# Algebraic $K$-Theory of Stable $C^{*}$-Algebras 

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#### Abstract

Let $A$ be a unital $C^{*}$-algebra and let 2 denote the Calkin algebra (the bounded operators on a separable Hilbert space, modulo the compact operators $\mathscr{K}$ ). We prove the following conjecture of M . Karoubi: the algebraic and topological $K$-theory groups of the tensor product $C^{*}$-algebra $A \otimes 2$ are equal. The algebra $A \otimes \not 2$ may be regarded as a "suspension" of the more elementary $C^{*}$-algebra $A \otimes \mathscr{K}$; thus Karoubi's conjecture asserts, roughly speaking, that the algebraic and topological $K$-theories of stable $C^{*}$-algebras agree. 1988 Academic Press, Inc.


## Introduction

Let $A$ be a $C^{*}$-algebra and suppose, for the sake of simplicity, that $A$ is unital. The general linear group of $A$ of dimension $n$, denoted $G L_{n} A$, is the group of invertible elements in the $n \times n$ matrix algebra $M_{n}(A)$ over $A$. This paper is about comparing invariants of $G L_{n} A$, considered as a topological space (with the norm topology, inherited from $A$ ), to invariants of $G L_{n} A$ considered as a discrete group. On the topological side, we are going to study the topological $K$-theory of $A$, denoted $K_{*}^{\prime}(A)$, which is nothing more than the homotopy, $\pi_{*}\left(G L_{n} A\right)$, of $G L_{n} A$. To be precise, $K_{*}^{\prime}(A)$ is the homotopy of the $G L_{\alpha} A$, the "limit" as $n \rightarrow \infty$ of the $G L_{n} A$ : it turns out to be a great conveneince to study this "stable" group, rather that the nonstable groups $G L_{n}$. This has been the object of quite intense scrutiny by operator algebraists in recent years. The result of this attention has been the development of powerful techniques to compute $K_{*}^{t}(A)$ for a great variety of $C^{*}$-algebras $A$, and numerous applications, both to the theory of operator algebras, and perhaps more importantly and more significantly, to various other disciplines, notably differential topology. The algebraic invariant of $G L_{n} A$ is called the algebraic $K$-theory of $A$, denoted $K_{*}(A)$. It can be defined for any ring, and from our point of view, it is an analogue of the homotopy of $G L_{n} A$ in a purely algebraic context. (Again, to be precise, we consider $G L_{\infty} A$ rather than $G L_{n} A$.) For instance, in the topological setting, $\pi_{0}\left(G L_{\infty} A\right)=G L_{\infty} A / G L_{x}^{0} A$, where $G L_{\infty}^{0} A$ denotes the connected component of the identity; in the algebraic setting, the corresponding group is $G L_{\infty} A / P G L_{\infty} A$, where $P G L_{r} A$ denotes the maximal perfect sub-
group, which plays the role of the connected component of the identity. Next is the fundamental group $\pi_{1}$ : in the topological case this is obtained from the universal covering group of $G L_{\infty}^{0} A$; we obtain the algebraic group from the analog of this- the universal central extension of $P G L_{\infty} A$.

Stated informally, the main theorem of this paper is as follows. (The statement is imprecise due to the fact that we will consider not stable $C^{*}$-algebras, but "suspensions" of stable $C^{*}$-algebras. We do not need to go into the details of this immediately.)

If $A$ is a stable $C^{*}$-algebra then the algebraic $K$-theory of $A$ is equal to the topological $K$-theory of $A$.

This is, we think, an interesting result for the following reasons. First, the algebraic $K$-theory of a ring is in general quite inaccessible. For example, the algebraic $K$-theory of say the integers $\mathbf{Z}$ is not yet known (although the first several groups $K_{n}(\mathbf{Z})$ are $)$. Again, the groups $K_{*}(\mathbf{C})$ have only recently been determined, as a result of very deep computations. Yet for a stable $C^{*}$-algebra, the algebraic $K$-theory is the same as the relatively accesible topological $K$-theory. The second, and we think more interesting point regards the quite distinct natures of topological and algebraic $K$-theory. Topological $K$-theory is constructed by considering $G L_{n} A$ solely as a topological space; on the other hand, algebraic $K$-theory is constructed from $G L_{n} A$, considering it solely as a discrete group, without regard to topology at all. Yet these two different approaches lead to exactly the same group, in the case of a stable $C^{*}$-algebra.

This last point brings us to an interesting parallel with the Brown Douglas Fillmore theory of extensions of $C^{*}$-algebras. Since this is in many ways the foundation of our work, we want to spend a few lines now acquainting the reader with the broad outlines of it. An essentially normal operator $N$ on a Hilbert space is an operator for which the self-commutator $\left[N, N^{*}\right.$ ] is compact (as opposed to zero, in which case $N$ would be normal). It is quite clear from the definition that every compact perturbation of a normal operator is essentially normal, and the question arises: does this exhaust the class? It is not hard to see that the answer is "no." For example, the unilateral shift is essentially normal but not of this form. Two operators $N_{1}, N_{2}$ are essentially unitarily equivalent if there exists a unitary $U$ such that $U N_{1} U^{*}-N_{2}$ is compact. An obvious invariant of this equivalence relation is the essential spectrum $\sigma_{e}(N)$, that is, the spectrum of the image of $N$ in the quotient $C^{*}$-algebra $\mathscr{B} / \mathscr{K}$, and the natural question to ask is: what are the possibilities (up to essential unitary equivalence) for an essentialy normal operator with given essential spectrum $X$ ? It turns out that they are classified by elements of an abelian group denoted $\operatorname{Ext}^{-1}(C(X))$. The construction of $\operatorname{Ext}^{-1}(C(X))$ and the fact that it is a group is in itself remarkable, but it is the even more remarkable description of $\operatorname{Ext}^{-1}(C(X))$ given by Brown, Douglas, and Fillmore that we are
interested in. Recall for a moment the classification of normal operators up to unitary equivalence on a separable Hilbert space. A complete list of invariants is: the spectrum $X$ of the operator, an equivalence class of measures on $X$, and a "multiplicity function" $X \rightarrow\{1,2, \ldots, \infty\}$. (If the multiplicity function is constantly 1 then the operator is simply multiplication by $x$ on $L^{2}(X)$; in general it is a direct sum of pieces of this, as dictated by the multiplicity function.) In contrast, if $N$ is essentially normal then the essential unitary equivalence class of $N$ is the set of all essentially normal operators $N^{\prime}$ with the same essential spectrum $X$ as $N$, for which

$$
\operatorname{index}(\lambda-N)=\operatorname{index}\left(\lambda-N^{\prime}\right)
$$

for every complex number $\lambda$ in the complement of $X$. Thus $N$ is characterized by $X$ and the "multiplicity function" $\lambda \mapsto \operatorname{index}(\lambda-N)$, defined on the complement of $X$. This is a result of the following beautiful fact:

## Ext ${ }^{1}(C(X))$ is equal to the (odd-dimensional) K-homology of $X$.

It is not important for us to describe exactly what the $K$-homology of a space $X$ is. The point we want to make is that the purely algebraically defined group Ext ${ }^{-1}(C(X))$ turns out to be completely topological in character. This is a very intriguing and remarkable phenomenon, and the results of this paper are offered as another illustration of it.

Besides the theorem mentioned, we present a number of other results, mostly on the same theme of comparing algebraic and topological $K$-theory, but occasionally as minor digressions from it. Most sections begin with a brief summary of their contents; however, let us give here an outline of the contents of the work as a whole.

## Section I

Almost the whole of the paper relies in a very crucial way on the technical underpinings of extension theory, as developed in its general form by Kasparov. These are results on the structure of multiplier algebras and the outer multiplier algebras, or as we shall call them, "generalized Calkin algebras" $\mathscr{M}(\mathscr{K} \otimes B) / \mathscr{K} \otimes B$. The two main results are a separation theorem of Kasparov, concerning orthogonal subalgebras of a Calkin algebra (Theorem 1.1.11), and a type of stabilization theorem, which compares the multiplier algebras $\mathscr{U}(\mathscr{K} \otimes B)$ and $\mathscr{H}(\mathscr{K} \otimes J)$, where $J$ is an ideal of $B$ (Theorem 1.3.14). Various other $C^{*}$-algebra preliminaries are also given.

## Section II

There are basically three topics covered. The first is the introduction of topological $K$-theory, about which we need say nothing here. The second is
the introduction of algebraic $K$-theory. The definition of the higher algebraic $K$-theory groups is due to Quillen; it is a beautiful illustration of the interplay possible between algebra and algebraic topology. A certain amount of familiarity with topology is necessary to work with it, and in an attempt to make the paper accessible to non-topologists we have included most of the background needed. The third topic of the section is the extension theory of $C^{*}$-algebras. We have included it partly because of the close parallels between our results, as we have described; partly because extension theory provides a good illustration of some of the techniques we develop; and partly because, by means of these techniques, we are able to contribute a little to the simplification of the subject.

## Section III

The main topic of the section is a homotopy invariance theorem, proved in a general context. The techniques in the proof are for the most part borrowed from Kasparov's treatment of the homotopy invariance of the extension groups. However, we use ideas due to Cuntz to put Kasparov's work in an abstract setting, and the result is quite surprising: any functor from $C^{*}$-algebras to abelian groups which is "matrix stable" and which preserves split exact sequences is homotopy invariant.

Section IV
We prove that if $A$ is a stable $C^{*}$-algebra then the following three objects are equal:
(i) The universal connected covering group of $G L_{\infty}^{0} A$.
(ii) The Steinberg extension of the group $E_{\infty} A$ of elementary matrices in $G L_{x_{1}} A$.
(iii) The universal central extension of the maximal perfect subgroup of $G L_{\infty} A$.

As a result, the algebraic $K_{2}$-group of $A$ is equal to topological $K_{2}$.

## Section V

This is the main section in the paper. We prove the theorem already stated that the topological and algebraic $K$-thcory groups for Calkin algebras are equal.

## Section VI

There are two main topics. The first is what might be called non-stable $K$-theory-the study of the group $G L_{1} A$ instead of the stable version $G L_{\infty} A$. As we mentioned earlier, it is a considerable simplification to work with $G L_{\infty}$ rather than $G L_{n}$ for some fixed $n$ : this section should illustrate
the point. However, we are able to show that if $B$ is unital then the nonstable algebraic $K$-theory of $\mathscr{M}(\mathscr{K} \otimes B) / \mathscr{K} \otimes B$ is equal to its non-stable topological $K$-theory. By results already known in topological $K$-theory, this implies that the non-stable and stable algebraic $K$-theories agree for these algebras. The other topic is the Karoubi-Villamayor algebraic $K$-theory, which is another possible definition for the homotopy of the discrete group $G L A$. We prove that for a stable $C^{*}$-algebra, this too is equal to the topological $K$-theory.

This paper is a modification of the author's Ph.D. thesis (Dalhousie University, 1985). He would like to take this opportunity to thank his supervisor, Peter Fillmore, for his patience and support, as well as Dick Kadison for his encouragement and his interest in this work.

## I. Multiplier Algebras and Tensor Products

### 1.1. Multiplier Algebras

Let $A$ be a $C^{*}$-algebra. There are a number of definitions of the multiplier algebra, $\mathscr{M}(A)$, of $A$, of which the following (the original one, due to Johnson [28]) is perhaps the most concrete.

Definition 1.1.1. A (double) centralizer of $A$ is a pair $(L, R)$ of linear maps from $A$ to itself, such that: $L$ is a right $A$-module homomorphism (i.e., $L(x y)=L(x) y$ ); $R$ is a left $A$-module homomorphism; and $R(x) y=x L(y)$ for all $x$ and $y$ in $A$. The composition of two centralizers ( $L_{1}, R_{1}$ ) and ( $L_{2}, R_{2}$ ) is given by

$$
\begin{equation*}
\left(L_{1}, R_{1}\right)\left(L_{2}, R_{2}\right)=\left(L_{1} L_{2}, R_{2} R_{1}\right) \tag{1.1.1}
\end{equation*}
$$

(which is easily seen to be a centralizer itself), and the resulting algebra is the multiplier algebra of $A$, denoted $\mathscr{M}(A)$.

This rather odd definition is made with a simple idea in mind: if $A$ is an ideal in an algebra $B$, and if $b \in B$, then $b$ defines a multiplier ( $L, R$ ), by

$$
\begin{equation*}
L(x)=b x, \quad R(x)=x b . \tag{1.1.2}
\end{equation*}
$$

We note that the composition law (1.1.1) corresponds to multiplication of elements in $B$. Of course, $A$ is contained as a trivial ideal in itself, and since for any $x \in A$ there is some $y \in A$ (namely $y=x^{*}$, for example) such that $x y \neq 0$, the element of $\mathscr{M}(A)$ corresponding to $x$ is non-zero. Thus $A$ is embedded as a subalgebra of $\mathscr{M}(A)$, and it is easily verified that $A$ is in fact an ideal in $\mathscr{M}(A)$. (From this we see that every element $(L, R)$ of $\mathscr{M}(A)$ is
obtained in the manner of (1.1.2), namely take $b=(L, R)$ and $B=\mathscr{M}(A)$.) Continuing along these lines, we arrive at the following useful (and, of course, well known) characterization of $\mathscr{M}(A)$.

THEOREM 1.1.2. If $B$ is any algebra containing $A$ as an ideal then there is $a$ unique homomorphism from $B$ to $\mathscr{M}(A)$ which extends the inclusion of $A$ into $\mathscr{M}(A)$.

The algebra $\mathscr{M}(A)$ is in fact a $C^{*}$-algebra, for it turns out that $L$ and $R$ are both bounded linear maps, of equal norm (see [14]), and we may set

$$
\begin{equation*}
\|(L, R)\|=\|L\| \quad(=\|R\|) \tag{1.1.3}
\end{equation*}
$$

which makes $\mathscr{M}(A)$ into a Banach algebra. We define an involution on $\mathscr{M}(A)$ by $(L, R)^{*}=\left(L^{*}, R^{*}\right)$, where $L^{*}(x)=\left(R\left(x^{*}\right)\right)^{*}$ and $R^{*}(x)=$ $\left(L\left(x^{*}\right)\right)^{*}$. With respect to this and the norm (1.1.3), $\mathscr{M}(A)$ is a $C^{*}$-algebra. In Theorem 1.1.2, if $A$ is a closed ideal in a $C^{*}$-algebra $B$ then the canonical homomorphism from $B$ to $\mathscr{A}(A)$ is a $*$-homomorphism.

Apart from obtaining elements of $\mathscr{M}(A)$ via algebras containing $A$ as an ideal, the main source of supply is from certain limits. For this it is useful to introduce the strict topology on $\mathscr{M}(A)$, which is characterized by the following: a net $\left\{x_{\alpha}\right\}$ in $\mathscr{M}(A)$ converges in the strict topology to $x \in \mathscr{M}(A)$ if and only if for every $a \in A$ the nets $\left\{x_{x} a\right\}$ and $\left\{a x_{x}\right\}$ converge in the norm topology to $x a$ and $a x$, respectively. For details, see [14]. The following simple fact is very useful: the strict topology on $\mathscr{M}(A)$ is complete, in the sense that if for every $a \in A$, the sequences $\left\{x_{\alpha} a\right\}$ and $\left\{a x_{\alpha}\right\}$ are Cauchy (in the norm topology), then $\left\{x_{x}\right\}$ converges in the strict topology.

Example 1.1.3. Denote by $\mathscr{K}$ the $C^{*}$-algebra of compact operators on a separable Hilbert space. Then the $C^{*}$-algebra $\mathscr{B}$ of all bounded operators on the Hilbert space contains $\mathscr{K}$ as an ideal, and so by Theorem 1.1.2 there is a canonical $*$-homomorphism $\mathscr{B} \rightarrow \mathscr{M}(\mathscr{K})$. This map is in fact a *-isomorphism since it follows from elementary representation theory that $\mathscr{B}$ enjoys the universal property described in Theorem 1.1.2. The notations $\mathscr{B}$ and $\mathscr{K}$ for bounded and compact operators on a separable Hilbert space will be used throughout the rest of the paper without further explanation.

There is an interesting genralization of this, which is useful to bear in mind (for counterexamples, and so on). If $C(X, \mathscr{K})$ denotes the $C^{*}$-algebra of norm continuous functions from a compact space $X$ to $\mathscr{K}$ then $\mathscr{M}(C(X, \mathscr{K}))$ is equal to $C_{* s t}(X, \mathscr{B})$, the $C^{*}$-algebra of bounded functions $A(x)$ from $X$ to $\mathscr{B}$ which are continuous in the $*$-strong topology (i.e., both
$A(x)$ and $A^{*}(x)$ are strongly continous; the $*$-strong topology on $\mathscr{B}$ is equal to the strict topology on bounded subsets); see [1]. Note that $C(X, \mathscr{K})$ and $C_{* s t}(X, \mathscr{B})$ are, respectively, the (pointwise) compact and the bounded endomorphisms of the trivial field of Hilbert spaces over the space $X$. In general, it is very convenient to regard elements of $A$ as "compact operators" and elements of $\boldsymbol{H}(A)$ as "bounded operators."

We consider now the functorial properties of the multiplier algebra, beginning with what we shall call "restriction." Suppose that $A^{\prime}$ is an ideal in $A$. Then from the fact that $A^{\prime} \cdot A^{\prime}=A^{\prime}$, it follows that $A^{\prime}$ is also an ideal in $\mathscr{M}(A)$.

Definition 1.1.4. The restriction homomorphism $r: \mathscr{H}(A) \rightarrow \mathscr{A}\left(A^{\prime}\right)$ is the unique $*$-homomorphism that extends the inclusion of $A^{\prime}$ into $\|\left(A^{\prime}\right)$.

