H^G\colon E_H(G/J, B) \to E_G(G/J, \mathrm{Ind}_H^G B)$$

is an isomorphism.

Proof. Observe that the compression of $A = C_0(G/J)$ to a J-C^*-algebra is $C(\mathrm{Pt})$, where 'Pt' denotes the one point space comprised of the identity coset. Thus there is a compression isomorphism

$$\mathrm{Comp}_J^G\colon E_G(G/J, \mathrm{Ind}_H^G B) \underset{\cong}{\longrightarrow} E_J(\mathrm{Pt}, \mathrm{Ind}_H^G B).$$

We shall consider the composition of induction with the compression isomorphism,

$$(\diamondsuit) \qquad E_H(G/J, B) \xrightarrow{\mathrm{Ind}_H^G} E_G(G/J, \mathrm{Ind}_H^G B) \xrightarrow[\cong]{\mathrm{Comp}_J^G} E_J(\mathrm{Pt}, \mathrm{Ind}_H^G B),$$

and show that the result is an isomorphism.

It will simplify the notation a bit if we write $D = \mathrm{Ind}_H^G B$ and note that B is the compression of the G-C^*-algebra D to H. Thus we will write $D = D[G/H]$ and $B = D[\mathrm{Pt}]$. With this understood, we can rewrite the above composition as a homomorphism

$$(\diamondsuit) \qquad E_H(G/J, D[\mathrm{Pt}]) \longrightarrow E_J(\mathrm{Pt}, D[G/H]).$$

Let us introduce the further notation

$$\begin{aligned} A_1[X] &= \Sigma C(X) \otimes \mathcal{K}(\mathcal{H}) \qquad (X \subset G/J) \\ D_1[Y] &= \Sigma D[Y] \otimes \mathcal{K}(\mathcal{H}) \qquad (Y \subset G/H). \end{aligned}$$

A class in $E_H(G/J, D[\mathrm{Pt}])$ is then represented by an H-equivariant $*$-homomorphism

$$\varphi\colon A_1[G/J] \to \mathfrak{A}^n(D_1[\mathrm{Pt}]),$$

which factors through some $A_1[X]$, where X is a finite, H-invariant subset of G/J:

$$\begin{tikzcd} A_1[G/J] \arrow[rr, "\varphi"] \arrow[dr] & & \mathfrak{A}^n(D_1[\mathrm{Pt}]) \\ & A_1[X] \arrow[ur] & \end{tikzcd}$$

Let us say that such a factorizable $*$-homomorphism φ is *H-finitely supported.* Two H-finitely supported $*$-homomorphisms determine the same element of the group $E_H(G/J, D[\mathrm{Pt}])$ if and only if they are homotopic via an H-finitely supported homotopy. The composition $(\diamondsuit)$ of induction and compression takes an H-finitely supported $*$-homomorphism $\varphi\colon A_1[G/J] \to \mathfrak{A}^n(D_1[\mathrm{Pt}])$ to the class of the $*$-homomorphism $\psi\colon A_1[\mathrm{Pt}] \to \mathfrak{A}^n(D_1[G/H])$ defined by

$$\psi(a) = \sum_{g \in G/H} g(\varphi(g^{-1}(a))),$$

where we are regarding $A_1[\mathrm{Pt}]$ as a subalgebra of $A_1[G/H]$, which is legitimate since Pt is not only a closed but also an open subset of G/H. Observe that the sum is finite, so there is no problem giving it a meaning.

Having described the composition ($\diamondsuit$) in suitable terms we are now going to to define an inverse map

$$(\triangle) \qquad E_J(\mathrm{Pt}, D[G/H]) \to E_H(G/J, D[\mathrm{Pt}]).$$

To do so, note first that the C^*-algebra $D[G/H]$, viewed as a J-C^*-algebra, is a direct limit of J-C^*-subalgebras $D[Y]$, as Y ranges over the finite, J-invariant subsets of G/H. The Green-Julg theorem asserts that $E_J(\mathrm{Pt}, D[Y])$ is isomorphic to $E(\mathbb{C}, C^*(J, D[Y]))$, which is the K-theory group $K_0(C^*(J, D[Y]))$. Since K-theory commutes with direct limits [**6**, Section 5.2] it follows that

$$E_J(\mathrm{Pt}, D[G/J]) \cong \varinjlim_{Y \subset G/H} E_J(\mathrm{Pt}, D[Y]),$$

where the direct limit is over the finite, J-invariant subsets of G. Thus every element of $E_J(\mathrm{Pt}, D[G/J])$ may be represented by a $*$-homomorphism $\psi\colon A_1[\mathrm{Pt}] \to \mathfrak{A}^n(D_1[G/J])$ which is *J-finitely supported,* in the sense that it factors through one of the C^*-subalgebras $D_1[Y] \subset D_1[G/J]$:

$$\begin{tikzcd} A_1[\mathrm{Pt}] \arrow[rr, "\psi"] \arrow[dr] & & \mathfrak{A}^n(D_1[G/J]) \\ & \mathfrak{A}^n(D_1[Y]) \arrow[ur] & \end{tikzcd} .$$

