

## The Tangent Groupoid and the Index Theorem

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ABSTRACT. We present a proof of the index theorem for Dirac operators that is drawn from Connes' tangent groupoid approach, as described in his book *Non-commutative Geometry*.

### 1. Introduction

The algebra of linear partial differential operators on a smooth manifold is filtered by the usual concept of order. The *principal symbol* of an operator of order  $k$  is its class in the degree  $k$  part of the associated graded algebra, which might therefore be called the *principal symbol algebra*. The set of polynomials

$$p(x) = D_0 + xD_1 + \cdots + x^n D_n$$

with partial differential operator coefficients for which the order of the coefficient  $D_k$  is no more than  $k$  (for  $0 \leq k \leq n$ ) is an algebra over  $\mathbb{C}[x]$ . The quotient by the ideal of polynomials that vanish at a nonzero  $t \in \mathbb{R}$  is the algebra of partial differential operators. When  $t$  is zero the quotient is the principal symbol algebra. In this standard algebraic way, the algebra of linear partial differential operators is exhibited as a deformation of the principal symbol algebra.

The convolution  $C^*$ -algebra of Alain Connes' tangent groupoid is an analytic counterpart of this deformation. It exhibits the  $C^*$ -algebra generated by the smoothing operators on a smooth closed manifold as a deformation of the  $C^*$ -algebra generated by the smooth, compactly supported functions on the cotangent bundle (in comparison, the principal symbol algebra is the algebra of smooth functions on the cotangent bundle that are polynomial in each fiber).

There is a simple way to relate *elliptic* partial differential operators to Connes'  $C^*$ -algebra using spectral theory, and this makes the  $C^*$ -algebra of the tangent groupoid available for use as a tool in the index theory of elliptic operators. In a short section of his famous book [Con94, Section II.5], Connes sketches a proof of the Atiyah-Singer index theorem using the tangent groupoid and groupoid techniques. As he notes, his proof is closely related to the  $K$ -theory proof of Atiyah and Singer [AS68a], but it has the advantage of extending easily to more elaborate settings, for example to foliations.

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Connes uses the tangent groupoid to build the analytic index map of Atiyah and Singer. Moreover Connes, like Atiyah and Singer, uses an embedding into Euclidean space to describe the index map topologically. But although these ingredients of the two proofs are the same, they are combined differently.

The aim of this paper is to explore this difference a bit further by presenting a proof of the index theorem that reduces Connes' proof to what is arguably its essence. The result is to my mind pleasingly simple, and although the argument does not actually mention groupoids at all, I hope it will help advertise the worth of Connes'  $C^*$ -algebraic and groupoid-theoretic points of view.

My friendship with Alain Connes goes back to our joint work on asymptotic morphisms and E-theory [CH90]. In some sense E-theory starts from the tangent groupoid, since the tangent groupoid is the basic example of an asymptotic morphism in the same way that the Dirac operator is the basic example of a Kasparov module. So it seems to me fitting to return to the subject here.

It goes without saying that Alain has taught me an immense amount over the years. Moreover, working with Alain gave my self-confidence a great boost at an early stage of my career, and it surely raised my standing in the eyes of others as well. So I owe him a great debt, and I am grateful that I am able to acknowledge that debt here.

## 2. Index for Families

In this section and the next I shall review some basic constructions in  $C^*$ -algebra K-theory and elliptic operator theory. Many of the likely readers of this paper will be familiar with these things, and they ought to proceed directly to Section 4, or even to the proof of the index theorem in Section 6. For those who are not so well-versed in these areas, I have tried to write enough to at least suggest that  $C^*$ -algebras are a very convenient setting in which to work out the analytic foundations of index theory.<sup>1</sup>

Let  $Y$  be a locally compact Hausdorff space and let  $\mathcal{H}$  be a countably generated, continuous field of Hilbert spaces over  $Y$ . The reader is referred to [Dix77, Ch 10] or [DD63] for the definition, but the essence of it is to specify a family of sections deemed to be continuous, rather than derive the concept of continuous section from an overall topology on the bundle of Hilbert space fibers.

A *bounded (and adjointable) operator* on  $\mathcal{H}$  is a uniformly bounded family  $T = \{T_y : \mathcal{H}_y \rightarrow \mathcal{H}_y\}_{y \in Y}$  of operators on the fibers of  $\mathcal{H}$  that, along with its adjoint family, maps continuous sections to continuous sections. The bounded operators form a  $C^*$ -algebra.

2.1. DEFINITION. The  $C^*$ -algebra of *compact operators* on  $\mathcal{H}$  is generated by bounded operators of the form

$$T_y : v_y \mapsto \langle v'_y, v_y \rangle v''_y,$$

where  $v'$  and  $v''$  are compactly supported continuous sections of  $\mathcal{H}$ .

The compact operators form a closed ideal within the bounded operators. Note that the compactness condition goes beyond compactness of the individual operators  $T_y$ . In the case of a constant field, bounded operators are bounded

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<sup>1</sup>John Roe and I are preparing a book-length account of groupoids and index theory that has very much influenced what is written here.

\*-strongly continuous families of operators, whereas compact operators are norm-continuous families of compact operators that vanish at infinity.

Suppose that the continuous field  $\mathcal{H}$  is  $\mathbb{Z}_2$ -graded, and that on each  $\mathcal{H}_y$  there is given an unbounded,<sup>2</sup> odd-graded, self-adjoint operator  $D_y$  such that:

- (a) each resolvent operator  $(D_y + iI)^{-1}$  is compact, and
- (b) the family of resolvents is a compact operator on  $\mathcal{H}$ .

We aim to construct an index for the family  $D = \{D_y\}_{y \in Y}$  in  $K(Y)$  that reduces to the Fredholm index when  $Y$  is a point (in the graded context the Fredholm index is defined to be the dimension of the even-graded part of the kernel of  $D$  minus the dimension of the odd-graded part).

We aim to do so because throughout the proof of the index theorem we shall be working with families of Fredholm operators. But we shall begin by considering a single Hilbert space operator.

2.2. DEFINITION. Let  $D$  be an unbounded, odd-graded, self-adjoint operator on a  $\mathbb{Z}_2$ -graded Hilbert space  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ . Its *graph projection* is the orthogonal projection onto the graph of the part of  $D$  that maps  $\mathcal{H}_0$  into  $\mathcal{H}_1$ .