Notice that if $A^{\prime}$ is an essential ideal of $A$, in other words, if the annihilator ideal

$$
\begin{equation*}
\operatorname{Ann}\left(A^{\prime}\right)=\left\{a \in A \mid a A^{\prime}=A^{\prime} a=0\right\} \tag{1.1.4}
\end{equation*}
$$

is zero, then the restriction homomorphism is injective. Because of this, whenever $\operatorname{Ann}\left(A^{\prime}\right)=0$ we will regard $\mathscr{M}(A)$ as a ( ${ }^{*}$-subalgebra of $\mathscr{M}\left(A^{\prime}\right)$. It is useful to have the following characterization of this subalgebra: if $x \in \mathscr{H}\left(A^{\prime}\right)$ then $x \in \mathscr{M}(A)$ if and only if

$$
\begin{equation*}
x \cdot A \subset A \quad \text { and } \quad A \cdot x \subset A \tag{1.1.5}
\end{equation*}
$$

Indeed, if $x$ satisfies (1.1.5) then $x$ defines a double centralizer of $A$, as in Definition 1.1.1, and the image of this centralizer in $\mathscr{U}\left(A^{\prime}\right)$ under restriction returns $x$.

Let us turn from the restriction homomorphism to a discussion of the covariant functoriality of $\mathscr{M}(A)$. Unfortunately it is not true that every *-homomorphism $f: A_{1} \rightarrow A_{2}$ extends to a $*$-homomorphism from $\mathscr{A}\left(A_{1}\right)$ to $\mathscr{M}\left(A_{2}\right)$. However, by means of the next three results we are able to get by.

Lemma 1.1.5. (See [43, Proposition 3.12.12].) If $f\left[A_{1}\right]$ contains an approximate unit for $A_{2}$ then the map $f: A_{1} \rightarrow A_{2}$ extends uniquely to a $*$-homomorphism $f: \mathbb{H}\left(A_{1}\right) \rightarrow \mathbb{H}\left(A_{2}\right)$.

All the approximate units that we deal with in this paper are assumed to be positive and increasing. The condition that $f\left[A_{1}\right]$ contain an approximate unit for $A_{2}$ is equivalent to $f\left[A_{1}\right] A_{2}$ being dense in $A_{2}$, or in the other words, it is equivalent to the hereditary subalgebra generated by $f\left[A_{1}\right]$ in $A_{2}$ being equal to $\mathrm{A}_{2}$.

Proof. First, if an extension $f: \mathscr{M}\left(A_{1}\right) \rightarrow \mathscr{M}\left(A_{2}\right)$ exists, then it is certainly unique because, for $x \in \mathscr{M}\left(A_{1}\right)$ and $a \in A_{2}$ we have

$$
\begin{align*}
f(x) a & =f(x) \lim _{\lambda \rightarrow \infty} f\left(u_{\lambda}\right) a  \tag{1.1.6}\\
& =\lim _{\lambda \rightarrow \infty} f\left(x u_{\lambda}\right) a,
\end{align*}
$$

and similarly,

$$
\begin{align*}
a f(x) & =\lim _{\lambda \rightarrow \infty}(a f(x)) f\left(u_{\lambda}\right)  \tag{1.1.7}\\
& =\lim _{\lambda \rightarrow \infty} a f\left(x u_{\lambda}\right),
\end{align*}
$$

where $\left\{u_{i}\right\}_{\lambda \in A}$ is an approximate unit for $A_{1}$ (and so $\left\{f\left(u_{i}\right)\right\}$ is an approximate unit for $A_{2}$ ). In other words, $f(x)$ is the limit in the strict topology of the net $\left\{f\left(x u_{i}\right)\right\}$. Since $x u_{\lambda} \in A_{1}$, the bottom lines of (1.1.6) and (1.1.7) do not depend on the extension of $f$. On the other hand, it is readily verified that the limits (1.1.6) and (1.1.7) always exist: if $a \in f\left[A_{1}\right] A_{2}$ then this is clear, while the case of a general $a \in A_{2}$ is dealt with by approximating with elements of $f\left[A_{1}\right] A_{2}$. It is easily seen that the limits define an extension of $f$ from $\mathscr{M}\left(A_{1}\right)$ into $\mathscr{M}\left(A_{2}\right)$.

The following definition gives a class of $*$-homomorphisms which is large enough for our purposes, and all of whose elements are extendible.

Definition 1.1.6. A quasi-unital $*$-homomorphism $f$ from $A_{1}$ to $A_{2}$ is a *-homomorphism with the property that the hereditary subalgebra of $A_{2}$ generated by $f\left[A_{1}\right]$ is of the form $p A_{2} p$, where $p$ is a projection in $\mathscr{M}\left(A_{2}\right)$.

Notice that the projection $p$ in this definition is unique, if it exists at all, since, for example, $1-p$ may be recovered from $f$ as the unit of the $C^{*}$-algebra of $x \in \mathscr{M}\left(A_{2}\right)$ such that $x f\left[A_{1}\right]=0=f\left[A_{1}\right] x$.

Proposition 1.1.7. A quasi-unital map $f: A_{1} \rightarrow A_{2}$ extends to $a$ *-homomorphism from $\mathscr{M}\left(A_{1}\right)$ to $\mathscr{M}\left(A_{2}\right)$.

Proof. Let $\left\{u_{i}\right\}$ be an approximate unit for $A_{1}$. Define $f: \mathscr{M}\left(A_{1}\right) \rightarrow \mathscr{M}\left(A_{2}\right)$ by the formulas

$$
\begin{align*}
& f(x) a=\lim _{\lambda \rightarrow \infty} f\left(x u_{i}\right) a  \tag{1.1.8}\\
& a f(x)=\lim _{i \rightarrow \infty} a f\left(x u_{\lambda}\right)
\end{align*}
$$

where $a \in A_{2}$ and $x \in \mathscr{M}\left(A_{1}\right)$. By writing $a$ as $p a+(1-p) a$, with $p$ the projection of Definition 1.1.6, since $f\left(x u_{i}\right)(1-p)=0$, we see from Lemma 1.1.5 that the limits exist and define a $*$-homomorphism from $\mathscr{M}\left(A_{1}\right)$ to $\mathscr{M}\left(A_{2}\right)$ as required.

Of course, the extension of $f: A_{1} \rightarrow A_{\text {, }}$ is not necessarily unique, for we can add to the map defined by (1.1.8) any *-homomorphism from the quotient $\mathscr{M}\left(A_{1}\right) / A_{1}$ to $(1-p) \mathscr{M}\left(A_{2}\right)(1-p)$. However, when we speak of the extension of a quasi-unital map we will always mean the one given by (1.1.8); we will denote it simply by $f$.

Proposition 1.1.8. (i) The composition of two quasi-unital maps is quasi-unital.
(ii) Extension of quasi-unital maps to multiplier algebras is functorial.

Proof. (i) Suppose that $f_{1}: A_{1} \rightarrow A_{2}$ and $f_{2}: A_{2} \rightarrow A_{3}$ are quasi-unital. Let $p A_{2} p$ be the hereditary subalgebra of $A_{2}$ generated by $f_{1}\left[A_{1}\right]$, and let $q A_{3} q$ be the hereditary subalgebra of $A_{3}$ generated by $f_{2}\left(A_{2}\right)$. If $\left\{u_{\lambda}\right\}_{i \in A}$ is an approximate unit for $A_{1}$ then $\left\{f_{1}\left(u_{\lambda}\right)\right\}_{i_{\in A}}$ is an approximate unit for $p A_{2} p$. Therefore, if $\left\{v_{\lambda^{\prime}}\right\}_{\lambda^{\prime} \in \mathcal{A}}$ is an approximate unit for $(1-p) A_{2}(1-p)$ then $\left\{f_{1}\left(u_{\lambda}\right)+v_{\nu^{\prime}}\right\}_{\left(, \lambda^{\prime}\right) \in A \times A}$ is an approximate unit for $A_{2}$, and so $\left\{f_{2} f_{1}\left(u_{\lambda}\right)+f_{2}\left(v_{\lambda^{\prime}}\right)\right\}_{\left(\lambda, \lambda^{\prime} \in A \times A^{\prime}\right.}$ is an approximate unit for $q A_{3} q$. Finally, from the fact that $f_{2}(p) f_{2}\left(v_{i}\right)=0$, it follows that $\left\{f_{2} f_{1}\left(u_{i}\right)\right\}_{i \in A}$ is an approximate unit for $f_{2}(p) A_{3} f_{2}(p)$. Hence $f_{2} f_{1}\left[A_{1}\right]$ generates $f_{2}(p) A_{3} f_{2}(p)$ as a hereditary subalgebra.
(ii) We have to show that the extension of $f_{2} f_{1}$ is equal to the extension of $f_{1}$ composed with the extension of $f_{2}$. Thus if $x \in \mathscr{M}\left(A_{1}\right)$ we must show that $f_{2} f_{1}(x)$ and $f_{2}\left(f_{1}(x)\right)$ are the same element of $\mathscr{M}\left(f_{2}(p) A_{3} f_{2}(p)\right)$. Since $f_{2} f_{1}\left[A_{1}\right]$ generates $f_{2}(p) A_{3} f_{2}(p)$ as a hereditary subalgebra, it suffices to show that $f_{2} f_{1}(x) b=f_{2}\left(f_{1}(x)\right) b$ for any $b \in f_{2} f_{1}\left[A_{1}\right]$. But if $b=f_{2} f_{1}(a)$ then

$$
\begin{aligned}
f_{2} f_{1}(x) b & =f_{2} f_{1}(x a) & & \text { (since } f_{2} f_{1} \text { is a homomorphism) } \\
& =f_{2}\left(f_{1}(x a)\right. & & \text { (since } \left.x a \in A_{1}\right) \\
& =f_{2}\left(f_{1}(x) f_{1}(a)\right) & & \text { (since } f_{1} \text { is a homomorphism) } \\
& =f_{2}\left(f_{1}(x)\right) f_{2}\left(f_{1}(a)\right) & & \text { (since } f_{2} \text { is a homomorphism). }
\end{aligned}
$$

We close our discussion of functioriality properties by making note of a theorem of Akemann, Pedersen, and Tomiyama [1]. The following terminology is due to Pedersen [44].

Definition 1.1.9. A $C^{*}$-algebra is said to be $\sigma$-unital if it possesses a countable approximate identity.

Theorem 1.1.10. If $A$ is a $\sigma$-unital $C^{*}$-algebra then any surjective *-homomorphism $A \rightarrow A / J$ extends to a surjective *-homomorphism $\mathscr{M}(A) \rightarrow \mathscr{M}(A / J)$.

Actually, the original version of this is for separable $C^{*}$-algebras $A$ only; the cxtension to $\sigma$-unital algebras is carried out by Pedersen in [44].

Next, we state a separation theorem for subalgebras of $\mathscr{M}(D)$ which will be used a great deal in the sequel.

Theorem 1.1.11. Let $D$ be a $C^{*}$-algebra; let $E_{1}$ and $E_{2}$ be $C^{*}$-subalgebras of $M(D), E_{1} \sigma$-unital and $E_{2}$ separable; let $E$ be a (closed, two-sided) ideal in $E_{1}$; and let $\mathscr{F}$ be a separable, linear subspace of $\mathscr{M}(D)$. If $E_{1} \cdot E_{2} \subset E,\left[\mathscr{F}, E_{1}\right] \subset E_{1}$ and $D \subset E_{1}+E_{2}$, then there exists an element $N \in \mathscr{M}(D) \quad$ such that $\quad 1 \geqslant N \geqslant 0, \quad(1-N) \cdot E_{1} \subset E, \quad N \cdot E_{2} \subset E, \quad$ and $[N, \mathscr{F}] \subset E$.
(The symbol $\left[\mathscr{F}, E_{1}\right]$, for example, denotes the set of all commutators $[F, e]=f e-e f$, where $f \in \mathscr{F}$ and $e \in E_{1}$.) This extremely useful result is due to Kasparov [35] (for a shorter proof, see [26]). It is a basic tool in Kasparov's bivariant $K$-theory, where it is the principal technical component in the construction of the Kasparov product map $K K\left(A_{1}, B_{1} \otimes D\right) \times K K\left(A_{2} \otimes D, B_{2}\right) \rightarrow K K\left(A_{1} \otimes A_{2}, B_{1} \otimes B_{2}\right)$ : the operator $N$, together with its "complement" $M=(1-N)$ appear as weights in the averaging of two operators, and the theorem asserts that these weights can be chosen as to produce an average with certain desirable properties (which, for example, make it amenable to study from the point of view of index theory). For details, see Kasparov's papers (see [33; 35, especially Remark 3, p. 773] for a motivation of the construction). Our uses of Theorem 1.1.11 will on the whole be more algebraic in nature. We will appeal to it in Section 1.3 when we discuss exact sequences related to multiplier algebras. In Section 3.5 we will use it to prove various excision properties of extension groups, and in Section 5.2 we will use it to deduce the existence of local units in certain $C^{*}$-algebra ideals (see Theorem 5.2.1). Finaly, it makes an appearance in a technical result in Section 6.1.

### 1.2. Tensor Products

Let $A$ and $B$ be $C^{*}$-algebras, and denote by $A \odot B$ the (algebraic) tensor product of $A$ and $B$. We put a $C^{*}$-norm on $A \odot B$ as follows. Pick faithful representations $\rho_{A}$ and $\rho_{B}$ of $A$ and $B$, on Hilbert spaces $\mathscr{H}_{A}$ and $\mathscr{H}_{B}$. Using $\rho_{A}$ and $\rho_{B}$ we embed $A \odot B$ in $\mathscr{B}\left(\mathscr{H}_{A}\right) \odot \mathscr{B}\left(\mathscr{H}_{B}\right)$, and since $\mathscr{B}\left(\mathscr{H}_{A}\right) \odot \mathscr{B}\left(\mathscr{H}_{B}\right)$ embeds in an obvious way into $\mathscr{B}\left(\mathscr{H}_{A} \otimes \mathscr{H}_{B}\right)$, we obtain a faithful representation of $A \odot B$ in $\mathscr{B}\left(\mathscr{H}_{A} \otimes \mathscr{H}_{B}\right)$ : the operator norm on $\mathscr{B}\left(\mathscr{H}_{A} \otimes \mathscr{H}_{B}\right)$ then gives the $C^{*}$-norm on $A \odot B$. It is not hard to show that
the norm does not depend on which particular representations, $\rho_{A}$ and $\rho_{B}$, are used in its definition. In fact, the norm is given by the formula

$$
\begin{equation*}
\|x\|^{2}=\sup \left\{\frac{\varphi \odot \psi\left(y^{*} x^{*} x y\right)}{\varphi \odot \psi\left(y^{*} y\right)}\right\}, \tag{1.2.1}
\end{equation*}
$$

where the supremum is taken over all states $\varphi$ of $A$ and $\psi$ of $B$, and all $y \in A \odot B$. (For a survey of the theory of $C^{*}$-algebra tensor products the reader is referred to [36].) The completion of $A \odot B$ in this norm is called the spatial tensor product of $A$ and $B$. We will denote it by $A \otimes B$ and refer to it simply as the tensor product, since we will not be using any others.

Example 1.2.1. The tensor product of $C_{0}(X)$ with a $C^{*}$-algebra $A$ is *-isomorphic to $C_{0}(X, A)$, the $C^{*}$-algebra of $A$-valued functions on $X$ which vanish at infinity. There is a canonical $*$-isomorphism from $C_{0}(X) \otimes A$ to $C_{0}(X, A)$, namely that which maps $f \otimes a$ to the function $x \mapsto f(x) a$.

Example 1.2.2. A $C^{*}$-algebra $A$ is said to be stable if it is $*$-isomorphic to the tensor product $\mathscr{K} \otimes A$. Most of the paper will be devoted to the study of these algebras. The $C^{*}$-algebra $\mathscr{K}$ is stable. Indeed $\mathscr{K} \otimes \mathscr{K}$ is canonically isomorphic to the $C^{*}$-algebra of compact operators on the tensor product Hilbert space $\mathscr{H} \otimes \mathscr{H}$, and a unitary isomorphism $\mathscr{H} \cong \mathscr{H} \otimes \mathscr{H}$ induces a $*$-isomorphism $\mathscr{K} \cong \mathscr{H} \otimes \mathscr{K}$. By forming the tensor product with the identity map on $A$ (see below) we get that $\mathscr{K} \otimes A \cong$ $\mathscr{K} \otimes \mathscr{K} \otimes A$ : thus $\mathscr{K} \otimes A$ is stable.

We stated earlier that $\mathscr{M}(D)$ should be thought of as "bounded operators," and the ideal $D$ in $\cdot \mu(D)$ should be thought of as the ideal of "compact operators." This is especially profitable for stable $C^{*}$-algebras. The algebra $\mathscr{K} \otimes A$ is the algebra of compact operators on the standard Hilbert module $\ell^{2} A$ (see [34] for a proof, together with an explanation of in what sense "compact" is to be taken-roughly speaking it is "closure of finite rank"). The algebra of all bounded operators is indeed $\mathscr{H}(\mathscr{K} \otimes A)$. We note that $\mathscr{M}(\mathscr{K}) \otimes \mathscr{M}(A)$ is contained in $\mathscr{M}(\mathscr{K} \otimes A)$ (because $\mathscr{K} \otimes A$ is an essential ideal in $\mathscr{M}(\mathscr{H}) \otimes \mathscr{M}(A)$ ), but it is not equal to it. If we think of elements in $\mathscr{H}(\mathscr{K} \otimes A)$ as infinite matrices with entries in $A$, then the elements in $\mathscr{H}(\mathscr{K}) \odot \mathscr{H}(A)$ correspond to those matrices whose entries are selected from a finite dimensional linear subspace of $A$. For more information on the difference between $\mathscr{M}(A) \otimes \mathscr{M}(B)$ and $\mathscr{M}(A \otimes B)$, see [1].