Furthermore two such J-finitely supported $*$-homomorphisms determine the same element in $E_J(\mathrm{Pt}, D[G/J])$ if and only if they are n-homotopic via a J-finitely supported homotopy. We define our inverse map by forming from a J-finitely supported $*$-homomorphism $\psi\colon A_1[\mathrm{Pt}] \to \mathfrak{A}^n(D_1[G/H])$ the $*$-homomorphism

$$\varphi\colon A_1[G/J] \to \mathfrak{A}^n(D_1[\mathrm{Pt}] \otimes \mathcal{K}(\ell^2 G/J))$$
$$\varphi(a) = \sum_{g \in G/J} g(\psi(g^{-1}(a))) \otimes e_{gJ},$$

where by $\psi(g^{-1}(a))$ we mean: transform $a \in A_1[G/J]$ by g^{-1}; restrict to $A_1[\mathrm{Pt}]$ (a quotient of $A_1[G/J]$); then apply ψ to obtain an element of $\mathfrak{A}^n(D_1[G/H])$; then transform by g; then finally restrict to $\mathfrak{A}^n(D_1[\mathrm{Pt}])$, which is a quotient of $\mathfrak{A}^n(D_1[G/H])$. Note that the sum is actually finite.

If we take an H-finitely supported $*$-homomorphism $\varphi\colon A_1[G/J] \to \mathfrak{A}^n B_1[\mathrm{Pt}]$, defining a class in $E_H(G/J, D[\mathrm{Pt}])$, and apply to it first ($\diamondsuit$) and then ($\triangle$) we obtain the $*$-homomorphism

$$A_1[G/J] \xrightarrow{\kappa} A_1[G/J] \otimes \mathcal{K}(\ell^2 G/J) \xrightarrow{\varphi \otimes 1} \mathfrak{A}^n(D_1[\mathrm{Pt}] \otimes \mathcal{K}(\ell^2 G/J)),$$

where κ is the stabilization homomorphism. On the other hand, if we apply first ($\triangle$), and then the map ($\diamondsuit$), to a J-finitely supported $*$-homomorphism $\psi\colon A_1[\mathrm{Pt}] \to \mathfrak{A}^n(D_1[G/H])$ we obtain the $*$-homomorphism

$$A_1[\mathrm{Pt}] \to \mathfrak{A}^n(D_1[G/H]) \xrightarrow{1 \otimes e_J} \mathfrak{A}^n(D_1[G/H] \otimes \mathcal{K}(\ell^2 G/J)).$$

It therefore follows from the results in Chapter 9 that ($\diamondsuit$) and ($\triangle$) are inverse to one another. $\square$

Proof of Proposition 12.9. We shall use the fact that $\mathcal{E}G$ may be represented as a simplicial complex, and that in forming $E_G(\mathcal{E}G, \operatorname{Ind}_H^G B)$ we need only consider the direct limit of the groups $E_G(Z, \operatorname{Ind}_H^G B)$ where Z is a G-finite subcomplex of $\mathcal{E}G$. In view of these things it suffices to prove that if Z is any G-finite proper simplicial complex then the induction map

$$\operatorname{Ind}_H^G \colon E_H(Z, B) \to E_G(Z, \operatorname{Ind}_H^G B)$$

is an isomorphism. This we shall do by induction (in the other sense of the word) on the dimension of Z.

If Z is zero dimensional then it is a disjoint union of coset spaces G/J, and so by Lemma 12.12 the induction map is an isomorphism.

Now let $n \geq 1$. Suppose we have shown induction to be an isomorphism for any C^*-algebra which is proper over a G-finite simplicial complex of dimension no more than $n-1$, and suppose that A is proper over an n-dimensional complex Z_n. Let Z_{n-1} be the $(n-1)$-skeleton of Z and let $U = Z_n \backslash Z_{n-1}$. Observe that U is a disjoint union of open n-simplices and that $C_0(U)$ is proper over the zero-dimensional complex Z_0 formed of the barycenters of the n-simplices. In fact $C_0(U) \cong \Sigma^n C_0(Z_0)$. Now let $A = C_0(Z_n)$ and $J = C_0(U)$, so that $A/J = C_0(Z_{n-1})$. The C^*-algebra J is proper over both Z_0 and Z_n, and the two induction homomorphisms

$$\varinjlim_{X \subset Z_0} E_H(J[X], B) \to E_G(J[Z_0], \operatorname{Ind}_H^G B)$$
$$\varinjlim_{X \subset Z_n} E_H(J[X], B) \to E_G(J[Z_n], \operatorname{Ind}_H^G B)$$