The graph is a closed subspace of  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ , so the graph projection  $P_D$  is an operator on  $\mathcal{H}$ . If  $P_1$  is the orthogonal projection onto  $\mathcal{H}_1$ , then it is easy to check that

$$P_D - P_1 = D(I + D^2)^{-1} + \gamma(I + D^2)^{-1},$$

where  $\gamma$  is the grading operator on  $\mathcal{H}$ . As a result, if the resolvent of  $D$  is compact, then  $P_D - P_1$  is compact too.

The index of  $D$  can be recovered from  $P_D$  and  $P_1$  using K-theory. In general, if the difference of two projections in a  $C^*$ -algebra lies in an ideal, then their formal difference determines an element in the K-theory of that ideal. In the case at hand, the K-theory of the compact operators is isomorphic to  $\mathbb{Z}$ , and the integer obtained from the formal difference  $P_D - P_1$  is the index of  $D$ .

If  $D$  is an operator on a continuous field  $\mathcal{H}$  as in (a) and (b) above, and if  $P_D$  and  $P_1$  are obtained by applying the graph projection construction fiberwise, then the difference  $P_D - P_1$  is compact, and so the formal difference determines an element in the K-theory of the compact operators on the continuous field  $\mathcal{H}$ .

If  $\mathcal{H}$  is a *trivial* field, then this K-theory group is canonically isomorphic to  $K(Y)$ , and we therefore obtain an index class in  $K(Y)$  as required. The case where  $\mathcal{H}$  is not trivial can be handled in many ways. For example we can embed  $\mathcal{H}$  as a continuous subfield of a trivial field  $\mathcal{H}'$  (even as a summand; see [DD63, p. 259]) and then proceed as follows.

2.3. LEMMA. *Every compact operator  $T$  on  $\mathcal{H}$  extends to a compact operator  $T'$  on  $\mathcal{H}'$  by defining  $T'_y$  to be zero on the orthogonal complement of  $\mathcal{H}_y$  in  $\mathcal{H}'_y$ .  $\square$*

The lemma gives a homomorphism from the compact operators on  $\mathcal{H}$  to the compact operators on  $\mathcal{H}'$  and hence a map from the K-theory of the compact operators on  $\mathcal{H}$  into  $K(Y)$ .

2.4. DEFINITION. Let  $D = \{D_y\}_{y \in Y}$  be an odd-graded operator on a  $\mathbb{Z}_2$ -graded continuous field of Hilbert spaces that satisfies conditions (a) and (b) above. Define the index class

$$\text{Index}(D) \in K(Y)$$

<sup>2</sup>In accordance with standard usage, “unbounded” means “possibly unbounded.”

by pushing forward the formal difference  $P_D - P_1$  into  $K(Y)$ , as above.

2.5. LEMMA. *The index class is independent of the embedding into a trivial field that is used in its construction. It has the following properties:*

- (a) *Index(D) is functorial: if Z is a closed subset of Y, then  $\text{Index}(D)$  maps to  $\text{Index}(D|_Z)$  under the restriction map from  $K(Y)$  to  $K(Z)$ .*
- (b) *Index(D) is additive: if  $D = D' \oplus D''$  on  $\mathcal{H}' \oplus \mathcal{H}''$ , then*

$$\text{Index}(D) = \text{Index}(D') + \text{Index}(D'').$$

- (c)  *$\text{Index}(D) = 0$  if the operators  $D_y$  are invertible.*
- (d) *If  $D^{\text{op}}$  denotes the operator on the field  $\mathcal{H}^{\text{op}}$  obtained by reversing the grading on  $\mathcal{H}$ , then  $\text{Index}(D^{\text{op}}) = -\text{Index}(D)$ .  $\square$*

2.6. REMARK. If  $Y$  is compact, and if  $\mathcal{H}$  is a continuous field of constant and finite fiber dimension, then

$$\text{Index}(D) = [\mathcal{H}_0] - [\mathcal{H}_1] \in K(Y),$$

where  $\mathcal{H}_0$  and  $\mathcal{H}_1$  are the even and odd subfields of  $\mathcal{H}$  (they are locally trivial, and hence are vector bundles). As a matter of fact, this additional property, together with the others, actually characterizes the index, although we shall not use this fact.

### 3. Functional Analysis for First Order Elliptic Operators

Let  $\pi: X \rightarrow Y$  be a submersion between smooth manifolds. The manifolds may have boundaries, but if so, then we require that the boundary of  $X$  be the inverse image of the boundary of  $Y$ . The fibers  $X_y = \pi^{-1}\{y\}$  are then smooth manifolds without boundary.

We shall assume that each fiber  $X_y$  is equipped with a smooth measure  $\mu_y$ , and that if  $f$  is a smooth, compactly supported function on  $X$ , then the quantity  $\int_{X_y} f(x) d\mu_y(x)$  is a smooth function of  $y$ .

Let  $S$  be a smooth Hermitian vector bundle over  $X$  and let  $S_y$  be its restriction to  $X_y$ . The Hilbert spaces  $\mathcal{H}_y = L^2(X_y, S_y)$  form a continuous field of Hilbert spaces  $\mathcal{H}$  over  $Y$  whose continuous sections are generated (in the sense of [Dix77, Proposition 10.2.3]) by the smooth compactly supported sections of  $S$ .

Let  $D_y$  be a first-order linear partial differential operator acting on the sections of  $S_y$  and suppose that the family  $D = \{D_y\}$  is *smooth* in the sense that if  $u$  is a smooth section of  $S$  on  $X$ , then the section  $Du$  defined by

$$(Du)|_{X_y} = D_y(u|_{X_y})$$

is also smooth.

We shall assume that each  $D_y$  is formally self-adjoint. To apply the index construction of the last section we shall need to obtain from  $D_y$  an operator that is self-adjoint in the sense of Hilbert space theory (see for example [Kat76, Ch. 5]). For this the following concept is useful.

3.1. DEFINITION (See [HR00, Definition 10.2.8]). The manifold  $X$  is *complete with respect to D* if there is a smooth, proper function  $g: X \rightarrow [0, \infty)$  such that the commutator  $[D, g]$  is a uniformly bounded endomorphism of  $S$ .