We turn to the functorial properties of the tensor product. If $f: A_{1} \rightarrow A_{2}$ and $g: B_{1} \rightarrow B_{2}$ are $*$-homomorphisms then the map $f \odot g: A_{1} \odot A_{2} \rightarrow$ $B_{1} \odot B_{2}$ extends to a $*$-homomorphism $f \otimes g: A_{1} \otimes A_{2} \rightarrow B_{1} \otimes B_{2}$. This is clear from the representation description of the tensor product if $f$ and $g$
are injective, and it is clear from the formula (1.2.1) for the norm if $f$ and $g$ are surjective. Since we can factor any $f$ and $g$ into surjections followed by injections the result follows. The tensor product of two injective maps is injective; the tensor product of two surjective maps is surjective.

Lemma 1.2.3. The tensor product of two quasi-unital *-homomorphisms is quasi-unital.

Proof. If $f\left[A_{i}\right]$ contains an approximate unit $\left\{u_{\lambda_{i}}\right\}_{\lambda_{i} \in \Lambda_{i}}$ for $p_{i} B_{i} p_{i}$ $(i=1,2)$, then $f\left[A_{1} \otimes A_{2}\right]$ contains an approximate unit for $p_{1} \otimes p_{2}\left(B_{1} \otimes B_{2}\right) p_{1} \otimes p_{2}$, namely $\left\{u_{\lambda_{1}} \otimes u_{\lambda_{2}}\right\}_{\left(\lambda_{1}, \lambda_{2}\right) \in A_{1} \times A_{2}}$.

The most important application of this lemma will be in the case of the map $1 \otimes f: \mathscr{K} \otimes A \rightarrow \mathscr{K} \otimes B$. For example, if $A$ is unital then $f: A \rightarrow B$ is certainly quasi-unital (the projection $p$ in Definition 1.1.6 is just $f(1)$ ) and so the tensor product $1 \otimes f$ is quasi-unital.

### 1.3. Completely Positive Mappings

Definition 1.3.1. A linear map $f: A \rightarrow B$ between two $C^{*}$-algebras is completely positive if it is positive, that is, $f\left(x^{*} x\right) \geqslant 0$ for every $x \in A$, and if for every $n$ the map $1 \otimes f: M_{n} \otimes A \rightarrow M_{n} \otimes B$ is positive, where $M_{n}$ denotes the $n \times n$ complex matrices.

This definition is due to Stinespring, as is the following result, the first in the subject.

Theorem 1.3.2. (See [51].) A linear map $f: A \rightarrow \mathscr{B}(\mathscr{H})$ is completely positive if and only if it has the form

$$
\begin{equation*}
f(a)=V^{*} g(a) V \quad(a \in A) \tag{1.3.1}
\end{equation*}
$$

where $g$ is a representation of $A$ on a Hilbert space $\mathscr{H}_{1}$, and $V$ is a bounded operator from $\mathscr{H}$ to $\mathscr{H}_{1}$.

Let us make a few comments on this. To begin with, any completely positive map $f: A \rightarrow B$ is bounded, for otherwise there would exist elements $a_{n} \geqslant 0$ in $A(n=1,2, \ldots)$, such that say $\left\|a_{n}\right\|<2^{-n}$ and $f\left(a_{n}\right) \geqslant n$ : but then since $f\left(\sum_{n=1}^{\infty} a_{n}\right) \geqslant f\left(\sum_{n=1}^{N} a_{n}\right) \geqslant N(N+1) / 2$ for all $N$, the value of $f$ at $\sum_{n=1}^{\infty} a_{n}$ would be infinite. Second, Stinespring proved Theorem 1.3.2 for unital $C^{*}$-algebras $A$. That is true in general is an observation of Lance; the following lemma suffices (compare [16, Lemma 3.9]).

Lemma 1.3.3. Let $f: A \rightarrow B$ be a completely positive map and denote by $\widetilde{f}: \widetilde{A} \rightarrow \widetilde{B}$ the map from the $C^{*}$-algebra obtained from $A$ by adding a unit, to
the $C^{*}$-algebra obtained from $B$ by adding a unit, which is equal to $f$ on $A$, and for which $f(1)=\|f\| \cdot 1$. Then $f$ is completely positive.

Proof. We must show that if $a \in M_{n} \otimes A$ then $1 \otimes \widetilde{f}\left(a^{*} a\right) \geqslant 0$. Write $a=a_{0}+a_{1} \otimes 1$, where $a_{0} \in M_{n} \otimes A$ and $a_{1} \otimes 1 \in M_{n} \otimes \mathbf{C}$, and let $\left\{e_{\lambda}\right\}_{\lambda \in A}$ be an approximate unit for $A$. Then

$$
\begin{aligned}
1 \otimes \tilde{f}\left(a^{*} a\right)= & 1 \otimes \tilde{f}\left(a_{0}^{*} a_{0}+a_{0}^{*}\left(a_{1} \otimes 1\right)+\left(a_{1}^{*} \otimes 1\right) a_{0}+a_{1}^{*} a_{1} \otimes 1\right) \\
= & 1 \otimes f\left(a_{0}^{*} a_{0}+a_{0}^{*}\left(a_{1} \otimes 1\right)+\left(a_{1}^{*} \otimes 1\right) a_{0}+a_{1}^{*} a_{1} \otimes e_{j}\right) \\
& +a_{1}^{*} a_{1} \otimes\left(\|f\|-f\left(e_{\lambda}\right)\right) \\
\geqslant & 1 \otimes f\left(a_{0}^{*} a_{0}+a_{0}^{*}\left(a_{1} \otimes 1\right)+\left(a_{1}^{*} \otimes 1\right) a_{0}+a_{1}^{*} a_{1} \otimes e_{\lambda}\right) \\
= & 1 \otimes f\left(\left(a_{0}+a_{1} \otimes e_{\lambda}^{1 / 2}\right)^{*}\left(a_{0}+a_{1} \otimes e_{\lambda}^{1 / 2}\right)\right) \\
& +1 \otimes f\left(a_{0}^{*}\left(a_{1} \otimes\left(1-e_{\lambda}^{1 / 2}\right)\right)+\left(a_{1}^{*} \otimes\left(1-e_{\lambda}^{1 / 2}\right)\right) a_{0}\right) \\
\geqslant & 1 \otimes f\left(a_{0}^{*}\left(a_{1} \otimes\left(1-e_{\lambda}^{1 / 2}\right)\right)+\left(a_{1}^{*} \otimes\left(1-e_{\lambda}^{1 / 2}\right)\right) a_{0}\right) .
\end{aligned}
$$

But as $\lambda \rightarrow \infty$, the argument of $1 \otimes f$ on the last line converges in norm to zero. Hence $1 \otimes f\left(a^{*} a\right) \geqslant 0$.

Returning to Theorem 1.3.2, we may extend $f$ to a completely positive map from $\bar{A}$ to $\mathscr{B}(\mathscr{H})$. Note that if $\|f\|=1$ then $f(1)=1$ and so from (1.3.1) we obtain

$$
\begin{equation*}
\mathrm{I}=V^{*} g(1) V \tag{1.3.2}
\end{equation*}
$$

If we define $V^{\prime}=g(1) V$ then (1.3.1) remains true with $V^{\prime}$ replacing $V$, and from (1.3.2) we obtain $V^{*} V^{\prime}=1$, or in the other words, $V^{\prime}$ is an isometry. Thus every completely positive map from $f$ into $\mathscr{B}(\mathscr{H})$ such that $\|f\|=1$ may be dilated to a representation $g$ of $A$ on a larger Hilbert space $\mathscr{H}_{1}$ (we may use the isometry $V^{\prime}$ to identify $\mathscr{H}$ as a subspace of $\mathscr{H}_{1}$ ), and conversely all maps which can be so dilated are completely positive contractions.

The first example of a completely positive map is a state $p: A \rightarrow \mathbf{C}$ on $A$. The fact that $p$ can be dilated to a $*$-homomorphism into some $\mathscr{B}\left(\mathscr{H}_{1}\right)$ follows also from the Gelfand-Naimark-Segal (GNS) construction of a representation from a state. Thus Stinespring's theorem is a generalization of the GNS construction (and the same can be said of the proof).

We will be using completely positive maps for two purposes. First, they play an important role in $C^{*}$-algebra extension theory (as was first pointed out by Arveson [3]): they are used to answer the question of when an extension is invertible. We will discuss this in Section II. Second, there is a close relationship between the theories of tenor products and of completely
positive maps. For our purposes it suffices to deal only with the elementary aspects of this, beginning with

Lemma 1.3.4. A linear map $f: A \rightarrow B$ is completely positive if and only if for every $C^{*}$-algebra $C$, the tensor product $f \odot 1: A \odot C \rightarrow B \odot C$ extends to a positive $\operatorname{map} f \otimes 1: A \otimes C \rightarrow B \otimes C$.

Remark. We note that, given this, if $f$ is a completely positive map then the map $f \otimes 1: A \otimes C \rightarrow B \otimes C$ is not only positive but completely positive, since if we form the tensor product of $f$ with the identity map on $C \otimes M_{n}$, then by the lemma, we obtain a positive map.

Proof. In view of the definition of complete positivity, the "if" part of the lemma is trivial. For the other half we will use Stinespring's theorem. Embed $B$ in some $\mathscr{B}\left(\mathscr{H}_{1}\right)$ so that $f$ may be written in the form $f(a)=V^{*} g(a) V$, where $g: A \rightarrow \mathscr{B}\left(\mathscr{H}_{1}\right)$ is a *-homomorphism and $V$ is an operator in $\mathscr{B}\left(\mathscr{H}_{1}\right)$. Since $B \otimes C$ embeds in $\mathscr{B}\left(\mathscr{H}_{1}\right) \otimes C$ it suffices to show that $f \odot 1: A \odot C \rightarrow \mathscr{B}\left(\mathscr{H}_{1}\right) \otimes C$ extends to a positive map on $A \otimes C$. However, we can extend $f \odot 1: A \odot C \rightarrow \mathscr{B}\left(\mathscr{H}_{1}\right) \otimes C$ by the formula

$$
f \otimes 1(x)=\left(V^{*} \otimes 1\right) g \otimes 1(x)(V \otimes 1) \quad(x \in A \otimes C)
$$

It is clear that $f \otimes 1$ so defined is positive.
Our interest is in the relationship between completely positive maps and exactness properties of the tensor product. Suppose that

$$
0 \rightarrow A_{1} \xrightarrow{j} A_{2} \xrightarrow{p} A_{3} \rightarrow 0
$$

is a short exact sequence of $C^{*}$-algebras and $*$-homomorphisms. That is, $j$ is injective, $p$ is surjective, and the kernel of $p$ is equal to the image of $j$. Consider the sequence

$$
\begin{equation*}
0 \rightarrow A_{1} \otimes C \xrightarrow{j \otimes \mathrm{I}} A_{2} \otimes C \xrightarrow{p \otimes \mathrm{I}} A_{3} \otimes C \rightarrow 0 . \tag{1.3.3}
\end{equation*}
$$

Under what conditions is it, too, a short exact sequence? Certainly $j \otimes 1$ is injective and $p \otimes 1$ is surjective, and also, the image of $j \otimes 1$ is contained in the kernel of $p \otimes 1$. However, the kernel of $p \otimes 1$ need not equal the image of $j \otimes 1$ (for a counterexample see [2]). The following positive result is suitable for our purposes.

TheOrem 1.3.5. If there exists a completely positive map s: $A_{3} \rightarrow A_{2}$ such that the composition ps: $A_{3} \rightarrow A_{3}$ is the identity on $A_{3}$ then the sequence (1.3.3) is exact for every $C^{*}$-algebra $C$.

Proof. The map $s \otimes 1: A_{3} \otimes C \rightarrow A_{2} \otimes C$ is defined, by Lemma 1.3.4, and te composition $(p \otimes 1)(s \otimes 1)=p s \otimes 1$ is the identity on $A_{3} \otimes C$. Also, the map

$$
(1-s p \otimes 1): A_{2} \otimes C \rightarrow A_{2} \otimes C
$$

takes $A_{2} \otimes C$ into the image of $A_{1} \otimes C$ (this is verified on $A_{2} \odot C$ and then on all of $A_{2} \otimes C$ by continuity). Thus if $x \in \operatorname{kernel}(p \otimes 1)$ then

$$
x=x-(s \otimes 1)(p \otimes 1)(x)=(1-s p \otimes 1)(x),
$$

and so, $x \in \operatorname{image}(j \otimes 1)$.
Remarks. Effros and Haagerup have recently show that the converse of Theorem 1.3.5 holds if $A_{1}$ is a nuclear $C^{*}$-algebra and $A_{3}$ is separable (this, is, needless to say, a much deeper result). Nuclearity, that is, the condition on a $C^{*}$-algebra $A$ that there be a unique $C^{*}$-norm on $A \odot C$ for all $C$, gives another sufficient condition: if $A_{3}$ is nuclear then (1.3.4) is exact. The reason is that $A_{3} \odot C$ embeds densely in $A_{2} \otimes C / A_{1} \otimes C$, and so if there is a unique completion of $A_{3} \odot C$ to a $C^{*}$-algebra then $A_{2} \otimes C / A_{1} \otimes C$ must be it. However, some of the algebras we consider will not be nuclear.

Let us give a simple illustration of conditions under which the hypotheses of the theorem are satisfied. The following two results are more or less a diversion and will only be used in remarks (or future diversions).

TheOrem 1.3.6. If $X$ is a second countable, locally compact space then any *-homomorphism $p$ from a $C^{*}$-algebra $A$ onto $C_{0}(X)$ has a completely' positive contractive section.

Sketch of Proof. Let $\left\{f_{1}, f_{2}, \ldots\right\}$ be a dense subset of $C_{0}(X)$, and for each $k=1,2, \ldots$, choose non-negative functions $\varphi_{k 1}, \ldots, \varphi_{k n_{k}}$ and points $x_{1}, \ldots, x_{n_{k}}$ in $X$ such that:
(i) $\sum_{n=1}^{n_{k}} \varphi_{k n} \leqslant 1$ and
(ii) $\left\|f_{j}-\sum_{n=1}^{n_{k}} f_{j}\left(x_{n}\right) \varphi_{k n}\right\|<1 / n(j=1, \ldots, k)$.
(The $\varphi_{k n}$ 's are a subselection of an appropriately chosen partition of unity for $X$.) Choose elements $\tilde{\varphi}_{k n}$ in $A$ such that $\tilde{\varphi}_{k n} \geqslant 0$ and $p\left(\tilde{\varphi}_{k n}\right)=\varphi_{k n}$, and define maps $\varphi_{k}: C_{0}(X) \rightarrow A$ by

$$
\varphi_{k}(f)=\sum_{n=1}^{n_{k}} f\left(x_{n}\right) \tilde{\varphi}_{k n} \quad\left(f \in C_{0}(X)\right) .
$$

These are certainly completely positive, since each component $f \mapsto f\left(x_{n}\right) \tilde{\varphi}_{k n}=\tilde{\varphi}_{k n}^{1 / 2} f\left(x_{n}\right) \tilde{\varphi}_{k n}^{1 / 2}$ of $\varphi_{k}$ is completely positive. Consider the
compositions $p \circ \varphi_{k}(k=1,2, \ldots)$. By condition (i), $\left\|p \circ \varphi_{k}\right\| \leqslant 1$ and, by this and condition (ii), $p \circ \varphi_{k}$ converges pointwise to the identity on $C_{0}(X)$. This is verified first for the $f_{j}$, and then for arbitrary $f$ by an approximation argument. We now appeal to a result of Arveson [4, Theorem 6] that if $p: A \rightarrow B$ is a surjective $*$-homomorphism, and $B$ is separable, then the set of completely positive, contractive maps $\alpha: B \rightarrow B$ which factor- $\alpha=p \cup \tilde{\alpha}$, with $\tilde{\alpha}$ completely positive and contractive-is closed in the topology of pointwise convergence. It follows that the identity $C_{0}(X) \rightarrow C_{0}(X)$ factors, or in other words, $p$ has a completely positive section.

The next result is a simple extension, the proof of which is left to the reader.

Theorem 1.3.7. Let $C_{0}(X)$ and $B$ be separable and let $p: A \rightarrow C_{0}(X) \otimes B$ be a surjective *-homomorphism. Suppose that for every $x \in X$ the composite *-homomorphism

$$
A \xrightarrow{p} C_{0}(X) \otimes B \xrightarrow{\varepsilon_{x}} B \quad\left(\varepsilon_{x}=\text { evaluation at } x\right)
$$

has a completely positive section. Then $p$ has a completely positive section.
We turn now to our main application of Theorem 1.3.5, which concerns multiplier algebras and exact sequences. Let $B$ be a $\sigma$-unital $C^{*}$-algebra and let $J$ be a (closed, two-sided) ideal in $B$. By Theorem 1.1.10, the extension $\mathscr{M}(B) \rightarrow \mathscr{M}(B / J)$ of the canonical projection $B \rightarrow B / J$ is surjective. The kernel of this map is the ideal given by the following notation.

Definition 1.3.8. Let $J$ be an ideal in a $C^{*}$-algebra $B$. The ideal $\mathscr{M}(B ; J)$ in $\mathscr{M}(B)$ is the set of those elements $x \in \mathscr{M}(B)$ for which $x b \in J$ and $b x \in J$ for every $b \in B$.

Thus we have a short exact sequence

$$
\begin{equation*}
0 \rightarrow \mathscr{M}(B ; J) \rightarrow \mathscr{M}(B) \rightarrow \mathscr{M}(B / J) \rightarrow 0 \tag{1.3.4}
\end{equation*}
$$

Actually, we are not so much interested in $\mathscr{M}(B) \rightarrow \mathscr{M}(B / J)$ as in the induced map $p: \mathscr{M}(B) / B \rightarrow \mathscr{M}(B / J) / B / J$. Since $B$, of course, maps onto $B / J$, the kernel of $p$ is the image of $\mathscr{M}(B ; J)$ in $\mathscr{M}(B) / B$. Thus

$$
\operatorname{kcrncl}(p)=\mathscr{M}(B ; J)+B / B \cong \mathscr{M}(B ; J) /(B \cap \mathscr{M}(B ; J)),
$$

and since $B \cap \mathscr{M}(B ; J)=J$, we obtain the short exact sequence

$$
\begin{equation*}
0 \rightarrow \mathscr{M}(B ; J) / J \rightarrow \mathscr{M}(B) / B \xrightarrow{p} \mathscr{M}(B / J) / B / J \rightarrow 0 . \tag{1.3.5}
\end{equation*}
$$

Lemma 1.3.9. Let $\mathscr{A}$ be a separable $C^{*}$-subalgebra of $\mathscr{M}(B / J) / B / J$ (where $B$ is a $\sigma$-unital $C^{*}$-algebra). There exists a completely positive map $s: \mathscr{A} \rightarrow \mathscr{M}(B) / B$ such that the composition $p \circ s: \mathscr{A} \rightarrow \mathscr{M}(B / J) / B / J$ is the identity on $\mathscr{A}$.