are equal. Since $J \cong \Sigma^n C_0(Z_0)$, the first induction map is an isomorphism by Bott periodicity and Proposition 12.12. Therefore so is the second. Similarly, we can regard A/J as proper over either Z_n or Z_{n-1}. The two induction homomorphisms

$$\varinjlim_{X \subset Z_{n-1}} E_H(A/J[X], B) \to E_G(A/J[Z_{n-1}], \operatorname{Ind}_H^G B)$$
$$\varinjlim_{X \subset Z_n} E_H(A/J[X], B) \to E_G(A/J[Z_n], \operatorname{Ind}_H^G B)$$

are equal, and since by hypothesis the first is an isomorphism, so is the second. Consider now the short exact sequence

$$0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0.$$

If we view J, A and A/J all as proper over Z_n then since the maps ι and π are $C_0(Z_n)$-linear, it follows from the description of the long exact sequence in Proposition 6.14 that for every H-compact $X \subset Z_n$ there is a commuting diagram

of long exact sequences

$$\begin{array}{ccccccc}
\longleftarrow & E_G(J[Z_n], \mathrm{Ind}_H^G B) & \longleftarrow & E_G(A[Z_n], \mathrm{Ind}_H^G B) & & & \\
 & \uparrow{\scriptstyle \mathrm{Ind}_H^G} & & \uparrow{\scriptstyle \mathrm{Ind}_H^G} & & & \\
\longleftarrow & E_H(J[X], B) & \longleftarrow & E_H(A[X], B) & & & \\
 & & & \longleftarrow & E_G(A/J[Z_n], \mathrm{Ind}_H^G B) & \longleftarrow & \\
 & & & & \uparrow{\scriptstyle \mathrm{Ind}_H^G} & & \\
 & & & \longleftarrow & E_H(A/J[X], B) & \longleftarrow & .
\end{array}$$

If we take a direct limit in the bottom row over $X \subset Z_n$ then the induction maps for J and A/J become isomorphisms. Therefore by the five lemma the induction map for A becomes an isomorphism too. $\square$

CHAPTER 13

A Generalized Green–Julg Theorem

We are now ready to tackle our generalization of the Green-Julg theorem to discrete groups.

13.1. THEOREM. *Let G be a countable discrete group. If D is a proper G-C^*-algebra then the assembly map*

$$\mu\colon E_G(\mathcal{E}G, D) \to E(\mathbb{C}, C^*(G, D))$$

is an isomorphism.

The theorem is true for other classes of groups. For instance it is true for Lie groups and totally disconnected groups, although we shall not consider these cases in the present work. Conjecturally the theorem is true for all groups, but in this generality there arise some awkward issues concerning the equivalence of various notions of proper action. In addition our definition of $E_G(\mathcal{E}G, D)$ as a direct limit over G-compact sets becomes a little problematic. For these reasons we shall concentrate on discrete groups, which are in any case those of the greatest importance in applications.

We are going to reduce the proof of Theorem 13.1 to the results about induction we obtained in the previous chapter. Let H be a finite subgroup of G and let B be an H-C^*-algebra. By virtue of the H-equivariant inclusion $B \subset \operatorname{Ind}_H^G B$ there is a map

$$j\colon C^*(H, B) \to C^*(G, \operatorname{Ind}_H^G B)$$

which sends $\sum_{h\in H} b_h[h] \in C^*(H, B)$ to the element of $C^*(G, \operatorname{Ind}_H^G B)$ given by the same formula. Its induced map in E-theory has the following property:

13.2. LEMMA. *The diagram*

$$\begin{array}{ccc} E_G(\mathcal{E}G, \operatorname{Ind}_H^G B) & \xrightarrow{\mu} & E(\mathbb{C}, C^*(G, \operatorname{Ind}_H^G B)) \\ {\scriptstyle \operatorname{Ind}_H^G}\uparrow & & \uparrow{\scriptstyle j_*} \\ E_H(\mathcal{E}G, B) & \xrightarrow{\mu} & E(\mathbb{C}, C^*(H, B)) \end{array}$$

is commutative.

PROOF. Let $\varphi\colon \Sigma C(X) \to \mathfrak{A}^n(\Sigma B \otimes \mathcal{K}(\mathcal{H}))$ be a $*$-homomorphism representing an element in $E_H(\mathcal{E}G, B)$. By Corollary 9.8 we can omit a copy of $\mathcal{K}(\mathcal{H})$ when writing down a typical representative of the group $E_H(\mathcal{E}G, B)$, and it will help simplify our notation to do so. Applying first the induction homomorphism Ind_H^G

and then the assembly map μ to φ gives the $*$-homomorphism

$$\psi_{11}\colon \Sigma \to \mathfrak{A}^n(C^*(G, \Sigma \operatorname{Ind}_H^G B \otimes \mathcal{K}(\mathcal{H})))$$
$$\psi_{11}(f) = \sum_g \Phi(f \otimes \theta g(\theta))[g],$$