3.2. PROPOSITION (See [HR00, Proposition 10.2.10]). *If  $X$  is complete with respect to  $D$ , then each formally self-adjoint operator  $D_y$  is essentially self-adjoint on the smooth, compactly supported sections of  $S_y$ .*  $\square$

Recall that an operator is essentially self-adjoint if its operator-theoretic closure is self-adjoint (see for example [Kat76, Ch. 5] again). From now on we shall assume that  $X$  is complete for  $D$ , and in a slight abuse of notation we shall write  $D_y$  when in operator-theoretic contexts we actually mean the closure of  $D_y$ . Form the resolvent family

$$r(D) := (D + iI)^{-1} = \{(D_y + iI)^{-1}\}_{y \in Y}.$$

It is certainly a bounded operator on the continuous field  $\mathcal{H}$ . To say more, we shall suppose from here on that each operator  $D_y$  is *elliptic*. We can then draw the following conclusion.

3.3. PROPOSITION. *If  $f$  is a smooth, compactly supported function on  $X$ , acting on the continuous field  $\mathcal{H}$  as a family of multiplication operators, then  $f \cdot r(D)$  is a compact endomorphism of  $\mathcal{H}$ .*

This is standard fare, but let us sketch a proof based on a  $C^*$ -algebra calculation.

3.4. LEMMA. *Let  $A$  be a  $C^*$ -algebra that includes  $C_0(X)$  as a  $C^*$ -subalgebra and let  $a$  be an element of  $A$  that commutes with  $C_0(X)$ . Suppose that for every  $x \in X$  and every  $\varepsilon > 0$  there is some  $f \in C_0(X)$  such that  $f(x) = 1$  and  $\|f \cdot a\| < \varepsilon$ . Then  $f \cdot a = 0$  for every  $f \in C_0(X)$ .*  $\square$

Let  $B$  be the  $C^*$ -algebra of bounded continuous functions from  $(0, 1]$  into the bounded operators on  $\mathcal{H}$ , and let  $J$  be the ideal of functions whose distance to the compact operators converges to zero at 0. Let  $A = B/J$ . The operator-valued function  $\alpha: t \mapsto (tD + iI)^{-1}$  determines an element of  $A$ , and so does every constant operator-valued function  $f: t \mapsto f$ , for every  $f \in C_0(X)$ . It suffices to show that the product  $f \cdot \alpha$  in  $A$  is zero since

$$\lim_{t \rightarrow 0} f \cdot (tD + iI)^{-1} (D + iI)^{-1} = -if \cdot (D + iI)^{-1}.$$

The elements  $\alpha$  and  $f$  commute in  $A$ , so it suffices to verify the estimates in Lemma 3.4 for each given  $x$  and  $\varepsilon > 0$ . The product  $f \cdot \alpha \in A$  depends only on the restriction of  $D$  to a neighborhood of the support of  $f$ , so we may as well assume that  $D$  is compactly supported and elliptic near the support of  $f$ . Furthermore, by choosing  $f$  to have sufficiently small support, we may assume that  $p: X \rightarrow Y$  is actually a trivial vector bundle (since any submersion is locally isomorphic to a trivial vector bundle).

Using the basic estimate for constant coefficient elliptic operators we can find  $f$  with sufficiently small support so that  $f \cdot \alpha$  is  $\varepsilon$ -close to  $f \cdot \alpha'$ , where  $\alpha'$  is defined in the same way as  $\alpha$ , but using an operator  $D'$  that restricts to the same constant coefficient elliptic operator in each vector space fiber of  $p$ . A Fourier transform calculation then shows that for every  $t \in (0, 1]$  the operator  $f \cdot (tD' + iI)^{-1}$  is compact, and the proof of Proposition 3.3 is complete.

If  $X$  is compact, then we may choose  $f \equiv 1$  in Proposition 3.3, and conclude from Section 2 that  $D$  has a well-defined index in the  $K$ -theory group  $K(Y)$ . Thus a smooth family of elliptic operators on the fibers of a submersion with compact

fibers and compact base has a well-defined *families index* in  $K(Y)$  (compare [AS71]). But in the proof of the index theorem that we shall present here the manifold  $X$  will not be compact.

As a substitute for compactness we shall work with operators of the form  $D + E$ , where  $E$  is a suitable smooth self-adjoint endomorphism of  $S$ . The operators in the family  $D + E$  are still essentially self-adjoint because  $X$  is complete with respect to  $D + E$ . The compactness of the resolvent  $r(D + E)$  is guaranteed (in the cases of concern to us) by the following calculation.

**3.5. PROPOSITION.** *Assume that  $S$  is  $\mathbb{Z}_2$ -graded and that  $D$  is odd-graded. Let  $E$  be a smooth, odd-graded self-adjoint endomorphism of the Hermitian bundle  $S$  over  $X$ . Assume that*

- (a) *The square of  $E$  is a proper scalar function from  $X$  to  $[0, \infty)$ .*
- (b) *The anticommutator  $DE + ED$  is a uniformly bounded smooth endomorphism of  $S$ .*

*Then  $r(D + E)$  is a compact operator on the continuous field  $\mathcal{H}$ .*

**PROOF.** We shall show that for every  $\varepsilon > 0$  the family  $r(D + E)$  lies within  $\varepsilon$  of a compact operator.

Choose a smooth, compactly supported real function  $f$  such that if  $F = \gamma f$ , where  $\gamma$  is the grading operator on  $S$ , then

$$\|(D + E + F)s\| \geq \varepsilon^{-1} \|s\|$$

for every compactly supported smooth section  $s$ . This is possible because first of all

$$(D + E + F)^2 = D^2 + (DE + ED) + [D, f]\gamma + E^2 + f^2.$$

It therefore suffices to choose  $f$  such that

$$E^2 + f^2 \geq \varepsilon^{-2} + \|DE + ED\| + \|[D, f]\|,$$

and this may be done because  $X$  is complete with respect to  $D$ .

The estimate implies that  $\|r(D + E + F)\| < \varepsilon$ . But then

$$r(D + E) - r(D + E + F) = r(D + E + F) \cdot \gamma f \cdot r(D + E),$$

and by Proposition 3.3 the right hand side is compact. □

**3.6. REMARK.** Obviously the hypotheses can be relaxed in various ways. But they are adequate for our purposes as they stand.