Proof. Let $\mathscr{A}^{\prime}$ be a separable $C^{*}$-subalgebra of $\mathscr{M}(B)$ which maps onto $\mathscr{A}$ via the obvious projection (we use Theorem 1.1.10 to ensure that this projection is surjective). Define $C^{*}$-subalgebras of $\mathscr{M}(B)$ as

$$
E_{1}=B, \quad E=J, \quad \mathscr{F}=\mathscr{A}^{\prime}
$$

Let $E_{2}$ be a separable $C^{*}$-subalgebra of $\mathscr{M}(B ; J)$ such that

$$
\mathscr{A}^{\prime} \cap(\mathscr{M}(B ; J)+B) \subset \overline{E_{2}+B}
$$

Then the hypotheses of Theorem 1.1.11 are satisfied, and there exists a positive operator $N \in \mathscr{M}(B)$ such that $N \cdot E_{2} \subset J,(1-N) \cdot B \subset J$, and $\left[N, \mathscr{A}^{\prime}\right] \subset J$. Now, denote by $\dot{x}$ the image of an element $x \in \mathscr{M}(B)$ in $\mathscr{M}(B) / B$, and define $s: \mathscr{A} \rightarrow \mathscr{M}(B) / B$ by

$$
s\left(p\left(\dot{a}^{\prime}\right)\right)=\dot{N} \dot{a}^{\prime} \quad\left(a^{\prime} \in A^{\prime}\right)
$$

In other words, given $a \in \mathscr{A}$, choose $a^{\prime} \in \mathscr{A}{ }^{\prime}$ such that $p\left(\dot{a}^{\prime}\right)=a$, and then define $s(a)=\dot{N} \dot{a}^{\prime}$. To see that this is a well defined, note that the kernel of the map $x \mapsto p(\dot{x})$ from $\mathscr{M}(B)$ to $\mathscr{M}(B / J) / B / J$ is equal to $\mathscr{M}(B ; J)+B$. Therefore, if $p\left(\dot{a}_{1}^{\prime}\right)=p\left(\dot{a}_{2}^{\prime}\right)$ then

$$
a_{1}^{\prime}-a_{2}^{\prime} \in \mathscr{A}^{\prime} \cap(\mathscr{M}(B ; J)+B) \subset \overline{E_{2}+B}
$$

But if $x \in \overline{E_{2}+B}$ then $\dot{N} \dot{x}=0$, by definition of $N$. So $\dot{N}\left(\dot{a}_{1}^{\prime}-\dot{a}_{2}^{\prime}\right)=0$. To see that $p \circ s$ is the identity on $\mathscr{A}$, note that $1-N \in \mathscr{M}(B ; J)$, because $1-N$ multiplies $E_{1}=B$ into $J$, and so $p(1-\dot{N})=0$. Therefore

$$
p s\left(p\left(\dot{a}^{\prime}\right)\right)=p(\dot{N}) p\left(\dot{a}^{\prime}\right)=p(1) p\left(\dot{a}^{\prime}\right)=p\left(\dot{a}^{\prime}\right) .
$$

Finally, $s$ is completely positive: given $x \in M_{n} \otimes \mathscr{A}$ with $x \geqslant 0$ we may choose $x^{\prime} \in M_{n} \otimes \mathscr{A}^{\prime}$ with $x^{\prime} \geqslant 0$ and $p\left(\dot{x}^{\prime}\right)=x$ (a simple application of functional calculus), and then

$$
(1 \otimes s)(x)=\left(1 \otimes \dot{N}^{1 / 2}\right) p \dot{x}^{\prime}\left(1 \otimes \dot{N}^{1 / 2}\right) \geqslant 0
$$

(note that $N$ is so chosen that $\dot{N}$ commutes with $\dot{\mathscr{A}}^{\prime}$ ).
It is quite a remarkable fact (or so the author thinks) that no hypothesis concerning the liftability of $B \rightarrow B / J$ is needed in the above lemma. The next result is proved in the same way.

Lemma 1.3.10. Let $\mathscr{A}$ be a separable $C^{*}$-subalgebra of $\mathscr{M}(B / J)$. There exists a completely positive map $s: \mathscr{A} \rightarrow \mathscr{M}(B) / J$, such that the composition $p \circ s: \mathscr{A} \rightarrow \mathscr{M}(B / J)$ is the identity on $\mathscr{A}$, where $p$ is the surjection in the short exact sequence

$$
\begin{equation*}
0 \rightarrow \mathscr{M}(B ; J) / J \rightarrow \mathscr{M}(B) / J \xrightarrow{p} \mathscr{M}(B / J) \rightarrow 0 . \tag{1.3.6}
\end{equation*}
$$

Theorem 1.3.11. The tensor product of the short exact sequence (1.3.5) with any $C^{*}$-algebra $C$ is again a short exact sequence.

Proof. Let $x \in(\mathscr{M}(B) / B) \otimes C$. Since $x$ is the limit of some sequence of the form $\left\{\sum_{i} x_{i n} \odot c_{i n}\right\}_{n=1}^{\infty}$, it is contained in some $\mathscr{B} \otimes C$, where $\mathscr{B}$ is a separable $C^{*}$-subalgebra of $\mathscr{M}(B) / B$ (the one generated by the $x_{i n}$, for example). Applying the lemma to the image $\mathscr{A}$ of $\mathscr{B}$ in $\mathscr{M}(B / J) / B / J$, and then applying Theorem 1.3.5 to the short exact sequence

$$
0 \rightarrow \mathscr{M}(B ; J) / J \rightarrow \mathscr{B}+\mathscr{M}(B ; J) / J \rightarrow \mathscr{A} \rightarrow 0
$$

we see that if $p \otimes 1(x)=0$ then $x$ is in the image of $(\mathscr{M}(B ; J) / J) \otimes C$. Consequently the sequence (1.3.5) tensored with $C$, is exact.

Similarly, using Lemma 1.3 .10 we obtain

Theorem 1.3.12. The tensor product of the short exact sequence (1.3.6) with any $C^{*}$-algebra $C$ is exact.

The next theorem again concerns $\mathscr{M}(B ; J)$. It is a sort of stabilization result, along the lines of [10,34]. For this particular formulation, see [25].

Theorem 1.3.13. Let $B$ be a $C^{*}$-algebra and let $J$ be an essential $\sigma$-unital ideal in $B$. There exists an isometry $v \in \mathscr{M}(\mathscr{K} \otimes J)$ such that

$$
v \mathscr{M}(\mathscr{K} \otimes J) v^{*} \subset \mathscr{M}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J) .
$$

(We remind the reader that we are regarding $\mathscr{M}(\mathscr{K} \otimes B)$ as a subalgebra of $\mathscr{M}(\mathscr{K} \otimes J)$.) We will need a somewhat technical strenghtening of this theorem.

Theorem 1.3.14. Let $J$ be a $\sigma$-unital, essential ideal in a $\sigma$-unital $C^{*}$ algebra $B$ and let $\mathscr{A}$ be a separable $C^{*}$-subalgebra of $\mathscr{M}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J)$. There exist isometries $v_{1} \in \mathscr{M}(K \otimes J)$ and $v_{2} \in \mathscr{M}(\mathscr{K} \otimes B)$ such that:
(i) $v_{1} \mathscr{M}(\mathscr{K} \otimes J) v_{1}^{*} \subset \mathscr{M}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J)$.
(ii) If $a \in \mathscr{A}$ then $v_{2} a v_{2}^{*}$ is equal to $v_{1} a v_{1}^{*}$, modulo $\mathscr{K} \otimes J$.

Proof. We begin by constructing various isometries to be used in the construction of $v_{1}$ and $v_{2}$. Let $w_{n m}$, where $n=1,2, \ldots$ and $m=0,1,2, \ldots$, be a collection of isometries in $\mathscr{B}=\mathscr{M}(\mathscr{K})$ with pairwise orthogonal final spaces. Let $\left\{d_{m}\right\}_{m=1}^{\infty}$ be a sequence in $\mathscr{K} \otimes J$ such that $\sum_{m=1}^{\infty} d_{m}^{*} d_{m}=1$, where the convergence is in the strict topology of $\mathscr{H}(\mathscr{K} \otimes J)$. We may construct $\left\{d_{m}\right\}_{m=1}^{\infty}$ by starting off with an approximate unit $\left\{e_{m}\right\}_{m=1}^{\infty}$ for $\mathscr{K} \otimes J$-a sequential one exists because $\mathscr{K} \otimes J$ is $\sigma$-unital- and then defining $d_{m}=\left(e_{m}-e_{m-1}\right)^{1 / 2}$ (we let $e_{0}=0$ ). We claim that the series

$$
\sum_{m=1}^{\infty}\left(u_{n m} \otimes 1\right) d_{m} \quad(n \text { is fixed })
$$

converges in the strict topology of $\mathscr{H}(\mathscr{K} \otimes J)$ to some isometry $S_{n}$, and furthermore, that $(\mathscr{K} \otimes B) \cdot S_{n} \subset \mathscr{K} \otimes J$. Note first that the partial sums

$$
S_{n}^{(M)}=\sum_{m=1}^{M}\left(w_{n m} \otimes 1\right) d_{m}
$$

are bounded by 1 , since

$$
S_{n}^{(M) *} S_{n}^{(M)}=\sum_{n_{t}=1}^{M} d_{m}^{*} d_{m} \leqslant 1
$$

If $x \in \mathscr{K} \otimes J$, then

$$
\begin{aligned}
\left\|S_{n}^{(M+K)} x-S_{n}^{(M)} x\right\|^{2} & =\left\|x^{*}\left(S_{n}^{(M+K) *}-S_{n}^{(M) *}\right)\left(S_{n}^{(M+K)}-S_{n}^{(M)}\right) x\right\| \\
& \leqslant\left\|x^{*}\right\|\left\|_{m=M+1}^{M+K} d_{m}^{*} d_{m} x\right\|
\end{aligned}
$$

and so, since $\sum_{m=1}^{\infty} d_{m}^{*} d_{m} x$ converges in norm (to $x$ ), we see that $\left\{S_{n}^{(M)} x\right\}_{M=1}^{\infty}$ is a Cauchy, and hence convergent, sequence. We must show also that $\left\{x S_{n}^{(M)}\right\}_{M=1}^{\infty}$ is norm-convergent for $x \in \mathscr{K} \otimes J$. Actually, this is true for $x \in \mathscr{K} \otimes B$ (and this will show that $\left.(\mathscr{K} \otimes B) \cdot S_{n} \subset \mathscr{K} \otimes J\right)$. To see this, note that the set of those $x \in \mathscr{K} \otimes B$ for which $x\left(w_{n m} \otimes 1\right)=0$, for all but finitely many $m$, is dense in $\mathscr{K} \otimes B$. For such $x$ as these, $\left\{x S_{n}^{(M)}\right\}_{M=1}^{x}$ certainly converges, and since they are dense in $\mathscr{K} \otimes B$, and since $\left\{S_{n}^{(M)}\right\}_{M=1}^{\infty}$ is bounded, the general case may be obtained by the obvious approximation argument. It follows that $\left\{S_{n}^{(M)}\right\}_{M=1}^{\infty}$ converges in the strict topology, and since multiplication is continuous on bounded subsets in the strict topology it follows from the computation

$$
S_{n}^{(M) *} S_{n}^{(M)}=\sum_{m=1}^{M} d_{m}^{*} d_{m}
$$

that the limit $S_{n}$ is an isometry. Now, let $\left\{z_{n}\right\}_{n=1}^{\infty}$ be a sequence in $\mathscr{K} \otimes B$ such that $\sum_{n=1}^{\infty} z_{n}^{*} z_{n}=1$ (convergence in the strict topology of $\mathscr{M}(\mathscr{H} \otimes B))$. The same arguments as those above show that the series

$$
\sum_{n=1}^{\infty}\left(w_{n 0} \otimes 1\right) z_{n}
$$

converges in the strict topology of $\mathscr{M}(\mathscr{K} \otimes B)$ to an isometry $v_{2} \in \mathscr{M}(\mathscr{K} \otimes B)$. The construction of $v_{1} \in \mathscr{M}(\mathscr{K} \otimes J)$ is a little more complicated. Let $J_{1}$ and $\mathscr{A}_{1}$ be compact subsets of the unit balls of $\mathscr{K} \otimes J$ and $\mathscr{A}$ respectively such that:
(i) $J_{1} \cdot \mathscr{K} \otimes J$ is dense in $\mathscr{K} \otimes J$ (note that $J_{1}$ exists because $\mathscr{K} \otimes J$ is $\sigma$-unital).
(ii) $\mathscr{A}_{1}$ is self-adjoint, and the closed linear span of $\mathscr{A}_{1}$ is $\mathscr{A}$.

Choose an approximate unit $\left\{u_{n}\right\}_{n=1}^{\infty}$ for $\mathscr{K} \otimes J$ such that:
( $\alpha$ ) For every $n$, and every $x \in J_{1} \cup \mathscr{A} \mathscr{A}_{1},\left\|x z_{n}^{*} z_{n}-x z_{n}^{*} z_{n} u_{n}\right\|<2^{n 11}$, and $\left\|u_{n}^{1 / 2} z_{n} x-z_{n} x\right\|<2^{-n+1}$.
( $\beta$ ) For every $n$ and every $j \in J_{1},\left\|u_{n}^{1 / 2} z_{n} j-z_{n} j\right\|<2^{-n}$.
( $\gamma$ ) For every $n,\left\|u_{n} z_{n}-z_{n} u_{n}\right\|<2^{-n+1}$.

We can satisfy $(\alpha)$ because the elements $x z_{n}^{*} z_{n}$ and $z_{n} x$ are in $\mathscr{K} \otimes J$, by the definition of $\mathscr{M}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J)$. That condition $(\gamma)$ may be satisfied follows from the existence of quasi-central approximate units (see [4 or 43]). Consider now the expression

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left(w_{n 0} \otimes 1\right) u_{n}^{1 / 2} z_{n}+\sum_{n=1}^{\infty} S_{n} r_{n}, \tag{1.3.7}
\end{equation*}
$$

where

$$
r_{n}=\left(z_{n}^{*} z_{n}-z_{n}^{*} u_{n} z_{n}\right)^{1 / 2}
$$

Both series are convergent in the strict topology of $\mathscr{M}(\mathscr{K} \otimes J)$. Let us consider the first series first: the partial sums are bounded in norm by 1 , and if $j \in J_{1}$ then

$$
\sum_{n=1}^{\infty}\left(w_{n 0} \otimes 1\right) u_{n}^{1 / 2} z_{n} j=\sum_{n=1}^{\infty}\left(w_{n 0} \otimes 1\right) z_{n} j-\sum_{n=1}^{\infty}\left(w_{n 0} \otimes 1\right)\left(1-u_{n}^{1 / 2}\right) z_{n} j
$$

where the first series is norm-convergent by the definition of $\left\{z_{n}\right\}_{n=1}^{\infty}$, while
the second is (absolutely) norm-convergent by ( $\beta$ ). It follows by an approximation argument that the first term of (1.3.7) is norm convergent when multiplied on the right by any $j \in \mathscr{K} \otimes J$. As for the multiplication of $\sum_{n=1}^{\infty}\left(w_{n 0} \otimes 1\right) u_{n}^{1 / 2} z_{n}$ on the left, the same argument that we used for $S_{n}$ shows that when multiplied on the left by any element of $\mathscr{K} \otimes B$ the series becomes norm-convergent. We argue that the series $\sum S_{n} r_{n}$ is convergent similarly, using the fact (which follows from ( $\alpha$ ) and $(\gamma)$ ) that $\left\|r_{n} j\right\|<2^{-n / 2}$ if $j \in J_{1}$. Using continuity of multiplication, the limit $v_{1} \in \mathscr{H}(\mathscr{K} \otimes J)$ is an isometry and, since $(\mathscr{K} \otimes B) v_{1} \subset \mathscr{K} \otimes J$, we have

$$
v_{1} \cdot \mathscr{H}(\mathscr{K} \otimes J) v_{1}^{*} \subset \mathscr{M}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J)
$$

as required. Now, the estimates $(\alpha)$ and $(\gamma)$ imply that if $a \in \alpha_{1}$ then $\left\|r_{n} a\right\|<2^{-n}$, and we see that

$$
v_{1} a-v_{2} a=\sum\left(u_{n 0} \otimes 1\right)\left(1-u^{1 / 2}\right) z_{n} a-\sum S_{n} r_{n} a
$$

which is the sum of two norm-convergent series with terms in $\mathscr{K} \otimes J$. Hence $v_{1} a-v_{2} a \in \mathscr{K} \otimes J$. However, we also have $a v_{1}^{*}-a v_{2}^{*} \in \mathscr{K} \otimes J$, since $\mathscr{A}_{1}$ is self-adjoint, and so $v_{1} a v_{1}^{*}-v_{2} a v_{2}^{*} \in \mathscr{K} \otimes J$.