where θ denotes a cut-off function for the saturation Z of X in $\mathcal{E}G$ and Φ denotes the induced $*$-homomorphism obtained from φ, as in the discussion following Definition 12.8. The reason for the double index in ψ_{11} will become apparent in a moment. Applying the assembly map μ first, and then composing with the $*$-homomorphism j, we obtain the $*$-homomorphism

$$\psi_{22}\colon \Sigma \to \mathfrak{A}^n(C^*(G, \Sigma \operatorname{Ind}_H^G B \otimes \mathcal{K}(\mathcal{H}))),$$
$$\psi_{22}(f) = \frac{1}{|H|} \sum_h \varphi(f \otimes 1_X)[h].$$

Denote by p the characteristic function of the identity coset in G/H, regarded as a multiplier of $\operatorname{Ind}_H^G B$, let

$$\psi_{21}(f) = \frac{1}{|H|^{1/2}} \sum_g p\Phi(f \otimes g(\theta))[g],$$

and let $\psi_{12}(f) = \psi_{21}(f^*)^*$. Then $\psi_{ij}(f')\psi_{jk}(f'') = \psi_{ik}(f'f'')$ and so the formula

$$f \mapsto \begin{pmatrix} s^2\psi_{11}(f) & s(1-s^2)^{1/2}\psi_{12}(f) \\ s(1-s^2)^{1/2}\psi_{21}(f) & (1-s^2)\psi_{22}(f) \end{pmatrix},$$

defines a homotopy of $*$-homomorphisms

$$\psi_s\colon \Sigma \to M_2(\mathfrak{A}^n(C^*(G, \Sigma \operatorname{Ind}_H^G B \otimes \mathcal{K}(\mathcal{H})))) \qquad (s \in [0,1])$$

connecting the $*$-homomorphisms ψ_{11} and ψ_{22}. □

13.3. Lemma. *The inclusion $j\colon C^*(H,B) \to C^*(G, \operatorname{Ind}_H^G B)$ induces an isomorphism $j_*\colon E(\mathbb{C}, C^*(H,B)) \underset{\cong}{\longrightarrow} E(\mathbb{C}, C^*(G, \operatorname{Ind}_H^G B))$.*

Proof. Denote by p the characteristic function of the identity coset in G/H, regarded as a multiplier of $A = C^*(G, \operatorname{Ind}_H^G B)$. Then the map j identifies $C^*(H,B)$ with pAp. Since $ApA = A$ the lemma follows from Morita invariance of K-theory [**6**]. □

Proof of Theorem 13.1. It follows from Lemmas 12.3, 13.2 and 13.3, along with Proposition 12.9, that if D is proper over G/H, where H is a finite subgroup of G, then assembly is an isomorphism. Thus if D is proper over Y, and if Y maps to some G/H, then assembly is an isomorphism for D. Suppose next that D is proper over a space Y which has a cover by a finite number, n, of G-invariant open sets, each of which maps to some G/H. If U is one of these open sets then there is a short exact sequence

$$0 \to D(U) \to D(Y) \to D(Y)/D(U) \to 0.$$

The quotient C^*-algebra is proper over $Y \setminus U$, which is covered by $n-1$ open sets, each admitting a map to some G/H. Thus an application of the long exact sequence in E-theory, combined with the five lemma and induction on n, shows

that assembly is an isomorphism for D. Observe that this argument applies to any D which is proper over a G-compact proper G-space.

If D is a general proper algebra then it is a direct limit of increasing sequence of C^*-subalgebras D_n (one can even take them to be ideals in D), each of which is proper over a G-compact proper G-space. So the general case of the theorem will be proved if we can show that the natural map

$$\varinjlim E_G(\mathcal{E}G, D_n) \to E_G(\mathcal{E}G, D)$$

is an isomorphism: observe that we already know that the analogous result for the group $E(\mathbb{C}, C^*(G, D))$ is true, since these groups identify with K-theory, which is known to commute with direct limits [**6**, Section 5.2]. The required isomorphism may be shown by another Mayer-Vietoris argument, based on the fact that $\mathcal{E}G$ is covered by sets of the form $G \times_H W$, where W is H-equivariantly contractible. Indeed by the five lemma again it suffices to show that for a single such space $G \times_H W$ the map

$$\varinjlim E_G(G \times_H W, D_n) \to E_G(G \times_H W, D)$$

is an isomorphism. For this, consider the diagram

$$\begin{array}{ccc} \varinjlim E_G(G \times_H W, D_n) & \longrightarrow & E_G(G \times_H W, D) \\ \mathrm{Comp}_H^G \big\downarrow \cong & & \cong \big\downarrow \mathrm{Comp}_H^G \\ \varinjlim E_H(W, D_n) & \longrightarrow & E_H(W, D). \end{array}$$