#### 4. Deformation Spaces

This section describes the deformation construction that underlies both the tangent groupoid and the proof of the index theorem.

**4.1. DEFINITION.** Let  $M$  be a smooth, closed submanifold of a smooth manifold  $V$  without boundary. The *deformation space*  $\mathbb{N}_V M$  associated to the inclusion of  $M$  into  $V$  is the set

$$\mathbb{N}_V M = N_V M \sqcup V \times (0, 1],$$

where  $N_V M$  is the normal bundle of  $M$  in  $V$ .

**4.2. REMARK.** The deformation space concept is taken from algebraic geometry, where its counterpart is called the *deformation to the normal cone*. See [BFM75] or [Fu98, Chapter 5].

We equip  $\mathbb{N}_V M$  with the weakest topology (i.e. the one with the fewest open sets) such that:

- (a) The natural map  $\mathbb{N}_V M \rightarrow V \times [0, 1]$  that on  $V \times (0, 1]$  is the inclusion map, while on  $\mathbb{N}_V M$  is the projection to  $M$ , followed by inclusion into  $V \times \{0\}$ , is continuous.
- (b) If  $f: V \rightarrow \mathbb{R}$  is a smooth function that vanishes on  $M$ , then the function

$$\delta f: \mathbb{N}_V M \longrightarrow \mathbb{R}$$

defined by the formulas

$$\delta f(X) = X(f) \quad \text{and} \quad \delta f(v, t) = \frac{f(v)}{t}$$

is continuous.

This topology is Hausdorff and also locally Euclidean:

4.3. LEMMA. *If  $x_1, \dots, x_p, y_1, \dots, y_q$  are smooth local coordinates on  $V$  such that the  $x_i$  restrict to local coordinates on  $M$ , whereas the  $y_j$  vanish on  $M$ , then the functions*

$$x_1, \dots, x_p, \delta y_1, \dots, \delta y_q, t$$

*are local coordinates on the deformation space: they determine a homeomorphism from the open subset where they are defined to an open subset of  $\mathbb{R}^{p+q} \times [0, 1]$ .*

PROOF. Let  $U$  be the open subset of  $V$  on which the coordinates are defined. The would-be coordinate functions are then defined on  $\mathbb{N}_U(M \cap U)$ , which by the item (a) above is an open subset. (Incidentally, by  $x_j$  and  $t$  we mean the functions obtained by first applying the map in item (a), then composing with  $x_j$  and  $t$ .)

The associated map

$$\mathbb{N}_U(M \cap U) \rightarrow \mathbb{R}^p \times \mathbb{R}^q \times [0, 1]$$

is continuous and one-to-one, and has open image. Viewing  $U$  as an open subset of  $\mathbb{R}^p \times \mathbb{R}^q$  via the given coordinates, and identifying the normal bundle with  $(M \cap U) \times \mathbb{R}^q$ , the inverse map is given by the formula

$$(u, v, t) \mapsto \begin{cases} (u, u + tv, t) & t \neq 0 \\ (u, v, 0) & t = 0. \end{cases}$$

To check its continuity one verifies that its compositions with the maps in (a) and (b) above are continuous.  $\square$

The coordinate charts given by the lemma in fact constitute an atlas for a smooth manifold structure on  $\mathbb{N}_V M$ . That smooth structure can be described a bit more invariantly as follows.

4.4. DEFINITION. Let  $X$  be a set and let  $\mathcal{F} = \{f_\alpha: X \rightarrow V_\alpha\}$  be a family of functions from  $X$  into smooth manifolds. Let us say that a function  $f: X \rightarrow \mathbb{R}$  is *smoothly composed from the family  $\mathcal{F}$*  if it has the form

$$X \xrightarrow{(f_{\alpha_1}, \dots, f_{\alpha_k})} V_{\alpha_1} \times \dots \times V_{\alpha_k} \xrightarrow{h} \mathbb{R},$$

where  $h$  is a smooth function on the product manifold.

4.5. PROPOSITION. *There is a unique smooth manifold structure on the deformation space  $\mathbb{N}_V M$  for which the smooth real-valued functions on  $\mathbb{N}_V M$  are precisely the functions that are locally smoothly composed from the functions in (a) and (b) above.*

PROOF. It suffices to show that if  $x_1, \dots, x_p, y_1, \dots, y_q$  are local coordinates, as in the previous lemma, then every function on  $\mathbb{N}_V M$  that is smoothly composed from the functions in (a) and (b) is locally smoothly composed from the functions

$$x_1, \dots, x_p, \delta y_1, \dots, \delta y_q, t.$$

In turn, it suffices to prove that every  $\delta f$  is so composed, and this is an immediate consequence of Taylor's theorem.  $\square$

4.6. EXAMPLE. Suppose that  $M$  is embedded in a smooth vector bundle  $W$  over  $M$  via a section  $s: M \rightarrow W$ . Each vector  $w \in W$  determines a tangent vector at each point in the fiber containing  $W$ , and in this way the normal bundle  $N_W M$  identifies with  $W$  itself. The product manifold  $W \times [0, 1]$  is mapped diffeomorphically onto the deformation space  $\mathbb{N}_W M$  by the formulas

$$(w, 0) \mapsto w \in N_W M \quad \text{and} \quad (m, w, t) \mapsto (m, s(m) + tw, t) \in W \times (0, 1].$$

### 5. Spinor Bundles, the Thom Class and the Dirac Operator

Let  $V$  be a smooth, oriented euclidean vector bundle of rank  $2k$  over a smooth manifold  $M$ . A *spinor bundle* for  $V$  is a smooth  $\mathbb{Z}_2$ -graded Hermitian vector bundle  $SV$  over  $M$  that is equipped with an  $\mathbb{R}$ -linear smooth bundle map

$$c: V \longrightarrow \text{End}(SV)$$

such that:

- (a) Each  $c(v)$  is odd-graded, skew-adjoint, and satisfies

$$c(v)^2 = -\|v\|^2 1.$$

- (b) The grading operator on  $SV$  is given by the local formula

$$\gamma = i^k v_1 \cdots v_{2k},$$

involving any oriented local orthonormal frame of  $V$ .