Finally, we wish to make note of a certain relation involving the ideals $\mathscr{M}(B ; J)$. Note first, that if $I$ and $J$ are disjoint ideals in $B$, then the ideals $\mathscr{M}(B ; I)$ and $\mathscr{M}(B ; J)$ of $\mathscr{M}(B)$ are disjoint. Indeed, if $x \in \mathscr{M}(B ; I) \cap$ $\mathscr{M}(B ; J)$ then $x \cdot B \subset I \cap J=0$ and so $x=0$. What we will need is the equality

$$
\begin{equation*}
\mathscr{M}(B ; I+J)=\mathscr{M}(B ; I)+\mathscr{M}(B ; J) \tag{1.3.8}
\end{equation*}
$$

The inclusion

$$
\mathscr{U}(B ; I+J) \supset \mathscr{U}(B ; I)+\mathscr{M}(B ; J)
$$

is obvious. For the reverse containment, suppose that $x \in \mathscr{M}(B ; I+J)$, which we regard as a double centralizer $(L, R)$. Define pairs of maps ( $L^{I}, R^{I}$ ) and ( $L^{J}, R^{J}$ ) by the conditions

$$
L b=L^{\prime} b+L^{J} b, \quad \text { where } \quad L^{\prime} b \in I \quad \text { and } \quad L^{J} b \in J
$$

and

$$
R b=R^{\prime} b+R^{J} b, \quad \text { where } \quad R^{I} b \in I \quad \text { and } \quad R^{\prime} b \in J
$$

It is easy to verify that both pairs are double centralizers, and of course $\left(L^{\prime}, R^{I}\right)$ defines an element of $\mathscr{M}(B ; I)$, while $\left(L^{J}, R^{J}\right)$ defines an element of $\mathscr{M}(B ; J)$.

## II. $K$-Theory of $C^{*}$-Algebras

This section is largely expository. Its purpose is to introduce the various $K$-theory groups that can be associated with a $C^{*}$-algebra. We begin with the usual topological $K$-theory groups in Section 2.1. These are by now well known to operator algebraists, and our goal is mainly to fix notations and definitions, since we will be viewing $K$-theory from a point of view which is a little different from the usual one. We give a new proof of the Bott periodicity theorem, using a simple computation in $C^{*}$-algebra extension theory, in Section 2.2. Fredholm modules (closely related to extensions) are the subject of Section 2.3. We discuss the basic facts about the algebraic $K_{1}$ and $K_{2}$ groups in Section 2.4. The material is again well known (almost all that we have to say can be found in Milnor's book [40]), so we will be brief and focus on our particular interest, which is the relationship between topological and algebraic $K$-theory. The final topic is the introduction of the higher $K$-theory groups of Quillen. Since some familiarity with algebraic topology is necessary to understand the definition, in Section 2.5 we give a survey of those concepts that we are going to need; we have tried to present them in a manner as accessible as possible to the non-specialist. Our algebraic $K$-theory/algebraic topology primer will be continued in Section V.

### 2.1. Topological K-Theory

Let $A$ be a $C^{*}$-algebra and denote by $\tilde{A}$ the $C^{*}$-algebra obtained by adjoining a unit to $A$ (thus if $A$ happens to be unital already then $\tilde{A}=A \oplus \mathbf{C})$. We define $G L_{n} A(n=1,2, \ldots)$, to be the group of invertible elements in the $n \times n$ matrix algebra $M_{n} \tilde{A}$ which are equal to the identity matrix, modulo the ideal $M_{n} A$ of $M_{n} \tilde{A}$. (Note that if $A$ is unital then $G L_{n} A$ so defined is isomorphic to the group of invertible elements in $M_{n} A$, via the map which sends $x \in M_{n} A$ to $x+(1-\underline{e}) \in M_{n} \tilde{A}$, where $\underline{e}$ denotes the diagonal matrix with diagonal entries equal to the unit $e$ of $A$.) We embed $G L_{n} A$ into $G L_{n+1} A$ as

$$
x \mapsto\left(\begin{array}{ll}
x & 0  \tag{2.1.1}\\
0 & 1
\end{array}\right)
$$

and denote by $G L_{\infty} A$ the union, or direct limit of the $G L_{n} A$, over all $n$.

This is the stable general linear group of $A$. In order to compress notation a bit, we will usually drop the symbol " $\propto$ " and write $G L$ for $G L_{\star}$.

Any *-homomorphism $f: A \rightarrow B$ induces a group homomorphism from $G L A$ to $G L B$, which we will also call $f$. Indeed, $f$ induces unital *-homomorphisms from $M_{n} \tilde{A}$ to $M_{n} \widetilde{B}$ for every $n$, and from these we obtain homomorphisms $f: G L_{n} A \rightarrow G L_{n} B$ which are compatible with the inclusions (2.1.1). In this way we make $G L$ into a functor from $C^{*}$-algebras (or for that matter, rings in general) to groups.

Observe that GLA is, in fact, a topological group: each $G L_{1 /} A$ is a topological group in the norm topology given by $A$, and we topologize $G L A$ as the direct limit of the $G L_{n} A$. Thus, a subset $X$ of $G L A$ is open (resp. closed) if for every $n$, the intersection $X \cap G L_{n} A$ is an open (resp. closed) subset of $G L_{n} A$. This topology, while being rather odd from some points of view (e.g.. it is not metrizable), has a useful property:

Lemma 2.1.1. If $X$ is a compact space then the image of any continuous map from $X$ into $G L A$ is contained in some $G L_{n} \mathcal{A}$.

The following fact is also sometimes useful.
Lemma 2.1.2. The space GLA is paracompact.
The proof of both of these lemmas are straightforward exercises in general topology. In any case, we will need neither of them in any essential way.

Of course, by its definition, the direct limit topology on GLA also has the property that a map $G L A \rightarrow X$ is continuous if and only if each of the restrictions $G L_{n} A \rightarrow X$ is continuous. Using this, we see that the homomorphism $f: G L A \rightarrow G L B$ induced from a *-homomorphism $f: A \rightarrow B$ is continuous.

Definition 2.1.3. For $n=1,2, \ldots$, the topological $K$-theory groups $K_{n}^{\prime}$ of a $C^{*}$-algebra $A$ are the homotopy groups of the topological space $G L A$ :

$$
\begin{equation*}
K_{n}^{t}(A)=\pi_{n-1}(G L A) \tag{2.1.2}
\end{equation*}
$$

So, for example, the group $K_{1}^{\prime}(A)$ is equal to the set of path components of $G L A$. This is isomorphic to the quotient group $G L A / G L^{\circ} A$, where $G L^{0} A$ denotes the path-connected component of the identity in $G L A$. For the higher groups, the base point of GLA is taken to be the identity of the group; it is reassuring to note that because of Lemma 2.1.1, $\pi_{k}(G L A)=\lim _{n \rightarrow \infty} \pi_{k}\left(G L_{n} A\right)$.

By a well-known argument, $K_{n}^{t} A$ is abelian: given two elements $x, y \in G L A$ (or maps into $G L A$ ) we have

$$
\begin{aligned}
x y & =\left(\begin{array}{ll}
x & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
y & 0 \\
0 & 1
\end{array}\right) \\
& \sim\left(\begin{array}{ll}
1 & 0 \\
0 & x
\end{array}\right)\left(\begin{array}{ll}
y & 0 \\
0 & 1
\end{array}\right) \\
& =\left(\begin{array}{ll}
y & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & x
\end{array}\right) \\
& \sim\left(\begin{array}{ll}
y & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
x & 0 \\
0 & 1
\end{array}\right) \\
& =y x,
\end{aligned}
$$

where $\left(\begin{array}{ll}x & 0 \\ 0 & 1\end{array}\right)$ is rotated to $\left(\begin{array}{ll}1 & 0 \\ 0 & x\end{array}\right)$ to obtain the above homotopies.
Recall the following notion from topology (see [50, p. 66]).
DEFINITION 2.1.4. A (continuous) map $p: E \rightarrow B$ between topological spaces is a fibration if it has the homotopy lifting property with respect to every space $X$. That is, given maps $F: I \times X \rightarrow B$ (where $I$ denotes the unit interval) and $\bar{f}: X \rightarrow E$, such that $p \bar{f}(x)=F(x, 0)$, there is a map $\bar{F}: I \times X \rightarrow E$ such that $p \bar{F}=F$ and $\bar{F}(x, 0)=\bar{f}(x)$.

Thus, the definition asserts the existence of the map $\bar{F}$ in the commutative diagram


Now, if $J$ is a (closed, two sided) ideal in a $C^{*}$-algebra $A$, then the map $G L A \rightarrow G L A / J$ is a fibration. In fact it is not difficult to show that somewhat more is true.

Theorem 2.1.5. Let $B$ denote the image of GLA in GLA/J under the map $p$ (this is an open and closed subgroup of GLA/J). Then the map $p: G L A \rightarrow B$ is a locally trivial principal GLJ-bundle.

For the definition, see, for example, [27]; for the proof see [41, Proposition 2.4]. Now, a theorem of Hurewicz (see [50, 2.8.14]) asserts
that a locally trivial fibre bundle over a paracompact base space is a fibration. It follows that the map $p: G L A \rightarrow G L A / J$ is a fibration, but rather than appeal to these results, we prove a weaker result which is sufficient for our purposes. The main reason for doing this is that we will want to compare the topological constructions given now with algebraic analogues to be discussed in later sections.

Definition 2.1.6. (See [50, p. 374].) A map $p: E \rightarrow B$ is a weak fibration if it has the homotopy lifting property as in (2.1.3) with respect to any cube $X=I^{n}$.

Lemma 2.1.7. Let $\beta^{\prime}$ be a map from $I^{m}$ to $G L A^{\prime} / J^{\prime}$. If $\beta^{\prime}(e)=1$, where e denotes the point $(0,0, \ldots, 0) \in I$, then there exists a map $\alpha^{\prime}: I^{m} \rightarrow G L A^{\prime}$ such that $p\left(\alpha^{\prime}\right)=\beta^{\prime}$.

Proof. By Lemma 2.1.1, the image of $\beta^{\prime}$ is contained in some $G L_{k} A^{\prime} / J^{\prime}$. Now, the space of maps from $I^{m}$ into $G L_{k} A^{\prime} / J^{\prime}$ is equal to $G L_{k} C\left(I^{m}, A^{\prime} / J^{\prime}\right)$. Deforming $I^{\prime \prime \prime}$ to the point $e$, and using the hypothesis $\beta^{\prime}(e)=1$, we see that $\beta^{\prime}$ is contained in the identity component of this group. So by standard Banach algebra theory, $\beta^{\prime}$ lifts to some $\alpha^{\prime} \in G L_{k} C\left(I^{\prime \prime}, A^{\prime}\right)$.

Theorem 2.1.8. The map $p: G L A \rightarrow G L A / J$ is $a$ weak fibration.
Proof. Note, first, that we may improve the previous lemma by choosing $\alpha$ such that $\alpha(e)=1$. Indeed, replace $\alpha$ with the map $x \mapsto \alpha(x) \alpha(e)$. Now, suppose we have a map $\beta: I^{n} \times I \rightarrow G L A / J$ and a lifting $\alpha_{0}: I^{n} \rightarrow G L A$ of the map $\beta(\cdot, 0): I^{n} \rightarrow G L A / J$. Apply the case $m=1$ of Lemma 2.1.7 to the algebras $A^{\prime}=C\left(I^{\prime}, A\right), J^{\prime}=C\left(I^{\prime}, J\right)$, and the map $\beta^{\prime}: I \rightarrow G L A^{\prime} / J^{\prime}$ given by $\beta^{\prime}(t)(x)=\beta(x, t) \beta(x, 0)^{-1}$. If $\alpha^{\prime}: I \rightarrow G L A^{\prime}$ is a lifting such that $\alpha^{\prime}(0)=1$ then the map $\alpha(t)=\alpha^{\prime}(t) \alpha_{0}$, considered as a map $\alpha: I^{\prime \prime} \times I \rightarrow G L A$, is the desired homotopy lifting.

Now, from any weak fibration $p: E \rightarrow B$, with fibre $F$ (which is by definition the space $F=p^{-1}\{e\}$, where $e \in B$ is the base point), we obtain a long exact homotopy sequence

$$
\begin{align*}
\cdots & \rightarrow \pi_{n}(F) \rightarrow \pi_{n}(E) \rightarrow \pi_{n}(B) \stackrel{\hat{c}}{\longrightarrow}  \tag{2.1.4}\\
& \pi_{n-1}(F) \rightarrow \pi_{n \quad 1}(E) \rightarrow \cdots .
\end{align*}
$$

For the purpose again of a later comparison with algebraic constructions, we recall the definition of the boundary map $\partial$, beginning with the simplest case: $n=1$. Given a loop $\gamma: S^{1} \rightarrow B$ defining an element $x$ of $\pi_{1}(B)$, lift it to a path $\tilde{\gamma}$ in $E$ such that $\tilde{\gamma}(0)$ is equal to the base point of $E$. Since $\gamma$ is a loop, $\tilde{\gamma}(1) \in p^{-1}\{e\}$; we define $\partial(x)$ to be the path component of $F$ contain-
ing $\tilde{\gamma}(1)$. For higher $n$ we can reduce to this case by introducing the loop space $\Omega X$. Recall that this is the space of all (base point preserving) loops $S^{1} \rightarrow X$, equipped with the compact open topology (see [50, p. 37]). There are obvious canonical isomorphisms

$$
\begin{equation*}
\pi_{n}(X) \cong \pi_{n-1}(\Omega X) \tag{2.1.5}
\end{equation*}
$$

and if $p: E \rightarrow B$ is a fibration then so is $p: \Omega E \rightarrow \Omega B$; the definition of $\hat{\sigma}: \pi_{n}(B) \rightarrow \pi_{n-1}(F)$ corresponds under these isomorphisms to the definition of $\partial: \pi_{n} \quad(\Omega B) \rightarrow \pi_{n-2}(\Omega F)$.

Returning to the spaces $G L A$, if $J$ is a closed two-sided ideal in $A$ then from the long exact homotopy sequence (2.1.4) we obtain the long exact K-theory sequence

$$
\begin{align*}
\cdots \quad & K_{n}^{\prime}(J) \rightarrow K_{n}^{\prime}(A) \rightarrow K_{n}^{t}(A / J) \xrightarrow{\partial}  \tag{2.1.6}\\
& K_{n-1}^{t}(J) \rightarrow K_{n-1}^{t}(A) \rightarrow \cdots .
\end{align*}
$$

Also, if $C_{0}((0,1), A)$ denotes the $C^{*}$-algebra of continuous functions from $[0,1]$ into $A$, which vanish at the endpoints of $[0,1]$, then by Lemma 2.1.1, $\Omega G L A=G L C_{0}((0,1), A)$. We follow convention and identify $(0,1)$ with $\mathbf{R}$, and then from (2.1.5) we obtain the isomorphism

$$
\begin{equation*}
K_{n+1}^{t}(A) \cong K_{n}^{\prime}\left(A \otimes C_{0}(\mathbf{R})\right) \tag{2.1.7}
\end{equation*}
$$

Next we list three well-known and fundamental properties of $K$-theory that will be of importance to us. See [46 or 20].

Definition 2.1.9. Two *-homomorphisms $f_{0}, f_{1}: A \rightarrow B$ are homotopic if there is a $*$-homomorphism from $A$ to $B \otimes C[0,1]$ which gives back $f_{0}$ and $f_{1}$ by composing with evaluation at the points 0 and 1 in [0,1]. A functor $F$ on $C^{*}$-algebras is homotopy invariant if $F\left(f_{0}\right)=F\left(f_{1}\right)$, whenever $f_{0}$ and $f_{1}$ are homotopic $*$-homomorphisms.

It is clear from the definition that the functors $K_{n}^{t}$ are homotopy invariant.

Definition 2.1.10. Let $e$ be a rank-one projection in $\mathscr{K}$, and if $A$ is any $C^{*}$-algebra then denote also by $e$ the $*$-homomorphism from $A$ to $\mathscr{K} \otimes A$ defined by $e(a)=e \otimes a$. A functor $F$ on $C^{*}$-algebras is stable if $F(e): F(A) \rightarrow F(\mathscr{K} \otimes A)$ is an isomorphism for every $A$.

Theorem 2.1.11. The functors $K_{n}^{t}$ are stable.
For an outline of the proof, see [46].

Definition 2.1.12. A functor $F$ from $C^{*}$-algebras to abelian groups is said to be half exact if for every short exact sequence of $C^{*}$-algebras and *-homomorphisms,

$$
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,
$$

the sequence of abelian groups $F(A) \rightarrow F(B) \rightarrow F(C)$ is exact at $F(B)$.
Given the long exact sequence (2.1.6) it is clear that the functors $K_{n}^{\prime}$ are half exact. We want to remind the reader that if $F$ is a half-exact functor which is in addition homotopy invariant, then we can build a natural long exact sequence

$$
\begin{equation*}
F_{n}(A) \rightarrow F_{n}(B) \rightarrow F_{n}(C) \stackrel{i}{\rightarrow} F_{n-1}(A) \rightarrow F_{n-1}(B) \rightarrow F_{n-1}(C), \tag{2.1.8}
\end{equation*}
$$

where $n=1,2, \ldots$, and $F_{0}=F$. The technique is borrowed from topology and for a detailed description of it, see [35, Sect. 7]. The functors $F_{n}$ are defined by $F_{n}(A)=F_{n-1}\left(A \otimes C_{0}(\mathbf{R})\right.$ ) (compare with (2.1.7)). The definition of $\partial$ involves the construction of the following auxiliary algebra.