By the equivariant contractibility of W, in the bottom of this diagram is the natural map

$$\varinjlim E_H(\mathrm{Pt}, D_n) \to E_H(\mathrm{Pt}, D),$$

and by the Green-Julg theorem this is the same as the map

$$\varinjlim E(\mathbb{C}, C^*(H, D_n)) \to E(\mathbb{C}, C^*(H, D_n)).$$

The latter identifies with the natural map in K-theory

$$\varinjlim K_0(C^*(H, D_n)) \to K_0(C^*(H, D)).$$

As we have already noted, this is an isomorphism. $\square$

CHAPTER 14

Application to the Baum-Connes Conjecture

We are now ready to state and prove our main results concerning the Baum-Connes assembly map. The following theorem is central to the argument in [**19**] that the assembly map is an isomorphism for discrete groups (such as countable amenable groups) which act isometrically and metrically properly on an infinite dimensional Euclidean space.

14.1. Theorem. *Let G be a countable discrete group and suppose that there is a proper G-C^*-algebra D and elements $\alpha \in E_G(D, \mathbb{C})$ and $\beta \in E_G(\mathbb{C}, D)$ whose composition is $\alpha \circ \beta = 1 \in E_G(\mathbb{C}, \mathbb{C})$. Then for any G-C^*-algebra B the assembly map*

$$\mu\colon E_G(\mathcal{E}G, B) \to E(\mathbb{C}, C^*(G, B))$$

is an isomorphism.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
E_G(\mathcal{E}G, B \otimes \mathbb{C}) \arrow[r, "\mu"] \arrow[d, "1\otimes\beta"'] & E(\mathbb{C}, C^*(G, B \otimes \mathbb{C})) \arrow[d, "1\otimes\beta"] \\
E_G(\mathcal{E}G, B \otimes D) \arrow[r, "\mu", "\cong"'] \arrow[d, "1\otimes\alpha"'] & E(\mathbb{C}, C^*(G, B \otimes D)) \arrow[d, "1\otimes\alpha"] \\
E_G(\mathcal{E}G, B \otimes \mathbb{C}) \arrow[r, "\mu"'] & E(\mathbb{C}, C^*(G, B \otimes \mathbb{C}))
\end{tikzcd}$$

Since D, and hence $B \otimes D$, is proper, it follows from Theorem 13.1 that the middle assembly map is an isomorphism. By hypothesis the vertical compositions are the identity. A trivial diagram chase now shows that the assembly map for B, which appears both at the top and the bottom of the diagram, is also an isomorphism. □

As we have already noted, it is of interest to ask when the assembly map is injective, if not necessarily an isomorphism. The following variant on Theorem 14.1 applies in many cases (for instance in the case of a discrete subgroup of a Lie group, although we shall not go into that here).

14.2. Theorem. *Let G be a countable discrete group. Let D be a proper G-C^*-algebra and suppose there are elements $\alpha \in E_G(D, \mathbb{C})$ and $\beta \in E_G(\mathbb{C}, D)$ whose composition $\alpha \circ \beta \in E_G(\mathbb{C}, \mathbb{C})$ maps to the identity $1 \in E_H(\mathbb{C}, \mathbb{C})$, upon restriction to any finite subgroup $H \subset G$. Then for any G-C^*-algebra B the assembly map*

$$\mu\colon E_G(\mathcal{E}G, B) \to E(\mathbb{C}, C^*(G, B))$$

is split injective.

The proof requires a lemma. The following is adequate for our present purposes, but we shall make a stronger assertion in a moment.

14.3. Lemma. *With the hypotheses of Theorem 14.2, if X is any G-compact proper G-space then the composition*

$$C_0(X) \otimes \mathbb{C} \xrightarrow{1\otimes\beta} C_0(X) \otimes D \xrightarrow{1\otimes\alpha} C_0(X) \otimes \mathbb{C}$$

is an isomorphism in $E_G(C_0(X), C_0(X))$.

Proof. We shall use the following simple criterion for a morphism in the equivariant E-theory category—or indeed in any category—to be an isomorphism: an element $\varphi \in E_G(A, B)$ is an isomorphism if and only if, for every G-C^*-algebra C, composition with φ induces an isomorphism from $E_G(B, C)$ into $E_G(A, C)$.