- (c) The map  $c$  induces an isomorphism

$$c: \text{Cliff}(V) \xrightarrow{\cong} \text{End}(SV),$$

where  $\text{Cliff}(V)$  denotes the bundle of complex Clifford algebras associated to  $V$ .

See [BD82] or [LM89], for example.

If  $M$  reduces to a single point, so that  $V$  is a single Euclidean vector space, then a spinor bundle for  $V$ , which is better named a *spinor vector space*  $S$  for  $V$ , exists and is unique up to isomorphism.

5.1. DEFINITION. Let  $V$  be an oriented, even-rank Euclidean vector bundle over a compact manifold  $M$  and let  $SV$  be a spinor bundle for  $V$ . View the pullback of  $SV$  to the total space of  $V$  as a continuous field of (finite-dimensional)  $\mathbb{Z}_2$ -graded Hilbert spaces over the manifold  $V$ . The *Thom class*  $\beta(SV) \in K(V)$  is the index class, in the sense of Section 2, of the family of operators

$$\{E_v: S_{\pi(v)} V \rightarrow S_{\pi(v)} V\}_{v \in V}$$

where  $E_v = c(v)\gamma$  is the *self-adjoint Clifford multiplication* operator associated to  $v \in V$  ( $\gamma$  is the grading operator on  $SV$ ).



5.2. REMARK. If  $V$  carries a Hermitian structure, then we can set  $SV = \bigwedge_{\mathbb{C}}^* V$  and

$$c(v) = d(v) - d(v)^*: S_{\pi(v)}V \longrightarrow S_{\pi(v)}V,$$

where  $d(v)(\omega) = v \wedge \omega$ . The family  $\{c(v)\gamma\}$  is homotopic to the family  $\{d(v) + d(v)^*\}$ , which agrees with the standard definition of Thom class.

There is a useful two-out-of-three principle for spinor bundles. We shall use it when we formulate the index theorem below.

5.3. LEMMA. *Let  $V'$  and  $V''$  be oriented, even-rank euclidean vector bundles over  $M$ , and let  $V = V' \oplus V''$ .*

(a) *If  $SV'$  and  $SV''$  are spinor bundles for  $V'$  and  $V''$ , then  $SV' \hat{\otimes} SV''$ , equipped with the action*

$$c(v', v'') = c(v') \hat{\otimes} I + I \hat{\otimes} c(v'')$$

*is a spinor bundle for  $V$ .*

(b) *If  $SV$  is a spinor bundle for  $V$ , and if  $SV'$  is a spinor bundle for  $V'$ , then there exists a spinor bundle  $SV''$  for  $V''$  such that  $SV' \hat{\otimes} SV''$ , viewed as a spinor bundle for  $V = V' \oplus V''$ , is isomorphic to  $SV$ .*

PROOF. Part (a) is straightforward. As for (b), one can take

$$S'V = \text{Hom}_{V''}(SV, SV''),$$

the  $\mathbb{Z}_2$ -graded Hermitian vector bundle of morphisms  $SV \rightarrow SV''$  that commute with the actions of  $V''$  on  $SV$  and  $SV''$ .  $\square$

We shall also use the following concepts of *conjugate* spinor bundle and *opposite* spinor bundle (these terms are introduced for our current purposes only and are not standard).

5.4. DEFINITION. Let  $SV$  be a spinor bundle for a Euclidean vector bundle  $V$  of rank  $2k$ , and denote by  $\bar{S}V$  the complex conjugate of the underlying Hermitian bundle. Since  $\text{End}(SV) = \text{End}(\bar{S}V)$ , we may equip  $\bar{S}V$  with the same map

$$c: V \rightarrow \text{End}(\bar{S}V)$$

with which  $SV$  is equipped. We shall also equip  $\bar{S}V$  with the same grading as  $SV$  if  $k$  is even, and the opposite grading if  $k$  is odd. We obtain a spinor bundle for  $V$ , called the *conjugate* spinor bundle.

5.5. DEFINITION. Suppose that  $V'$  and  $V''$  are oriented, even-rank euclidean vector bundles over  $M$  and that the direct sum

$$V = V' \oplus V''$$

is a trivial bundle. Let  $SV'$  be a spinor bundle for  $V'$ . A spinor bundle  $SV''$  for  $V''$  is *opposite* to the spinor bundle  $SV'$  if the tensor product

$$SV = \bar{S}V' \hat{\otimes} SV''$$

is isomorphic to the trivial spinor bundle for  $V$ .

5.6. REMARK. Lemma 5.3 guarantees that opposite spinor bundles exist.

Now let  $M$  be an oriented Riemannian manifold without boundary and let  $SM$  be a spinor bundle for the tangent bundle. A *Dirac operator* for  $SM$  is a first-order, formally self-adjoint, odd-graded linear partial differential operator

$$D: C_c^\infty(M, SM) \longrightarrow C_c^\infty(M, SM)$$

such that

$$[D, f]u = c(\text{grad } f)u.$$

for every smooth function  $f$  on  $M$  (viewed on the left as a multiplication operator on sections of  $SM$ ). Every Dirac operator  $D$  is elliptic and so if  $M$  is closed, then (the self-adjoint extension of)  $D$  is Fredholm.

5.7. EXAMPLE. If  $M$  is a Hermitian complex manifold, then the Dolbeault operator  $D = \bar{\partial} + \bar{\partial}^*$  (see [AS68b, §4]) is a Dirac operator in the above sense. The manifold need not be Kähler.

The K-theory formulation of the index theorem uses an embedding of  $M$  into a Euclidean space. Let us assume for simplicity that  $M$  is embedded into a Euclidean vector space  $V$  *isometrically*. We can always adjust the metric on  $M$  and spinor bundle so that this is so, without altering the index of the Dirac operator, so this is no real restriction.<sup>3</sup>

5.8. THEOREM (Atiyah and Singer). *Let  $M$  be a closed submanifold of dimension  $2k$  in an oriented, even-dimensional Euclidean space  $V$  and let  $S$  be a spinor vector space for  $V$ . If  $D$  is a Dirac operator on  $M$ , acting on sections of a spinor bundle  $SM$ , and if the normal bundle  $N_V M$  is equipped with a spinor bundle  $SN$  opposite to  $SM$ , then*

$$\text{Index}(D) \cdot \beta(S) = (-1)^k \iota_* (\beta(SN)) \in K(V),$$

where  $\iota_*: K(N_V M) \rightarrow K(V)$  is the map induced by a tubular neighborhood inclusion of  $N_V M$  into  $V$ .