Definition 2.1.13. The mapping cone of a *-homomorphism $f: D \rightarrow E$ is the $C^{*}$-algebra

$$
C_{f}=\left\{d \oplus \gamma \in D \oplus C_{0}(0,1] \otimes E \mid f(d)=\gamma(1)\right\} .
$$

Consider the mapping cone $C_{p}$ of the surjection $p: B \rightarrow C$. There are short exact sequences

$$
0 \rightarrow A \rightarrow C_{p} \rightarrow C_{0}(0,1] \otimes C \rightarrow 0
$$

and

$$
0 \rightarrow C_{0}(\mathbf{R}) \otimes C \rightarrow C_{p} \rightarrow B \rightarrow 0,
$$

where, for example, the inclusions of $A$ and $C_{0}(\mathbf{R}) \otimes B$ in $C_{\rho}$ are given by $a \mapsto a \oplus 0$ and $\gamma \mapsto 0 \oplus \gamma$, respectively. Now it turns out that the map $F(A) \rightarrow F\left(C_{p}\right)$ is an isomorphism (this is very plausible, in view of the fact that the quotient $C_{p} / A \cong C_{0}(0,1] \otimes C$ is contractible $)$. The boundary map is given by the composition of the inverse of this map with $F\left(C_{0}(\mathbf{R}) \otimes C\right) \rightarrow F\left(C_{p}\right)$,


We note that if $K_{n+1}^{t}(C)$ is identified with $K_{n}^{t}\left(C_{0}(\mathbf{R}) \otimes C\right)$ via the boundary map in the long exact sequence for the "path fibration"

$$
0 \rightarrow C_{0}(\mathbf{R}) \otimes C \rightarrow C_{0}(0,1] \otimes C \rightarrow C \rightarrow 0
$$

then the two available definitions for the boundary map $K_{n+1}^{t}(C) \rightarrow K_{n}^{t}(A)$ agree (compare [35, Lemma 7.6]).

To complete our discussion of topological $K$-theory, we state the main theorem of the subject-the Bott periodicity theorem.

Theorem 2.1.14. If $A$ is any $C^{*}$-algebra then for every $n$, the group $K_{n}^{t}\left(C_{0}\left(\mathbf{R}^{2}\right) \otimes A\right)$ is naturally isomorphic to $K_{n}^{t}(A)$.

The following remarkable generalization of the Bott Periodicity Theorem was discovered by Cuntz [20].

Theorem 2.1.15. Let $F$ be a homotopy invariant, stable, and half-exact functor from $C^{*}$-algebras (or separable $C^{*}$-algebras, or $\sigma$-unital $C^{*}$-algebras) to abelian groups. Then $F\left(C_{0}\left(\mathbf{R}^{2}\right) \otimes A\right)$ is naturally isomorphic to $F(A)$.

This will be proved using extension theory in the next section. We finish this section by describing the isomorphism $F\left(C_{0}\left(\mathbf{R}^{2}\right) \otimes A\right) \cong F(A)$. It will, in fact, be the boundary map associated with a short exact sequence of the form

$$
\begin{equation*}
0 \rightarrow \mathscr{K} \otimes A \rightarrow T_{0} \otimes A \rightarrow C_{0}(\mathbf{R}) \otimes A \rightarrow 0 \tag{2.1.10}
\end{equation*}
$$

which is constructed as follows. Let $T$ be the universal $C^{*}$-algebra generated by an isometry. In other words, $T$ contains a canonical isometry $v$, and the pair $(T, v)$ is characterized by the property that if $v^{\prime}$ is any isometry in a $C^{*}$-algebra $A$, then there exists a unique $*$-homomorphism $f: T \rightarrow A$ such that $f(v)=v^{\prime}$. It is easy to establish the existence of $(T, v)$ by category theory methods (we can realize $T$ as a subalgebra of the gigantic product of all (up to isomorphism) $C^{*}$-algebras generated by an isometry). On the other hand, Coburn [17] showed that all $C^{*}$-algebras generated by (non-unitary) isometries are isomorphic, and so any one will do for $T$. In particular, for example, we may take $T$ to be the Toeplitz algebra-the $C^{*}$-algebra generated by the unilateral shift. Given $(T, v)$, since the canonical unitary in $C\left(S^{1}\right)$ is in particular an isometry, we obtain a surjection $T \rightarrow C\left(S^{1}\right)$ mapping $v$ to this unitary. The kernel is, of course, generated as an ideal by the projection $1-v v^{*}$, and it is not hard to show that the kernel is $*$-isomorphic to $\mathscr{K}$, with $e=1-v v^{*}$ representing a rank one projection (see [17]). Thus we have a short exact sequence

$$
0 \rightarrow \mathscr{K} \rightarrow T \rightarrow C\left(S^{1}\right) \rightarrow 0
$$

Identifying $C_{0}(\mathbf{R})$ with the ideal in $C\left(S^{1}\right)$ of all functions which vanish at some fixed point, and letting $T_{0}$ be the pre-image of this ideal in $T$, we obtain the short exact sequence we want

$$
\begin{equation*}
0 \rightarrow \mathscr{K} \rightarrow T_{0} \rightarrow C_{0}(\mathbf{R}) \rightarrow 0 . \tag{2.1.11}
\end{equation*}
$$

We will call $T_{0}$ the reduced Toeplitz algebra. Let us verify "half" of Theorem 2.1.15.

Lemma 2.1.16. The boundary map

$$
\partial: \quad F\left(C_{0}\left(\mathbf{R}^{2}\right) \otimes A\right) \rightarrow F(\mathscr{K} \otimes A)
$$

associated with the short exact sequence (2.1.10) is surjective.
The following definition, and the next lemma, will prove to be very useful.

Definition 2.1.17. We shall call a functor $F$ from $C^{*}$-algebras to abelian groups additive if for every pair of $C^{*}$-algebras $B_{1}$ and $B_{2}$, the natural projections

$$
B_{1} \oplus B_{2} \rightarrow B_{1} \quad \text { and } \quad B_{1} \oplus B_{2} \rightarrow B_{2}
$$

induce an isomorphism $F\left(B_{1}\right) \oplus F\left(B_{2}\right) \cong F\left(B_{1} \oplus B_{2}\right)$.
Lemma 2.1.18 (Compare [20, Proposition 4.1]). Let $F$ be an additive functor and let $f_{1}, f_{2}: A \rightarrow B$ be two *-homomorphisms such that $f_{1}[A] f_{2}[A]=0$. Then $F\left(f_{1}\right)+F\left(f_{2}\right)=F\left(f_{1}+f_{2}\right)$.

Proof. Let $B_{i}=f_{i}[A]$, for $i=1,2$. We may consider $f_{1}, f_{2}$, and $f_{1}+f_{2}$ as maps into $B_{1} \oplus B_{2}$, and it suffices to prove the equation

$$
F\left(f_{1}\right)+F\left(f_{2}\right)=F\left(f_{1}+f_{2}\right)
$$

in this context. But by additivity, it suffices to prove that the two sides of the above equation are equal after we compose with the projection onto $B_{1}$ or $B_{2}$, and this much is obvious.

Proof of Lemma 2.1.16. It follows from the portion,

$$
F\left(C_{0}\left(\mathbf{R}^{2}\right) \otimes A\right) \xrightarrow{Z} F(\mathscr{K} \otimes A) \rightarrow F\left(T_{0} \otimes A\right),
$$

of the long exact sequence (2.1.8) that it suffices to show the map $F(\mathscr{K} \otimes A) \rightarrow F\left(T_{0} \otimes A\right)$ is zero. Furthermore, because of stability, and since (by virtue of the long exact sequence applied to $0 \rightarrow T_{0} \rightarrow T \leftrightarrows \mathbf{C} \rightarrow 0$ ) the
group $F\left(T_{0} \otimes A\right)$ is a summand of $F(T \otimes A)$, it suffices to show that the composite map

$$
F(A) \rightarrow F(\mathscr{K} \otimes A) \rightarrow F\left(T_{0} \otimes A\right) \rightarrow F(T \otimes A)
$$

is zero. A final simplification: we may embed $T \otimes A$ in $M_{2}(T \otimes A)$ and need only show that the resulting composition is zero. This map, call it $\bar{e}$, is given by

$$
\bar{e}(a)=\left(\begin{array}{cc}
e \otimes a & 0 \\
0 & 0
\end{array}\right) \quad(a \in A) .
$$

We may write this as $\bar{e}=\overline{1}-\operatorname{Ad}(V) \overline{1}$, where $V$ is the isometry $\left(\begin{array}{cc}\bullet \otimes 1 & 1 \\ 0 & 1 \\ 1\end{array}\right)$, and where the $*$-homomorphism $\overline{\mathrm{I}}: A \rightarrow M_{2}(T \otimes A)$ is given by

$$
\overline{\mathrm{I}}(a)=\left(\begin{array}{cc}
1 \otimes a & 0 \\
0 & 0
\end{array}\right) \quad(a \in A) .
$$

Now, rotate $V$ to the isometry $V^{\prime}=\left(\begin{array}{cc}1 & 0 \\ 0 & \bullet \otimes 1\end{array}\right)$. Since $\operatorname{Ad}\left(V^{\prime}\right) \overline{1}=\overline{1}$, it follows from homotopy invariance that $F(\operatorname{Ad}(V) \overline{1})=F(\overline{1})$. Therefore, by the preceding lemma,

$$
F(\bar{e})=F(\overline{1})-F(\operatorname{Ad}(V) \overline{1})=F(\overline{1})-F(\overline{1})=0 .
$$

### 2.2. C*-Algebra Extensions

We will start off with a number of definitions, follow with some discussion, and finish off with a proof of Theorem 2.1.15. For further information and references on the theory of $C^{*}$-algebra extensions, the reader is referred to the survey article of Rosenberg [47].

A $C^{*}$-algebra extension is a short exact sequence of $C^{*}$-algebras and *-homomorphisms.

$$
\begin{equation*}
0 \rightarrow A \xrightarrow{j} B \xrightarrow{n} C \rightarrow 0 . \tag{2.2.1}
\end{equation*}
$$

We say that (2.2.1) is an extension of C by $A$; the goal of extension theory is to classify such extensions (with $A$ and $C$ fixed) up to a suitable notion of equivalence. The extension (2.2.1) is said to be degenerate (also split) if there exists a $*$-homomorphism $s: C \rightarrow B$ such that $p s=1_{C}$. The sum of two extensions

$$
\begin{equation*}
0 \rightarrow A \rightarrow B_{i} \xrightarrow{p_{i}} C \rightarrow 0 \tag{2.2.2}
\end{equation*}
$$

(where $i=0,1$ ), is the extension

$$
\begin{equation*}
0 \rightarrow M_{2}(A) \rightarrow B \xrightarrow{p} C \rightarrow 0, \tag{2.2.3}
\end{equation*}
$$

where

$$
B=\left\{\left.\left(\begin{array}{cc}
b_{0} & a_{12} \\
a_{21} & b_{1}
\end{array}\right) \right\rvert\, b_{0} \in B_{0} ; b_{1} \in B_{1} ; a_{12}, a_{21} \in A ; \text { and } p_{0}\left(b_{0}\right)=p_{1}\left(b_{1}\right)\right\}
$$

and the $*$-homomorphism $p: B \rightarrow C$ is given by

$$
p\left(\begin{array}{cc}
b_{0} & a_{12} \\
a_{21} & b_{1}
\end{array}\right)=p_{0}\left(h_{0}\right) \quad\left(=p_{1}\left(h_{1}\right)\right) .
$$

Of course, the sum is no longer an extension of $C$ by $A$, but rather an extension of $C$ by $M_{2}(A)$. Two extensions (2.2.2) are said to be unitarily equivalent if there is a unitary $u \in \mathscr{H}(A)$ and a $*$-homomorphism $B_{0} \rightarrow B_{1}$ such that the diagram

commutes. Returning to the addition of extensions, we see that if $A$ is replaced by $\mathscr{K} \otimes A$, then since there is, up to conjugation by unitaries, a canonical isomorphism $\mathscr{K} \otimes A \rightarrow M_{2}(\mathscr{K} \otimes A)$ (namely, the tensor product of an isomorphism $\mathscr{K} \cong M_{2}(\mathscr{K})$ with id $_{A}$ ), the set of unitary equivalence classes of extensions of $C$ by $\mathscr{K} \otimes A$ has the structure of a commutative semigroup.

Before going on, we introduce the following important alternative point of view. Given an extension (2.2.1), since $B$ contains $A$ as an ideal, there is a canonical map from $B$ into $\mathscr{M}(A)$. Passing to quotients, we obtain a map from $B / A=C \rightarrow \mathscr{M}(A) / A$, and it is not hard to show (see [14]) that this sets up a one-to-one correspondence between extensions and maps from $C$ into $\mathscr{M}(A) / A$, as long as we identify extensions (2.2.2) for which there is an isomorphism $B_{0} \rightarrow \equiv B_{1}$ fixing $A$ and $C$. For most purposes it is technically more convenient to work with the map $C \rightarrow \mathscr{A}(A) / A$ than with the extension it came from. Let us note that an extension $\varphi: C \rightarrow, \mathcal{M}(A) / A$ is degenerate if $\varphi$ lifts to a $*$-homomorphism from $C$ into $\mathscr{M}(A)$; that the sum of two extensions $\varphi_{0}, \varphi_{1}$ is the extension

$$
\varphi_{0} \oplus \varphi_{1}=\left(\begin{array}{cc}
\varphi_{0} & 0 \\
0 & \varphi_{1}
\end{array}\right): \quad A \rightarrow \mathscr{M}\left(M_{2} A\right) / M_{2} A
$$

and that $\varphi_{0}$ and $\varphi_{1}$ are unitarily equivalent if there exists a unitary $u \in \mathscr{M}(A)$ which intertwines them.

Definition 2.2.1. Let $\operatorname{Ext}(C, A)$ denote the quotient of the semigroup of unitary equivalence classes of extensions $C \rightarrow \mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A$ by the sub-semigroup of unitary equivalence classes of degenerate extensions. (Thus $\operatorname{Ext}(C, A)$ is a commutative monoid, consisting of classes $[\varphi]$, where $\varphi$ is an extension; and $\left[\varphi_{0}\right]=\left[\varphi_{1}\right]$ if and only if there exist degenerate extensions $\psi_{0}$ and $\psi_{1}$ such that $\varphi_{0} \oplus \psi_{0}$ is unitarily equivalent to $\varphi_{1} \oplus \psi_{1}$.) Denote by $\operatorname{Ext}^{-1}(C, A)$ the group of invertible elements in $\operatorname{Ext}(C, A)$.

If the extensions $\varphi_{0}$ and $\varphi_{1}$ determine the same element of $\operatorname{Ext}(C, A)$, then we say that $\varphi_{0}$ and $\varphi_{1}$ are stably unitarily equivalent Let us note that the zero element of $\operatorname{Ext}(C, A)$ is the one represented by a degenerate extension (it is clear that all degenerate extensions represent the same element of $\operatorname{Ext}(C, A)$ ). Thus an extension $\varphi$ determines the zero element if and only if there is a degenerate extension $\psi$ such that $\varphi \oplus \psi$ is degenerate, in which case we say that $\varphi$ is stably split. It follows that an extension $\varphi$ determines an invertible element of $\operatorname{Ext}(C, A)$-an element of $\operatorname{Ext}^{-1}(C, A)$-if and only if there is an extension $\varphi^{\prime}$ such that $\varphi \oplus \varphi^{\prime}$ is degenerate.

The group $\operatorname{Ext}^{-1}(C, A)$ is the principal object of study in extension theory. Because $\operatorname{Ext}^{-1}(C, A)$ is a group it has an obvious advantage over the set of unitary equivalence classes, which is merely a semi-group. So it is remarkable fact that in many cases these two objects are essentially the same, and thus unitary equivalence classes of extensions are classified by the group $\operatorname{Ext}^{-1}(C, A)$. Let us sketch this. Following Arveson [3], we consider the question of when an element $[\varphi] \in \operatorname{Ext}(C, A)$ is invertible. If $[\varphi]$ is invertible then there exists an extension $\varphi^{\prime}$ such that the map

$$
\varphi \oplus \varphi^{\prime}: \quad C \rightarrow M_{2}(\mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A)
$$

lifts to some $*$-homomorphism $\psi$ from $C$ into $M_{2}(\mathscr{M}(\mathscr{K} \otimes A))$. The compression of $\psi$ by the projection $\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right) \in M_{2}(\mathscr{M}(\mathscr{K} \otimes A))$ is a completely positive, contractive map which lifts $\varphi$. Suppose, on the other hand, that $\varphi$ is an extension which lifts to a completely positive contractive map $\theta: C \rightarrow \mathscr{M}(\mathscr{K} \otimes A)$. Kasparov [34] proves the following generalization of Stinespring's theorem.

Theorem 2.2.2. If $C$ is separable and $A$ is $\sigma$-unital then every contractive, completely positive map from $C$ to $\mathscr{M}(\mathscr{K} \otimes A)$ can be dilated to a *-homomorphism from $C$ to $M_{2}(\mathscr{M}(\mathscr{K} \otimes A))$.

Therefore, assuming that $A$ is $\sigma$-unital and that $C$ is separable, we can dilate $\theta$ to a *-homomorphism $\left(\begin{array}{cc}\theta & \theta_{12} \\ \theta_{21} & \theta_{22}\end{array}\right.$. It is easy to check that $\theta_{22}$ is a *-homomorphism, modulo $\mathscr{K} \otimes A$ (since $\theta$ is) and also that $\theta_{12}$ and $\theta_{21}$ are zero, modulo $\mathscr{K} \otimes A$. It follows that $[\theta]$ is invertible, with $\theta_{22}$ determining the inverse. Hence

Theorem 2.2.3. Let $A$ be $\sigma$-unital and $C$ separable. An extension $\varphi$ is invertible (that is, it determines an invertible element of $\operatorname{Ext}(C, A))$ if and only if it has a completely positive, contractive lifting to $\mathscr{M}(\mathscr{K} \otimes A)$.

We remark that an observation of Haagerup-see [21]-shows that the hypothesis that the lifting be contractive can be dropped. In terms of short exact sequences, the existence of a completely positive contractive lifting amounts to the existence of a completely positive contractive section $s: C \rightarrow B$ (i.e., $s p=1_{C}$ ). So, for example, it follows from Theorem 1.3.6 that every extension of $C(X)$ by $\mathscr{K} \otimes A$ (where $C(X)$ is separable and $A$ is $\sigma$-unital) is invertible. More generally, Choi and Effos [16] show that if $C$ is any separable, nuclear $C^{*}$-algebra then completely positive, contractive lifting always exist, and so for such $C$, if $A$ is $\sigma$-unital then $\operatorname{Ext}(C, A)=$ $\operatorname{Ext}^{-1}(C, A)$.