Suppose first that X is a locally compact proper G-space (not necessarily G-compact) which admits a continuous G-map to a coset space G/H, where H is a finite subgroup of G. Then according to Lemma 12.11, there is a compression isomorphism

$$E_G(C_0(X), C) \cong E_H(C_0(W), C),$$

where W is the inverse image in X of the identity coset in G/H. In view of the commutative diagram

$$\begin{array}{ccc}
E_G(C_0(X) \otimes \mathbb{C}, C) & \xrightarrow[\text{with } 1_{C_0(X)}\otimes\alpha\beta]{\text{composition}} & E_G(C_0(X) \otimes \mathbb{C}, C) \\
\text{compression}\Big\downarrow\cong & & \cong\Big\downarrow\text{compression} \\
E_H(C_0(W) \otimes \mathbb{C}, C) & \xrightarrow[\text{with } 1_{C_0(W)}\otimes\alpha\beta]{\text{composition}} & E_H(C_0(W) \otimes \mathbb{C}, C)
\end{array}$$

and in view of the hypothesis that $\alpha \circ \beta = 1$ in $E_H(\mathbb{C}, \mathbb{C})$ we see that composition with $1_{C_0(X)} \otimes \alpha\beta$ is an isomorphism, in fact the identity, on $E_G(C_0(X), C)$. Suppose now that a proper G-space X (still not necessarily G-compact) may be covered by n open G-subsets, each admitting a map to some G/H. An induction on n, using the commuting diagram

$$\begin{array}{ccccccc}
\longleftarrow & E_G(C_0(U), C) & \longleftarrow & E_G(C_0(X), C) & & & \\
 & 1\otimes\alpha\beta\Big\downarrow & & 1\otimes\alpha\beta\Big\downarrow & & & \\
\longleftarrow & E_G(C_0(U), C) & \longleftarrow & E_G(C_0(X), C) & & & \\
 & & & & \longleftarrow & E_G(C_0(X \setminus U), C) & \longleftarrow \\
 & & & & & 1\otimes\alpha\beta\Big\downarrow & \\
 & & & & \longleftarrow & E_G(C_0(X \setminus U), C) & \longleftarrow
\end{array}$$

and the five lemma, shows that $1_{C_0(X)} \otimes \alpha\beta$ is an isomorphism. To conclude the proof, note that a G-compact, proper G-space X may be covered by a finite number of open G-sets, each of which admits a G-map to some G/H. $\square$

Proof of Theorem 14.2. Let us consider the same diagram we used in the proof of Theorem 14.1:

$$\begin{array}{ccc}
E_G(\mathcal{E}G, B\otimes\mathbb{C}) & \xrightarrow{\mu} & E(\mathbb{C}, C^*(G, B\otimes\mathbb{C})) \\
{\scriptstyle 1\otimes\beta}\downarrow & & \downarrow{\scriptstyle 1\otimes\beta} \\
E_G(\mathcal{E}G, B\otimes D) & \xrightarrow[\cong]{\mu} & E(\mathbb{C}, C^*(G, B\otimes D)) \\
{\scriptstyle 1\otimes\alpha}\downarrow & & \downarrow{\scriptstyle 1\otimes\alpha} \\
E_G(\mathcal{E}G, B\otimes\mathbb{C}) & \xrightarrow[\mu]{} & E(\mathbb{C}, C^*(G, B\otimes\mathbb{C}))
\end{array}$$

The middle assembly map is still an isomorphism, of course, but the vertical compositions require more study. In fact we cannot say anything of use about the right hand composition. But as for the left hand composition, if X is a G-compact subset of $\mathcal{E}G$, and if $\varphi\colon C_0(X)\to B$ is an E_G-theory morphism (i.e. an element of $E_G(X,B)$) then the vertical maps send it to the composition

$$C_0(X)\otimes\mathbb{C}\xrightarrow{\varphi\otimes 1} B\otimes\mathbb{C}\xrightarrow{1\otimes\alpha\beta} B\otimes\mathbb{C},$$

which is equal to the composition

$$C_0(X)\otimes\mathbb{C}\xrightarrow{1\otimes\alpha\beta} C_0(X)\otimes\mathbb{C}\xrightarrow{\varphi\otimes 1} B\otimes\mathbb{C}.$$

But it follows from Lemma 14.3 that $1\otimes\alpha\beta\colon C_0(X)\otimes\mathbb{C}\to C_0(X)\otimes\mathbb{C}$ is an isomorphism (in fact, as we will note below, it is the identity). As a result the left hand vertical composition in our commutative diagram is an isomorphism, and a diagram chase completes the proof. □

We continue by writing down a useful strengthening of Lemma 14.3, and an interesting consequence. We shall not prove the lemma here.

14.4. Lemma. *With the hypotheses of Theorem 14.2, if D_1 is any proper G-C^*-algebra then the composition*

$$D_1\otimes\mathbb{C}\xrightarrow{1\otimes\beta} D_1\otimes D\xrightarrow{1\otimes\alpha} D_1\otimes\mathbb{C}$$

is the identity in $E_G(D_1,D_1)$. □

Note that Lemma 14.4 is stronger than Lemma 14.3 in three respects: it applies to proper algebras which are not necessarily commutative; it applies to algebras which are proper over not necessarily G-compact sets; and it asserts that the composition is the identity, not merely an isomorphism. The first is not of any great consequence: indeed our proof of Lemma 14.3 carries over right away to the case of G-C^*-algebras which are proper over a G-compact proper G-space. We could also have generalized our argument to cover the non-G-compact case by invoking a direct limit argument (any proper G-C^*-algebra is a direct limit of algebras which are proper over G-compact proper G-spaces). But to prove that $\alpha\beta\otimes 1$ is the identity, and not merely an isomorphism, a more explicit argument is obviously needed.