5.9. REMARK. As it stands, the theorem is an identity in  $K(V)$ . An application of the Chern character yields the formula

$$\text{Index}(D) = (-1)^{l-k} \int_{N_V M} \text{ch}(\beta(SN)),$$

where  $\dim(V) = 2l$ , since  $\int_V \text{ch}(\beta(S)) = (-1)^l$ . Characteristic class computations then give the more familiar formula

$$\text{Index}(D) = \int_M e^{\frac{1}{2}c_1(L(SM))} \hat{A}(TM),$$

where  $L(SM) = \text{Hom}_V(\bar{S}M, SM)$  is the *canonical line bundle* of maps from  $\bar{S}M$  to  $SM$  commuting with the action of  $TM$ . Compare [AS68b] or [LM89, Appendix D].

## 6. Proof of the Index Theorem

Let  $M$  be a smooth closed manifold that is embedded as a submanifold of a finite-dimensional real vector space  $V$ . Define a map

$$p: M \times V \times [0, 1] \longrightarrow N_V M$$

where  $N_V M$  is the deformation space of Section 4, by the formulas

$$p(m, v, 0) = p_m(v),$$

<sup>3</sup>Even without appealing the existence of isometric embeddings.

where  $p_m$  is the projection from  $V$  onto  $V/T_m M$  (which is the fiber of the normal bundle  $N_V M$  over the point  $m \in M$ ), and, if  $t \neq 0$ ,

$$p(m, v, t) = (m + tv, t).$$

6.1. LEMMA. *The map  $p$  is a submersion.* □

6.2. REMARK. If we think of  $M \times V$  as a trivial vector bundle over  $M$ , and the diagonal embedding of  $M$  into  $M \times V$  as a section, then the diffeomorphism from  $M \times V \times [0, 1]$  to  $\mathbb{N}_{M \times V} M$ , given in Example 4.6, identifies the submersion  $p$  with the map from  $\mathbb{N}_{M \times V} M$  onto  $\mathbb{N}_V M$  that is induced from the projection of  $M \times V$  onto  $V$ .

Suppose now that  $SM$  is a Hermitian bundle on  $M$  (soon to be a spinor bundle) and that  $D$  is a first-order linear partial differential operator acting on the sections of  $SM$  (soon to be a Dirac operator, but for the moment not necessarily even elliptic).

Pull back the bundle  $SM$  to  $M \times V \times [0, 1]$ . Define a smooth family of operators on the fibers of  $p$ , acting on the sections of this pullback bundle, as follows.

(a) If  $(v, t) \in V \times (0, 1] \subseteq \mathbb{N}_V M$ , then the fiber of  $p$  over  $(v, t)$  is the manifold

$$\{(m, t^{-1}(v - m), t) : m \in M\} \subseteq M \times V \times [0, 1]$$

and so the coordinate projection onto  $M$  identifies the fiber with the manifold  $M$ . We define  $D_{(v,t)} = tD$ .

(b) If  $(m, v) \in N_V M \subseteq \mathbb{N}_V M$ , then the fiber of the map  $p$  over  $X$  is the manifold

$$\{m\} \times T_m M \times \{0\} \subseteq M \times V \times [0, 1].$$

Let  $D_m$  be the *model operator* on  $T_m M$ , obtained from  $D$  by freezing the coefficients at  $m$  and dropping order zero terms. We define  $D_{(m,v)} = -D_m$ .

6.3. LEMMA. *The operators above form a smooth family of elliptic operators. Moreover the manifold  $M \times V \times [0, 1]$  is complete with respect to this family.*

PROOF. If we embed the bundle  $S$  over  $M$  as a summand of a trivial bundle, then we can reduce the lemma to the case where  $S$  is trivial, in which case the original operator  $D$  is a system of operators on scalar functions. This allows us to further reduce to the cases where  $D$  is either a vector field or multiplication by a function  $f$  on  $M$ . In the latter case the family is multiplication by the smooth function  $(m, v, t) \mapsto tf(m)$  on  $M \times V \times [0, 1]$ . In the former case, if  $X$  is a vector field on  $M$ , then the associated family of operators is given by the smooth vector field

$$X_{(m,v,t)} = (tX_m, -X_m, 0)$$

on  $M \times V \times [0, 1]$ , where we identify the tangent space of the product manifold at  $(m, v, t)$  with  $T_m M \times V \times \mathbb{R}$  and we consider  $T_m M$  as a subspace of  $V$  via the given embedding of  $M$ .

As for completeness, if  $g: V \rightarrow [0, \infty)$  is any smooth proper function, then its composition with the second coordinate projection on  $M \times V \times [0, 1]$  is a smooth proper function on the product manifold whose commutator with  $D$  is uniformly bounded. □

Assume now that  $V$  is even-dimensional, oriented and Euclidean. Assume that  $M$  has dimension  $2k$ , that it is oriented, and that it is equipped with the Riemannian metric it inherits as a submanifold of  $V$ . Let  $SM$  be a spinor bundle for  $TM$  and  $D$  be a Dirac operator for  $SM$ .

Fix a spinor space  $S$  for  $V$  and let  $E: V \rightarrow \text{End}(S)$  be self-adjoint Clifford multiplication, as in Definition 5.1. Consider  $E$  as a function

$$E: M \times V \times [0, 1] \longrightarrow \text{End}(S)$$

via the coordinate projection onto  $V$ .

Form the tensor product  $SM \hat{\otimes} S$  with fibers  $S_m M \hat{\otimes} S$ . Form the operator  $D \hat{\otimes} I$  on sections of  $SM \hat{\otimes} S$  over  $M \times V \times [0, 1]$ , and also the self-adjoint endomorphism  $I \hat{\otimes} E$ .