Let us turn briefly to the relationship between unitary equivalence and stable unitary equivalence. First, they are not the same, since the kernel of a *-homomorphism $\varphi: C \rightarrow \mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A$ is an invariant under unitary equivalence, but not under stable unitary equivalence. Also, if $C$ happens to be unital then whether or not $\varphi$ is unital is a unitary invariant but not a stable invariant. However, at least in the special case where $A=C$, these are the only two "obstructions." The following theorem expresses this fact; it is due to Brown, Douglas, and Fillmore in the case where $C$ is abelian (see [12]) and to Voiculescu [53] in the general case; see also [4]. For a generalization to more or less arbitrary $C^{*}$-algebras $C$, where unfortunately as good a result is not possible, see [34].

Theorem 2.2.4. Let $C$ be a separable $C^{*}$-algebra and let $\varphi_{0}$, $\varphi_{1}: C \rightarrow \mathscr{B} / \mathscr{K}$ be extensions of $C$ by the compact operators which are injective (as maps into $\mathscr{B} / \mathscr{K}$ ) and unital, if $A$ is unital. They are unitarily equivalent if and only if they are stably unitarily equivalent.

The groups $\operatorname{Ext}^{-1}(C, A)$ are contravariantly functorial in the first variable: given a $*$-homomorphism $f: C \rightarrow C^{\prime}$ we define a homomorphism

$$
f^{*}: \operatorname{Ext}^{-1}\left(C^{\prime}, A\right) \rightarrow \operatorname{Ext}^{-1}(C, A)
$$

by $f^{*}[\varphi]=[\varphi f]$. Similarly, they are covariantly functorial in the second variable, with respect to quasi-unital maps $f: A \rightarrow A^{\prime}$, since these induce *-homomorphisms $f: \mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A \rightarrow \mathscr{M}\left(\mathscr{K} \otimes A^{\prime}\right) / \mathscr{K} \otimes A^{\prime}$ by 1.1.7.

We turn to some illustrations of extension theory, beginning with the relationship between extensions and the topological $K$-theory of the previous section. Any extension of $C$ by $\mathscr{K} \otimes A$ determines a map $K_{n+1}^{t}(C) \rightarrow K_{n}^{\prime}(A)$, namely the boundary map in the corresponding long
exact sequence. It is a fact that this map only depends on the stable unitary equivalence class of the extension, and we obtain a group homomorphism

$$
\operatorname{Ext}^{-1}(C, A) \rightarrow \operatorname{Hom}\left(K_{*}^{t}(C), K_{*-1}^{t}(A)\right)
$$

We shall use this result only to the following extent.
Lemma 2.2.5. Let $F$ be a stable, half exact, and homotopy invariant functor from $C^{*}$-algebras to abelian groups. If an extension is stably split then the boundary map (2.1.9) from $F_{n}(C)$ to $F_{n-1}(A)$ is zero.

Proof. If the extension

$$
0 \rightarrow A \rightarrow B_{0} \xrightarrow{p_{0}} C \rightarrow 0
$$

is stably split then we may add some degenerate extension

$$
0 \rightarrow A \rightarrow B_{1} \underset{p_{1}}{\stackrel{s_{1}}{\leftrightarrows}} C \rightarrow 0
$$

to it and obtain a degenerate extension as the result. The sum is given by the extension (2.2.3). In view of the commuting diagram

where $f(b)=\left(\begin{array}{cc}b & 0 \\ 0 & s, p_{0}(b)\end{array}\right)$ (and $s: C \rightarrow B$ is some splitting of the sum), it follows from the naturality of the boundary map in the long exact sequence that $\partial$ for the top sequence is zero, since it is zero for the bottom sequence (because it is degenerate), and since $F(A) \rightarrow F\left(M_{2} A\right)$ is an isomorphism (because $F$ is stable).

Actually, it is useful for us to strengthen this result a bit.
Lemma 2.2.6. If the extension

$$
\begin{equation*}
0 \rightarrow A \rightarrow B_{0} \xrightarrow{p_{0}} C \rightarrow 0 \tag{2.2.4}
\end{equation*}
$$

is stably split then

$$
\begin{equation*}
0 \rightarrow F_{n}(A) \rightarrow F_{n}\left(B_{0}\right) \rightarrow F_{n}(C) \rightarrow 0 \tag{2.2.5}
\end{equation*}
$$

is a split exact sequence of abelian groups.

Note that if (2.2.4) is actually split exact then it follows from the long exact sequence (2.1.8) that (2.2.5) is split exact. If (2.2.4) is only stably split then it follows from the long exact sequence, and Lemma 2.2.5, that (2.2.5) is exact everywhere except perhaps at $F(C)$. So all that needs to be proved is that the homomorphism $p_{0 *}: F\left(B_{0}\right) \rightarrow F(C)$ has a right inverse $q: F(C) \rightarrow F\left(B_{0}\right)$.

Proof. We will use the notations of the previous proof. Define two auxiliary $C^{*}$-algebras by

$$
D=\left\{\left.\left(\begin{array}{cc}
b_{0} & a_{12} \\
a_{21} & b_{1}
\end{array}\right) \right\rvert\, a_{i j} \in A \text { and } b_{i} \in B_{i}\right\}
$$

and

$$
J=\left\{\left.\left(\begin{array}{cc}
b_{0} & a_{12} \\
a_{21} & a_{22}
\end{array}\right) \right\rvert\, a_{i} \in A \text { and } b_{0} \in B_{0}\right\} .
$$

There is a split exact sequence

$$
\begin{equation*}
0 \rightarrow J \rightarrow D \underset{\pi}{\stackrel{\pi}{\leftrightarrows}} C \rightarrow 0 \tag{2.2.6}
\end{equation*}
$$

where the maps $\sigma$ and $\pi$ are defined as

$$
\sigma(c)=\left(\begin{array}{cc}
0 & 0 \\
0 & s_{1}(c)
\end{array}\right) \quad \text { and } \quad \pi\left(\begin{array}{cc}
b_{0} & a_{12} \\
a_{21} & b_{1}
\end{array}\right)=p_{1}\left(b_{1}\right) .
$$

Now, $\pi s=\mathrm{id}_{C}=\pi \sigma$, where $s: C \rightarrow B$ is a splitting of the sum (2.2.3) of the extensions

$$
0 \rightarrow A \rightarrow B_{i} \rightarrow C \rightarrow 0 \quad(i=0,1)
$$

(We are regarding $B$ as a subalgebra of $D$.) It follows that the image of the map $s_{*}-\sigma_{*}: F(C) \rightarrow F(D)$ is contained in the kernel of $\pi_{*}$, which, since (2.2.6) is a split exact sequence, is equal to the isomorphic image of $F(J)$ in $F(D)$. Denote by $j: J \rightarrow M_{2}\left(B_{0}\right)$ the inclusion map, and then define $q: F(C) \rightarrow F\left(B_{0}\right)$ to be the composition

$$
F(C) \xrightarrow{s_{*}-\sigma_{*}} F(J) \xrightarrow{j_{*}} F\left(M_{2}\left(B_{0}\right)\right) \stackrel{\cong}{\cong} F\left(B_{0}\right) .
$$

(Here and later on, an algebra is embedded in the ring of $2 \times 2$ matrices in the upper left-hand corner; by stability, if we apply the functor $F$ to this
embedding, then we obtain an isomorphism.) We claim that $q$ is right inverse to $p_{0 *}: F\left(B_{0}\right) \rightarrow F(C)$. The crucial observation is that the diagram

commutes, where $\pi^{\prime}: D \rightarrow M_{2}(C)$ is defined by

$$
\pi^{\prime}\left(\left(\begin{array}{cc}
b_{0} & a_{12} \\
a_{21} & b_{1}
\end{array}\right)\right)=\left(\begin{array}{cc}
p_{0}\left(b_{0}\right) & 0 \\
0 & p_{1}\left(b_{1}\right)
\end{array}\right) .
$$

Given this, from the commuting diagram

it follows immediately that $p_{0_{*}} q=\mathrm{id}_{F(C)}$. As for the commutativity of the above square, it follows from the definitions of $\sigma$ and $s$ that the maps $\pi^{\prime} s$, $\pi^{\prime} \sigma: C \rightarrow M_{2}(C)$ are

$$
\pi^{\prime} s(c)=\left(\begin{array}{ll}
c & 0 \\
0 & c
\end{array}\right), \quad \pi^{\prime} \sigma(c)=\left(\begin{array}{ll}
0 & 0 \\
0 & c
\end{array}\right)
$$

By Lemma 2.1.18, $\pi_{*}^{\prime} s_{*}-\pi_{*}^{\prime} \sigma_{*}=\left(\pi^{\prime} s-\pi^{\prime} \sigma\right)_{*}$, and since $\left(\pi^{\prime} s-\pi^{\prime} \sigma\right)$ is equal to the canonical embedding of $C$ into $M_{2}(C)$, this equation asserts that (2.2.7) commutes.

We close this section by proving, the Bott Periodicity Theorem (Theorem 2.1.15). The particular proof given here is an interesting example of how a topological result may be deduced from more or less purely $C^{*}$-algebraic considerations.

Lemma 2.2.7. Any extension of the reduced Toeplitz algebra $T_{0}$ by a stable $C^{*}$-algebra is stably split.

Proof. An extension $T_{0} \rightarrow \mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A$ corresponds to a unital map from $T$ into $\mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A$, and hence to an isometry
$v \in \mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A$. We must show that there exists an isometry $v_{1} \in \mathscr{M}(\mathscr{K} \otimes A)$, such that the isometry

$$
\left(\begin{array}{cc}
v & 0 \\
0 & \dot{v}_{1}
\end{array}\right) \in M_{2}(\mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A)
$$

lifts to an isometry in $M_{2}(\mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A)$. Choose $v_{1}$ such that the projection $1-v_{1} v_{1}^{*}$ is equivalent to 1 . Then since the projection $\left(\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right)$ is equivalent to the identity $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$, it follows that the projection $\left(\begin{array}{cc}0 & 0 \\ 0 & 1-v_{1} \\ v_{1}^{*}\end{array}\right)$ is equivalent to $\left(\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right)$. In other words, there exists an isometry $U \in M_{2}(\mathscr{M}(\mathscr{K} \otimes A))$ such that

$$
U U^{*}=\left(\begin{array}{cc}
0 & 0 \\
0 & 1-v_{1} v_{1}^{*}
\end{array}\right)
$$

Now, if $W$ is any lifting of $\left(\begin{array}{cc}v & 0 \\ 0 & i_{1}\end{array}\right)$ to $M_{2}(\mathscr{M}(\mathscr{K} \otimes A))$ such that $\|W\| \leqslant 1$ (for the existence of such a $W$ see $[43,(1.5 .10)])$, then the element

$$
\left(1-U U^{*}\right) W+U\left(1-W^{*}\left(1-U U^{*}\right) W\right)^{1 / 2}
$$

is an isometry in $M_{2}(\mathscr{M}(\mathscr{K} \otimes A))$ whose image in the quotient is $\left(\begin{array}{cc}1 & 0 \\ 0 & i_{1}\end{array}\right)$.
Proof of Theorem 2.1.15. We see from the portion

$$
F_{n}\left(T_{0} \otimes A\right) \rightarrow F_{n}\left(C_{0}(\mathbf{R}) \otimes A\right) \stackrel{\hat{a}}{\longrightarrow} F_{n-1}(\mathscr{K} \otimes A) \rightarrow F_{n-1}\left(T_{0} \otimes A\right),
$$

of the long exact sequence (2.1.8) that in order to show that the boundary map is an isomorphism, it suffices to show that $F_{n}\left(T_{0} \otimes A\right)=0$ for all $n$. It is convenient to suppress that $C^{*}$-algebra $A$ for the moment: let us show that $F_{*}\left(T_{0}\right)=0$. Consider the commuting diagram

where $E$ is the pullback: the $C^{*}$-subalgebra of $\mathscr{K} \otimes C_{0}(0,1] \otimes T_{0}$ which maps onto $e \otimes T_{0} \subset \mathscr{K} \otimes T_{0}$. According to the previous lemma, the top sequence is stably split, and so it follows from Lemma 2.2.6 that the corresponding sequence with $F_{n}$ applied is also split exact. In particular, $F_{n}(E)$ maps onto $F_{n}\left(T_{0}\right)$. It follows from this, and the commutativity of (2.2.8), that $F_{n}\left(\mathscr{K} \otimes C_{0}(0,1] \otimes T_{0}\right)$ maps onto $F_{n}\left(\mathscr{K} \otimes T_{0}\right)$. But $F_{n}\left(\mathscr{K} \otimes C_{0}(0,1] \otimes T_{0}\right)=0$, since the argument of $F_{n}$ is contractible; hence $F_{n}\left(\mathscr{K} \otimes T_{0}\right)=0$, and so by stability, $F_{n}\left(T_{0}\right)=0$. For the case of a general
$A$, we can take the tensor product of the $C^{*}$-algebras and *-homomorphism in diagram (2.2.8) with $A$ and $\mathrm{id}_{A}$, and then repeat the argument (the tensor product of a stably split sequence is again stably split). Alternatively, we may repeat the argument as above, but apply it to the functor $F_{*}(\cdot \otimes A)$.

Remarks. The idea behind the proof is illustrated by considering the task of showing that $K_{1}^{t}\left(T_{0}\right)=0$, or in other words, that the group $G L T_{0}$ is connected. The group $G L C_{0}(0,1] \otimes T_{0}$ is precisely the set of based paths in $G L T_{0}$, and so what we want to show is that the homomorphism $G L C_{0}(0,1] \otimes T_{0} \rightarrow G L T_{0}$, given by evaluation at the endpoint of a path, is onto. The obvious way to guarantee this is to show that the map $p: C_{0}(0,1] \otimes T_{0} \rightarrow T_{0}$ has a right inverse $s: T_{0} \rightarrow C_{0}(0,1] \otimes T_{0}$. Unfortunately such a map $s$ does not exist (as far as we know), but Lemma 2.2.7 shows that if we throw in the qualification "up to stability" where necessary, then it is very easy to find a suitable $s$. Because of the stability properties of $K$-theory, this lesser construction is sufficient.

Let us point out that instead of Lemma 2.2.6, we could have appealed to Lemma 2.1.16: indeed it follows from this lemma that the boundary map $\partial$ is onto, while Lemma 2.2.5, in conjunction with Lemma 2.2.7, is sufficient to show that $F_{n}\left(A \otimes T_{0}\right)=0$ for $n>0$ and hence that the boundary map $\partial$ is one to one.

### 2.3. Fredholm Modules

Definition 2.3.1. Let $A$ be a $C^{*}$-algebra. A Fredholm $A$-module is a triple $\left(\varphi_{+}, \varphi_{-}, F\right)$, where $\varphi_{+}$and $\varphi_{-}$are $*$-representations of $A$ on a Hilbert space $\mathscr{H}$, and $F$ is a Fredholm operator on $\mathscr{H}$ which is unitary, modulo compact operators (in other words, $F F^{*}-1$ and $F^{*} F-1$ are compact), and which essentially intertwines $\varphi_{+}$and $\varphi_{-}$, in the sense that if $a \in A$ then $F \varphi_{+}(a)-\varphi_{-}(a) F$ is a compact operator on $\mathscr{H}$.

There are a number of variations on this definition (we will encounter one below) all of which are equivalent for most practical purposes. The idea behind the definition is due to Atiyah [5]; the term "Fredholm module" is due to Connes [18].

Let us digress for a moment and make note of the close relationship which exists between Fredholm modules and $C^{*}$-algebra extensions. A Fredholm module of the form $(\varphi, \varphi, F)$ determines an extension $\varphi_{F}: C\left(S^{1}\right) \otimes A \rightarrow \mathscr{B} / \mathscr{K}$ by mapping $f \otimes a$ to $f(\dot{F}) \dot{\varphi}(a)$. By Theorem 1.3.7, this extension is invertible, since $\varphi_{F}$ has a completely positive lifting over each point of $S^{1}$, namely the $\operatorname{map} \varphi: A \rightarrow \mathscr{B}$. It is in fact possible to obtain in this manner every element of $\mathrm{Ext}^{-1}\left(C_{0}(\mathbf{R}) \otimes A, \mathbf{C}\right)$. (But note that not every element of $\operatorname{Ext}^{-1}\left(C\left(S^{1}\right) \otimes A, C\right)$ can be so obtained, since if $\alpha$ comes from a Fredholm module then $\alpha$ is mapped to 0 by the homomorphism of
extension groups induced from $A \rightarrow C\left(S^{1}\right) \otimes A$.) Furthermore; one can describe the equivalence relation on extensions of $C_{0}(\mathbf{R}) \otimes A$ by $\mathscr{K}$ in terms of Fredholm modules (see, e.g., [6]).

It is usual to denote $\operatorname{Ext}\left(C_{0}(\mathbf{R}) \otimes A, \mathbf{C}\right)$ by $\operatorname{Ext}^{0}(A, \mathbf{C})$.
Proposition 2.3.2. Let $A$ be a separable $C^{*}$-algebra. If every surjection $B \rightarrow A$ has a completely positive contractive section then $\operatorname{Ext}^{\circ}(A, \mathbf{C})$ is $a$ group.

Proof. We must show that any extension

$$
0 \rightarrow \mathscr{K} \rightarrow E \rightarrow C_{0}(\mathbf{R}) \otimes A \rightarrow 0
$$

has a completely positive, contractive section $s: C_{0}(\mathbf{R}) \otimes A \rightarrow E$. This follows from Theorem 1.3 .7 since by hypothesis each of the surjections $E \rightarrow C_{0}(\mathbf{R}) \otimes A \rightarrow A$ (where $x \in \mathbf{R}$ and $\varepsilon_{r}$ denotes evaluation at $x$ ) has a completely positive, contractive section.

The proposition applies, for instance, to the (non-reduced) $C^{*}$-algebra of a free group, and so it answers affirmatively a question of Rosenberg (see [22]) concerning whether or not $\operatorname{Ext}^{0}\left(C^{*} F_{n}, \mathrm{C}\right)$ is a group.