Granted Lemma 14.4, we have the following result:

14.5. Proposition. *Let G be a countable discrete group and let γ_1 and γ_2 be two elements of $E_G(\mathbb{C},\mathbb{C})$ such that:*

(i) *γ_1 and γ_2 are compositions $\mathbb{C} \xrightarrow{\beta_1} D_1 \xrightarrow{\alpha_1} \mathbb{C}$ and $\mathbb{C} \xrightarrow{\beta_2} D_2 \xrightarrow{\alpha_2} \mathbb{C}$ where D_1 and D_2 are proper G-C^*-algebras; and*

(ii) *if H is any finite subgroup of G then under the restriction homomorphism*

$$E_G(\mathbb{C},\mathbb{C}) \to E_H(\mathbb{C},\mathbb{C})$$

both γ_1 and γ_2 map to $1 \in E_H(\mathbb{C},\mathbb{C})$.

Then $\gamma_1 = \gamma_1\gamma_2 = \gamma_2$.

Proof. In $E_G(\mathbb{C},\mathbb{C})$ the composition $\gamma_1\gamma_2$ is the same as the tensor product $\gamma_1 \otimes \gamma_2$. Using the fact that $\gamma_1 = \alpha_1 \circ \beta_1$ we may write $\gamma_1 \otimes \gamma_2$ as

$$\mathbb{C} \otimes \mathbb{C} \xrightarrow{\beta_1 \otimes 1} D_1 \otimes \mathbb{C} \xrightarrow{1 \otimes \gamma_2} D_1 \otimes \mathbb{C} \xrightarrow{\alpha_1 \otimes 1} \mathbb{C} \otimes \mathbb{C}$$

then Lemma 14.4 implies that the middle morphism is the identity, and so the composition is $\alpha_1\beta_1$, which is γ_1. Hence $\gamma_1\gamma_2 = \gamma_1$. The proof that $\gamma_1\gamma_2 = \gamma_2$ is of course exactly the same. □

Thus if an element $\gamma \in E_G(\mathbb{C},\mathbb{C})$ as in the proposition exists then it is unique and is an idempotent. This is what in KK-theory is called the 'gamma-element' for G (compare [**23**, Section 5]). To summarize the theorems in this chapter, the existence of the gamma-element implies split injectivity of the assembly map, while isomorphism of the assembly map follows from the assertion $\gamma = 1$.

We conclude by writing down analogs of the Theorems 14.1 and 14.2 for the reduced assembly map. Given the extra hypothesis of exactness, the proofs are exactly the same as those we have just finished.

14.6. Theorem. *Let G be a countable discrete group which is C^*-exact, in the sense of Definition 4.13. Suppose that there is a proper G-C^*-algebra D and elements $\alpha \in E_G(D,\mathbb{C})$ and $\beta \in E_G(\mathbb{C},D)$ whose composition is $\alpha \circ \beta = 1 \in E_G(\mathbb{C},\mathbb{C})$. Then for any G-C^*-algebra B the reduced assembly map*

$$\mu_{\mathrm{red}}\colon E_G(\mathcal{E}G,B) \to E(\mathbb{C}, C^*_{\mathrm{red}}(G,B))$$

is an isomorphism. □

14.7. Theorem. *Let G be a countable discrete group which is C^*-exact. Let D be a proper G-C^*-algebra and suppose there are elements $\alpha \in E_G(D,\mathbb{C})$ and $\beta \in E_G(\mathbb{C},D)$ whose composition $\alpha \circ \beta \in E_G(\mathbb{C},\mathbb{C})$ maps to the identity $1 \in E_H(\mathbb{C},\mathbb{C})$, upon restriction to any finite subgroup $H \subset G$. Then for any G-C^*-algebra B the reduced assembly map*

$$\mu_{\mathrm{red}}\colon E_G(\mathcal{E}G,B) \to E(\mathbb{C}, C^*_{\mathrm{red}}(G,B))$$

is split injective. □

CHAPTER 15

A Concluding Remark on Assembly for Proper Algebras

We close by returning briefly to the generalized Green-Julg theorem of Chapter 13. We proved the result for discrete groups by a somewhat indirect argument, but it is worth noting that for *any* locally compact group G and any proper algebra D there is a homomorphism

$$\nu\colon E(\mathbb{C}, C^*(G,D)) \to E_G(\mathcal{E}G, D)$$

which ought to be inverse to the Baum-Connes assembly map.