6.4. LEMMA. *The anticommutator of  $D \hat{\otimes} I$  and  $I \hat{\otimes} E$  is a uniformly bounded endomorphism of  $SM \hat{\otimes} S$ , while  $(I \hat{\otimes} E)^2$  is a proper function on  $M \times V \times [0, 1]$ .*

PROOF. The square of  $I \hat{\otimes} E$  is the scalar function  $\|v\|^2$ , which is certainly a proper function on  $M \times V \times [0, 1]$ . The anticommutator of  $D \hat{\otimes} I$  and  $I \hat{\otimes} E$  is the same as the commutator of  $D \otimes I$  and  $I \otimes E$  on the sections of  $SM \otimes S$ .  $\square$

According to Proposition 3.5 there is therefore an index class

$$\text{Index}(D \hat{\otimes} I + I \hat{\otimes} E) \in K(\mathbb{N}_V M).$$

We shall prove the index theorem by computing the restriction of this index class to the closed subsets  $N_V M$  and  $V$  of  $\mathbb{N}_V M$ , where the latter is embedded as  $V \times \{1\}$ . We shall calculate that

$$\text{Index}(D \hat{\otimes} I + I \hat{\otimes} E)|_{N_V M} = (-1)^k \beta(N_V M) \in K(N_V M)$$

and

$$\text{Index}(D \hat{\otimes} I + I \hat{\otimes} E)|_V = \text{Index}(D) \cdot \beta(V) \in K(V),$$

and then the index formula will follow from the following calculation.

6.5. LEMMA. *Let  $\iota: N_V M \rightarrow V$  be a tubular neighborhood embedding associated to the embedding of  $M$  into  $V$  as a closed submanifold. The diagram*

$$\begin{array}{ccc} K(\mathbb{N}_V M) & \xlongequal{\quad} & K(\mathbb{N}_V M) \\ \downarrow & & \downarrow \\ K(N_V M) & \xrightarrow{\quad \iota_* \quad} & K(V), \end{array}$$

in which the vertical maps are given by the inclusions of  $N_V M$  and  $V$  into  $\mathbb{N}_V M$ , is commutative.

PROOF. Since  $\mathbb{N}_V M = N_V M \sqcup V \times (0, 1]$ , and since  $V \times (0, 1]$  is contractible in the sense of locally compact spaces, the inclusion of  $N_V M$  into  $\mathbb{N}_V M$  as a closed subset induces an isomorphism

$$K(\mathbb{N}_V M) \xrightarrow{\cong} K(N_V M)$$

by restriction of  $K$ -theory classes. It follows that the open inclusion of the tubular neighborhood  $W = \iota(N_V M)$  into  $V$  induces an isomorphism

$$K(\mathbb{N}_W M) \xrightarrow{\cong} K(N_V M).$$

The lemma therefore reduces to the case where  $V = W$ , and now the calculation in Example 4.6, plus the homotopy invariance of K-theory, completes the proof.  $\square$

We shall now calculate  $\text{Index}(D \hat{\otimes} I + I \hat{\otimes} E)|_{N_V M}$ . Recall from Definition 5.5 that the opposite spinor bundle  $SN$  for the normal bundle  $N_V M$  is defined so that there is an isomorphism of spinor bundles

$$M \times S \cong \bar{S}M \hat{\otimes} SN$$

for the trivial bundle  $M \times V$ . As a result there is an isomorphism

$$SM \hat{\otimes} S \cong (SM \hat{\otimes} \bar{S}M) \hat{\otimes} SN.$$

The continuous field of Hilbert spaces on which  $(D \hat{\otimes} I + I \hat{\otimes} E)|_{N_V M}$  acts can therefore be written as the field with fiber

$$L^2(T_m M, S_m M) \hat{\otimes} S \cong L^2(T_m M, S_m M \hat{\otimes} \bar{S}_m M) \hat{\otimes} S_m N$$

over  $(m, v) \in N_V M$ . If we define an operator  $B_m$  on the first tensor factor on the right hand side by

$$B_m = -D_m \hat{\otimes} I + I \hat{\otimes} E_m,$$

where the function  $E_m$  on  $T_m M$  is self-adjoint Clifford multiplication, and if we denote by  $E_v$  self-adjoint Clifford multiplication by a normal vector  $v$  on  $S_m N$ , then

$$(D \hat{\otimes} I + I \hat{\otimes} E)_{(m, v)} \cong B_m \hat{\otimes} I + I \hat{\otimes} E_v$$

(one should be aware that the descriptions on the left and right use different tensor product decompositions).

We shall now compute the index of the family on the right hand side. The first step is the following lemma, in which we shall use the canonical isomorphisms

$$S_m M \hat{\otimes} \bar{S}_m M \cong \text{End}(S_m M) \cong \text{Cliff}(T_m M),$$

so as to view  $B_m$  as an operator on  $L^2(T_m M, \text{Cliff}(T_m M))$ . Note that according to our conventions, the first isomorphism in the display is grading-preserving if  $k$  is even and grading-reversing if  $k$  is odd.

6.6. LEMMA. *The kernel of (the closure of)  $B_m$  is spanned by the function*

$$v \mapsto \exp(-\frac{1}{2}\|v\|^2)I \in \text{Cliff}(T_m M).$$

*On the orthogonal complement of the kernel,  $B_m^2$  is bounded below by 2.*

PROOF. We compute that

$$B_m^2 = \Delta + \|v\|^2 + (N - 2k)$$

where  $\Delta$  is the Laplace operator and  $N$  is the *number operator* that acts as  $pI$  on all monomials  $e_{i_1} \cdots e_{i_p}$ . The lemma therefore follows from the well-known eigenvalue theory of the quantum harmonic oscillator  $\Delta + \|v\|^2$ . See for example [GJ87, p. 12].  $\square$

Now form the one-dimensional continuous field of Hilbert spaces

$$\mathcal{K}_m = \ker(B_m) \subseteq L^2(T_m M, S_m M \hat{\otimes} \bar{S}_m M).$$

It is purely even-graded if  $k$  is even, and purely odd-graded if  $k$  is odd. The section given in the lemma trivializes  $\mathcal{K}$ , and as a result

$$\begin{aligned} L^2(T_m M, S_m M \hat{\otimes} \bar{S}_m M) \hat{\otimes} S_m N &\cong \mathcal{K}_m \hat{\otimes} S_m N \oplus \mathcal{K}_m^\perp \hat{\otimes} S_m N \\ &\cong S_m N \oplus \mathcal{K}_m^\perp \hat{\otimes} S_m N, \end{aligned}$$

where the isomorphism between the first summands is grading-preserving or grading-reversing, according as  $k$  is even or odd. In the final direct sum decomposition the operator  $B_m \hat{\otimes} I + I \hat{\otimes} E_v$  acts as the self-adjoint Clifford multiplication operator  $E_v$  on  $S_m N$  and as an invertible operator on  $\mathcal{K}_m^\perp \hat{\otimes} S_m N$ , since

$$(B_m \hat{\otimes} I + I \hat{\otimes} E_v)^2 = B_m^2 \hat{\otimes} I + I \hat{\otimes} E_v^2,$$

while  $B_m^2 \geq 2$  on  $\mathcal{K}_m^\perp$ . Using the additivity of the index, together with the triviality of the index of the second summand, we find that

$$\begin{aligned} \text{Index}(D \hat{\otimes} I + I \hat{\otimes} E) \Big|_{N_\vee M} &= \text{Index}(B \hat{\otimes} I + I \hat{\otimes} E) \\ &= (-1)^k \beta(SN) \in K(N_\vee M), \end{aligned}$$

as required.

It remains to compute  $\text{Index}(D \hat{\otimes} I + I \hat{\otimes} E)|_V$ . This is quite simple. The map

$$\begin{aligned} q: M \times V \times [0, 1] &\longrightarrow V \times [0, 1] \\ q: (m, v, t) &\mapsto (tm + v, t) \end{aligned}$$

is a submersion, and every fiber

$$q^{-1}\{(v, t)\} = \{(m, v - tm, t) : m \in M\} \subseteq M \times V \times [0, 1]$$

is isomorphic to  $M$  via the projection to  $M$ . Construct the smooth family  $D$  that is the Dirac operator on each fiber, and then form the family

$$D \hat{\otimes} I + I \hat{\otimes} E,$$

acting on sections of  $SM \hat{\otimes} S$  by using the same self-adjoint Clifford multiplication endomorphism as  $E$  as before. We are re-using notation, but this is not especially reckless because the restriction to  $V \cong V \times \{1\}$  of the new family is identical to the same restriction of the old one. However the restriction to  $V \cong V \times \{0\}$  of the new family, which has the same index as the restriction to  $V \times \{1\}$  by homotopy invariance of  $K$ -theory, is the family of operators

$$D \hat{\otimes} I + I \hat{\otimes} E_v: L^2(M, SM) \hat{\otimes} S \longrightarrow L^2(M, SM) \hat{\otimes} S$$

Decompose  $L^2(M, SM)$  into the kernel of  $D$ , direct sum its orthogonal complement, and decompose  $L^2(M, SM) \hat{\otimes} S$  accordingly. On the second summand the above operators are uniformly bounded below by the first positive eigenvalue in the spectrum of  $D$ . On the first summand the operators are

$$I \hat{\otimes} E_v: \ker(D) \hat{\otimes} S \longrightarrow \ker(D) \hat{\otimes} S$$

Taking into account the grading on  $\ker(D)$  we find that

$$\text{Index}(D \hat{\otimes} I + I \hat{\otimes} E) \Big|_V = \text{Index}(D) \cdot \beta(S),$$

as required.

### 7. Some Remarks on the Tangent Groupoid

The proof of the index theorem given in Section 6 generalizes in a number of simple ways. For instance we can introduce a coefficient vector bundle  $F$  on  $M$ , and if  $D_F$  is a Dirac-type operator acting on  $F \otimes SM$ , then we find that

$$\text{Index}(D_F) \cdot \beta(S) = (-1)^k \iota_*([F] \cdot \beta(SN)) \in K(V).$$

In addition, since the proof deals with families anyway, it extends directly to a proof of the Atiyah-Singer index theorem for families of Dirac-type operators.

Other cases can be handled too, but after a certain point it becomes more conceptual and otherwise more appropriate to invest in groupoid theory.

The tangent groupoid  $\mathbb{T}M$  is the deformation space associated to the diagonal embedding of  $M$  into  $M \times M$ . This embedding is the inclusion of the units into the pair groupoid of  $M$ , and same construction, when applied to the inclusion of the unit space into other groupoids, produces other tangent groupoids. For example when applied to the foliation groupoid of a foliated manifold it produces the leafwise tangent groupoid of the foliation.

But let us focus on the classical index theorem and indicate how the proof of the index theorem presented in Section 6 is drawn from Connes' argument in [Con94, Section II.5]. Assuming  $M$  is embedded in a vector space  $V$ , Connes constructs a homomorphism from the groupoid  $\mathbb{T}M$  into  $V$ , and hence an action of  $\mathbb{T}M$  on the manifold  $V$  by translations [Con94, p. 105]. Associated to this there is a crossed product groupoid  $\mathbb{T}M \rtimes V$ , and in fact a family of crossed product groupoids, since the translations can be scaled by  $s \in [0, 1]$ .

When  $s = 0$  the translation action is trivial and the groupoid  $\mathbb{T}M \rtimes V$  is the tangent groupoid  $\mathbb{T}M$  times the parameter space  $V$ . It exhibits an elliptic operator as a deformation of its symbol, more or less as indicated in the introduction.

Connes' key observation is that when  $s = 1$  the crossed product  $\mathbb{T}M \rtimes V$  is Morita equivalent to a *space*, that is, to a groupoid comprised entirely of units. The space in question is  $\mathbb{N}_V M$  and in fact if we set  $X = M \times V \times [0, 1]$ , then

$$\mathbb{T}M \rtimes V \cong \{ (x_1, x_2) \in X \times X : p(x_1) = p(x_2) \},$$

where  $p$  is the submersion from  $X$  onto  $\mathbb{N}_V M$  defined in Section 6. The proof we presented is based on the fact that if  $D$  is a Dirac operator, then the symbol-to-operator deformation at  $s = 0$  that is encoded by the tangent groupoid deforms as  $s$  varies from 0 to 1, to the family that we studied in Section 6.

For general operators the analysis of the deformation as  $s$  varies is more difficult, and a Bott periodicity argument is required, as in the final part of Connes' proof [Con94, p. 106].

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