Recall from Chapter 11 that we defined such an inverse map for compact groups using the fact that $C^*(G,D)$ identifies with the fixed-point subalgebra of the G-C^*-algebra $D \otimes \mathcal{K}(L^2(G))$. There is a similar identification for proper G-C^*-algebras:

15.1. Definition. (See [**23**, Definition 3.2].) Let D be a proper G-C^*-algebra over X. Let $\tilde{D}$ be the C^*-subalgebra of the multiplier algebra $M(D)$ comprised of the elements d such that $fd \in D$, for every $f \in C_0(X)$. Define $\{D\}^G$ to be the C^*-algebra generated by the G-fixed elements in $C_0(X/G)\tilde{D}$.

15.2. Proposition. *If D is a proper G-C^*-algebra then*

$$C^*(G,D) \cong \{D \otimes \mathcal{K}(L^2(G))\}^G. \quad \square$$

The proof of the proposition is, quite naturally, very similar to the proof of Proposition 11.2. We begin with a covariant representation π of D on a Hilbert space $\mathcal{H}_D$ and then consider the covariant representation σ of D on $L^2(G, \mathcal{H}_D)$ defined by

$$\begin{cases} (\sigma(d)\xi)(g_1) = \pi[g_1(d)](\xi(g_1)) \\ (\sigma(g)\xi)(g_1) = \Delta(g)^{\frac{1}{2}}\xi(g_1 g)), \end{cases}$$

where Δ is the modular function of G. One can check right away that σ maps $C^*(G,D)$ into the G-invariant part of $\mathcal{B}(L^2(G,\mathcal{H}_D))$ (as before we let G act on $L^2(G,\mathcal{H}_D)$ by the left regular representation $g\xi(g_1) = \pi[g](\xi(g^{-1}g_1))$). Since $C^*(G,D) = C^*_{\text{red}}(G,D)$ for proper C^*-algebras D (compare [**29,** Theorem 6.1]), this map is injective, and a further computation as in the proof of Proposition 11.2 proves surjectivity. The details are left to the reader, as is the following calculation:

15.3. Proposition. *Let D be proper over X and let $\mathcal{H}$ be a nondegenerate covariant representation Hilbert space for D. Define an action of $C_0(X)$ on $L^2(G,\mathcal{H}_D)$ by pointwise multiplication:*

$$(f \cdot \xi)(g_1) = \pi[f]\xi(g_1)$$

The representations of $C_0(X)$ and $C^*(G,D)$ on $L^2(G,\mathcal{H}_D)$ commute with one another and so combine to form a representation of $C_0(X)\otimes C^*(G,D)$. The image of this representation lies within $\mathcal{K}(L^2(G))\otimes D$ and the $*$-homomorphism

$$\psi\colon C_0(X)\otimes C^*(G,D)\longrightarrow \mathcal{K}(L^2(G))\otimes D$$

so obtained is equivariant for the given action on $C_0(X)$ and the trivial action on $C^*(G,D)$. $\square$

Suppose now that D is proper over a G-compact proper G-space X. Using Proposition 15.3 we define a map

$$\nu\colon E(\mathbb{C},C^*(G,D))\longrightarrow E_G(C_0(X),D)$$

by means of the commutative diagram

$$\begin{array}{ccc} E_G(C_0(X),C_0(X)\otimes C^*(G,B)) & \xrightarrow{\psi_*} & E_G(C_0(X),D\otimes\mathcal{K}(L^2(G))) \\ \uparrow & & \cong\uparrow\kappa \\ E(\mathbb{C},C^*(G,D)) & \xrightarrow[\nu]{} & E_G(C_0(X),D) \end{array}$$

where the vertical map on the left tensors a $*$-homomorphism defining an element of $E(\mathbb{C},C^*(G,D))$ with the identity on $C_0(X)$, and regards the tensor product as an equivariant $*$-homomorphism, for the trivial action on $C^*(G,B)$.

Since X maps equivariantly to the universal proper G-space $\mathcal{E}G$, the E-theory group $E_G(C_0(X),D)$ maps to $E_G(\mathcal{E}G,D)$, and we obtain a homomorphism

$$\nu\colon E(\mathbb{C},C^*(G,D))\longrightarrow E_G(\mathcal{E}G,D),$$

as required.

In the case where D is not proper over a G-compact proper G-space, we write it as a direct limit $D=\varinjlim D_n$, where each D_n *is* proper over a G-compact proper G-space, and use the continuity property

$$E(\mathbb{C},C^*(G,D))\cong\varinjlim E(\mathbb{C},C^*(G,D_n))$$

to define $\nu\colon E(\mathbb{C},C^*(G,D))\longrightarrow E_G(\mathcal{E}G,D)$ as a direct limit.

Unfortunately we are not able to prove in full generality that ν really is the inverse of μ. It would of course be extremely interesting to resolve this rather substantial loose end, one way or the other.

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