We turn now to another form of Definition 2.3.1.

Definition 2.3.3. A Fredholm pair for a $C^{*}$-algebra $A$ is a pair of *-representations ( $\varphi_{+}, \varphi$ ) of $A$ on a (separable) Hilbert space such that for every $a \in A$, the operator $\varphi_{+}(a)-\varphi(a)$ is compact.

A Fredholm pair is, of course, a Fredholm module, where the operator $F$ is taken to be the identity. We will close this section by indicating how to go the other way and obtain a Fredholm pair from a Fredholm module $\left(\varphi_{+}, \varphi_{-}, F\right)$. For the sake of simplicity, we will assume for now that the Fredholm operator $F$ is a partial isometry. The idea of reducing a Fredholm module is due to Cuntz, who uses it in his "quasihomomorphism" description of $K K$-theory (see [19, 20]). The utility of this construction, as we will see in Section III, is that since a Fredholm pair is just a pair of *-homomorphisms, it is relatively easy to construct a pairing between the set of Fredholm pairs and a functor on $C^{*}$-algebras, whereas general Fredholm modules are somewhat harder to deal with here.

Let $\left(\varphi_{+}, \varphi_{--}, F\right)$ be a Fredholm module such that $F$ is a partial isometry. We begin by manufacturing a Fredholm module with unitary operator. The construction is simply

$$
\left(\varphi_{+}, \varphi_{-}, F\right) \mapsto\left(\left(\begin{array}{cc}
\varphi_{+} & 0 \\
0 & 0
\end{array}\right),\left(\begin{array}{cc}
\varphi & 0 \\
0 & 0
\end{array}\right),\left(\begin{array}{cc}
F & 1-F F^{*} \\
1-F^{*} F & F^{*}
\end{array}\right)\right) .
$$

The essential features of the old and new modules are the same. For example, they determine the same element of $\operatorname{Ext}^{\circ}(A, C)$. Also, if $F$ was unitary to start with then the manufactured Fredholm module is a "direct sum" of $\left(\varphi_{+}, \varphi_{-}, F\right)$ with the "trivial module" $\left(0,0, F^{*}\right)$. Let us denote the new module by $\left(\varphi_{+}^{\prime}, \varphi_{-}^{\prime}, F^{\prime}\right)$. Then the Fredholm pair we wish to associate with $\left(\varphi_{+}^{\prime}, \varphi_{-}^{\prime}, F^{\prime}\right)$ is $\left(\varphi_{+}^{\prime}, \operatorname{Ad}\left(F^{*}\right) \varphi_{-}\right)$.

### 2.4. Low Dimensional Algebraic $K$-Theory

In this section we introduce the algebraic $K$-theory functors $K_{1}$ and $K_{2}$, for which there exist elegant, conceptual, and (as opposed to the higher $K$-theory functors) simple definitions. Our point of view, a reasonably standard one, is that the algebraic $K$-theory groups provide an algebraic analog of the homotopy theory of the stable general linear group of a ring. This is obviously convenient for our purposes since it allows for a direct comparison with topological $K$-theory, as defined in Section 2.1. Furthermore it is consistent with, and strengthened by, Quillen's definition of higher $K$-theory, to be discussed in Section 2.6.

Most of the material presented here is available in Milnor's book [40], except that Milnor treats, for the most part, only the case of unital rings. We will indicate what modifications are necessary as we go along. Usually only very minor changes need be made to the proofs, but we will encounter some rather more delicate points in Section IV.

Throughout the section, unless otherwise specified, $A, B, C, \ldots$ will denote discrete rings (i.e., rings with no topological structure assumed), not necessarily with units. The stable general linear group, $G L A$, of $A$ is defined just as in the $C^{*}$-algebra case considered in Section 2.1.

Definition 2.4.1. Let $a \in A$ and let $i$ and $j$ be distinct indices. The elementary matrix $e_{i j}^{i j}$ is the element of $G L A$ which is equal to the identity, except for the element $a$ in position ( $i, j$ ). Denote by $E A$ the subgroup of $G L A$ generated by all the elementary matrices.

Thus, for example, $e_{12}^{a}=\left(\begin{array}{ll}1 & a \\ 0 & 1 \\ 1\end{array}\right)$. It is a simple matter to verify the relations amongst the elementary matrices,

$$
\begin{align*}
e_{i j}^{a} e_{i j}^{b} & =e_{i j}^{a+b}, & &  \tag{2.4.1}\\
{\left[e_{i j}^{u}, e_{j l}^{b}\right] } & =e_{i l}^{a b} & & \text { if } \quad i \neq l,  \tag{2.4.2}\\
{\left[e_{i j}^{u}, e_{k l}^{b}\right] } & =1 & & \text { if } \quad j \neq k \quad \text { and } \quad i \neq l . \tag{2.4.3}
\end{align*}
$$

(Here, $[x, y]$ denotes the multiplicative commutator $x y x^{-1} y^{-1}$.) Relation (2.4.1) shows that each $e_{i j}^{a}$ is indeed an invertible matrix, since $\left(e_{i j}^{a}\right)^{-1}=e_{i j}^{-a}$. Relation (2.4.2), along with (2.4.1), shows that if $A^{2}=A$ then $E A$ is equal to its commutator subgroup [ $E A, E A$ ].

The following very useful result is known as the Whitehead lemma. (Actually, the usual Whitehead lemma deals with unital rings: for the proof of the marginally stronger result given here see [52].)

Theorem 2.4.2. Let $A$ be a ring such that $A^{2}=A$.
(i) Every element of the form

$$
\left(\begin{array}{cc}
x & 0 \\
0 & x^{11}
\end{array}\right) \in G L_{2 n} A
$$

is contained in EA.
(ii) The group EA is equal to the commutator subgroup [GLA, GLA] of GLA.

Definition 2.4.3. The group $K_{1}(A)$ is the quotient $G L A /[G L A, G L A]$.
All the rings that we deal with will meet the requirement $A^{2}=A$, and so by the Whitehead lemma, $K_{1} A=G L A / E A$. From our point of view, the idea behind Definition 2.4 .3 is as follows. First, the $e_{i j}^{a}$ should be thought of as being in some sense "algebraically connected to the identity"-for instance, if $t$ is an indeterminate then $e_{i j}^{t a}$ forms a "path" from $e_{i j}^{u}$ to the identity. (This idea is considered in more detail in Section VI, where the Karoubi-Villamayor $K$-theory is discussed.) More importantly, it follows from the Whitehead lemma that EA is the maximal perfect subgroup of GLA:

Definition 2.4.4. A group $G$ is said to be perfect if it is equal to its own commutator subgroup

$$
G=[G, G] .
$$

If $G$ is any group then the subgroup of $G$ generated by the union of all the perfect subgroups of $G$ is itself a perfect group. It is denoted $P G$, and called the maximal perfect subgroup of $G$.

The terminology "maximal" is standard but a bit misleading since the maximal perfect subgroup $P G$ of a group $G$ is by definition the largest-that is, maximum, not maximal-perfect subgroup of $G$.

This concept is very important in $K$-theory, particularly in the Quillen $K$-theory to be discussed presently. It is very profitable to think of $P G$ as the algebraic analog of the connected component of the identity in a topological group. This will become clearer when we discuss $K_{2}$, in a moment; we might note now that the maximal perfect subgroup has a number of advantages over say the commutator subgroup when it comes to
playing this role: for example, the "connected component" so defined will be connected.

The group $K_{1} A$ is obviously abelian, and functorial in $A$. It is also halfexact.

Proposition 2.4.5. If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of rings and if $C^{2}=C$ then the sequence

$$
K_{1} A \rightarrow K_{1} B \rightarrow K_{1} C
$$

is exact at $K_{1} B$.
Proof (Compare [40, Sect. 4]). If $X \in G L B$, and if $X$ maps to the zero element of $K_{1}(C)$, then the image of $X$ in $G L C$ is a product of commutators, and so, by the Whitehead lemma it is a product of elementary matrices. Since $E B$ clearly maps onto $E C$, we may modify $X$, without altering the element in $K_{1} B$ it defines, so that $X$ projects to the identity element in $G L C$. But then $X$ so altered is an element of $G L A$

Now, let $A$ be a $C^{*}$-algebra. Then every elementary matrix $e_{i j}^{a}$ is path connected to the identity, and so $E A \subset G L^{0} A$. It follows that there is a canonical map

$$
\alpha: \quad G L A / E A \rightarrow G L A / G L^{0} A
$$

or, in other words, a canonical map $\alpha: K_{1} A \rightarrow K_{1}^{t} A$. This map is, of course, always onto, but it need not be one to one: for example, $K_{1} \mathbf{C}=\mathrm{C}^{*}$ (the multiplicative group of $\mathbf{C}$ ) but $K_{1}^{\prime} \mathbf{C}=0$. However, $\alpha$ is an isomorphism if $A$ is a stable $C^{*}$-algebra. The result is due to de la Harpe and Skandalis [24], although various similar results predate it (see, for example, [13]).

Theorem 2.4.6. If $A$ is any $C^{*}$-algebra then the homomorphism

$$
\alpha: \quad K_{1}(\mathscr{K} \otimes A) \rightarrow K_{1}^{\prime}(\mathscr{K} \otimes A)
$$

is an isomorphism.
The theorem follows immediately from the following computation.
Theorem 2.4.7. If $x \in G L_{n}(\mathscr{K} \otimes A)$ and $x$ is connected to the identity then the matrix $\left(\begin{array}{cc}x & 0 \\ 0 & 1\end{array}\right) \in G L_{2 n}(\mathscr{K} \otimes A)$ is a product of commutators.

Proof. Since $M_{n}(\mathscr{K} \otimes A) \cong \mathscr{K} \otimes A$, we may as well assume that $n=1$ (which simplifies the notation a bit). Also, since $x$ is connected to the identity, it follows from elementary Banach algebra theory that it is a product of exponentials $e^{y}$. It suffices to show that an exponential $x=e^{y}$, where
$y \in \mathscr{K} \otimes A$, is a product of commutators. For the proof of this we use the following infinite dimensions trick, due to Pearcy and Topping [42]. Write $\left(\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right)$ as an infinite sum of orthogonal projections, each equivalent to $\left(\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right)$. Then we may express $\left(\begin{array}{cc}x & 0 \\ 0 & 1\end{array}\right)$ as the product of two matrices, $X_{1}, X_{2}$, both infinite diagonals with respect to this decomposition of $\left(\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right)$, where the diagonal of $X_{1}$ is the sequence

$$
\begin{equation*}
(x, x^{-1 / 2}, x^{-1 / 2}, \underbrace{x^{1 / 4}, x^{1 / 4}, x^{1 / 4}, x^{1 / 4}, \underbrace{x^{-1 / 8}, x^{-1 / 8}, \ldots, x^{-1 / 8}}_{8 \text { imes }}, x^{1 / 16}, \ldots))(\underbrace{2})}_{4 \text { times }} \tag{2.4.4}
\end{equation*}
$$

and where the diagonal of $X_{2}$ is the sequence

$$
\begin{equation*}
(1, x^{1 / 2}, x^{1 / 2}, \underbrace{x^{-1 / 4}, x^{1 / 4}, x^{-1 / 4}, x^{-1 / 4}}_{4 \text { times }}, \underbrace{x^{1 / 8}, x^{1 / 8}, \ldots, x^{1 / 8}}_{8 \text { times }}, x^{-1 / 16}, \ldots) \tag{2.4.5}
\end{equation*}
$$

Now, both $X_{1}$ and $X_{2}$ are of the form

$$
\left(\begin{array}{ccc}
w^{2} & 0 & 0 \\
0 & w^{1} & 0 \\
0 & 0 & w^{-1}
\end{array}\right)=\left(\begin{array}{ccc}
w^{1} & 0 & 0 \\
0 & w^{-1} & 0 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{ccc}
w^{1} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & w^{-1}
\end{array}\right),
$$

after suitable rearrangement. It follows from the Whitehead lemma that both $X_{1}$ and $X_{2}$ are products of commutators.

Let us pass on to the definition of the algebraic $K_{2}$-group of a ring. Consider first the topological $K$-theory group $K_{2}^{t}(A)$ (where $A$ is a $C^{*}$-algebra). By definition, this is the fundamental group of GLA. We may obtain this as the kernel of the projection from the universal covering group, $\widetilde{G L^{0}} A$, onto $G L^{0} A$. Recall that, as a topological space $\widetilde{G L^{0}} A$ is the connected, simply connected cover of $G L^{\circ} A$, elements of which are fixed endpoint homotopy classes of paths in $G L^{0} A$, based at the identity element $1 \in G L^{0} A$; the projection $\widetilde{G L^{0}} A \rightarrow G L^{0} A$ maps a class of paths to its endpoint. The group structure on $\widetilde{G L^{0}} A$ is inherited from $G L^{0} A$ : multiplication of paths is carried out pointwise, and this passes to homotopy classes. By construction, the kernel of the projection from $\widetilde{G L^{0}} A$ to $G L^{0} A$ is the set of homotopy classes of paths in $G L^{0} A$ which begin and end at $1 \in G L^{0} A$ or, other words, it is the fundamental group $\pi_{1}\left(G L^{\circ} A\right)\left(=\pi_{1}(G L A)\right)$. It is well known that the group structure on $\pi_{1}(G L A)$ given by pointwise multiplication of loops, is the same as the usual one on $\pi_{1}$ given by concatenation of loops. So we obtain the extension of groups

$$
\begin{equation*}
1 \rightarrow \pi_{1}(G L A) \rightarrow \widetilde{G L_{0}} A \rightarrow G L_{0} A \rightarrow 1 . \tag{2.4.6}
\end{equation*}
$$

(For a detailed discussion of covering spaces, the reader is referred to [50, Chap. 2].)

Definition 2.4.8. An extension of groups

$$
1 \rightarrow N \rightarrow G / N \rightarrow 1
$$

is said to be a central extension if $N$ is contained in the center of $G$.
It is easy to see that (2.4.6) is a central extension. The definition of the algebraic $K_{2}$, due to Milnor, produces $K_{2}$ as the kernel of a central extension analogous to (2.4.6).

Definition 2.4.9. The Steinberg group StA is the group generated by the symbols $x_{i j}^{c}$, where $i$ and $j$ are distinct natural number indices, and $a \in A$; subject to the Steinberg relations

$$
\begin{array}{rlrl}
x_{i j}^{a} x_{i j}^{b} & =x_{i j}^{a+b}, & \\
{\left[x_{i j}^{a}, x_{j l}^{b}\right]} & =x_{i l}^{a b} \quad & \text { if } \quad i \neq l, \\
{\left[x_{i j}^{a}, x_{k l}^{b}\right]} & =1 \quad \text { if } j \neq k \quad \text { and } \quad i \neq l . \tag{2.4.9}
\end{array}
$$

The generators $x_{i j}^{a}$ are obviously intended to correspond to the elementary matrices $e_{i j}^{a}$. In view of relations (2.4.1), (2.4.2), and (2.4.3), there is a group homomorphism $\pi: S t A \rightarrow E A$ such that $\pi\left(x_{i j}^{a}\right)=e_{i j}^{a}$. Just as $e_{i j}^{a}$ can be thought of as a matrix "algebraically connected to the identity," we can think of $x_{i j}^{a}$ as representing the "canonical path" $e_{i j}^{t a}$ connecting $e_{i j}^{a}$ to the identity. Indeed, suppose that $A$ is a $C^{*}$-algebra. Then of course $E A$ is contained in $G L^{0} A$, and we may lift this to a map from St $A$ to $G L^{0} A$, to obtain the commuting diagram


The map is given by sending $x_{i j}^{a}$ to the path $t \mapsto e_{i j}^{t a}$, where $t \in[0,1]$. It is ready verified that for each of the Steinberg relations, the right-hand side of the relation gives a path which is homotopic to that given by the left-hand side; so we obtain a group homomorphism. Seen in this light, an element of the kernel of $\pi: S t A \rightarrow E A$ is a "loop" based at $1 \in E A$, and so the following definition is natural.

Definition 2.4.10. The group $K_{2} A$ is defined to be the kernel of the homomorphism $\pi:$ StA $\rightarrow E A$.

By the above, if $A$ is a $C^{*}$-algebra then we obtain a canonical homomorphism $\alpha: K_{2} A \rightarrow K_{2}^{t} A$ :


This map is, in general, neither onto nor one to one: for example, $K_{2} \mathbf{C}$ is a rational vector space (of uncountable dimension), but $K_{2}^{t} \mathbf{C}=Z$. Therefore $\alpha=0$ in this case.

Proposition 2.4.11. If $A$ is a $C^{*}$-algebra then the Steinberg extension

$$
\begin{equation*}
K_{2} A \rightarrow S t A \rightarrow E A \tag{2.4.11}
\end{equation*}
$$

is a central extension, and in fact $K_{2} A$ is the center of StA.
Proof. The proof of Theorem 5.1 in [40], which treats the same result but for a unital ring, may be applied, mutatis mutandis. The hypotheses of our proposition are much stronger than necessary. What we need is that $A^{2}=A$ and for every $a \in A$, if $a \neq 0$ then $a \cdot A \neq 0 \neq A \cdot a$.

Definition 2.4.12. A central extension $U \rightarrow G$ is universal if, for every central extension $H \rightarrow G$, there is a unique homomorphism $f: U \rightarrow H$ over $G$; that is, such that the diagram

is commutative.
Clearly, there is a unique universal central extension up to canonical isomorphism, assuming that one exists at all. (It turns out that it exists if and only if $G$ is perfect--see [40, Theorem 5.7].)

Having made the link between covering groups in the topological category and central extensions in the discrete case, the following important theorem, due to Kervaire and Steinberg, obviously greatly bolsters Definition 2.4.10.

Theorem 2.4.13. If A is a unital ring then the Steinberg extension is the universal central extension of $E A$.

In Section IV we will prove that the same result is true if $A$ is a $C^{*}$-algebra. Thus in some sense, if $A$ is a $C^{*}$-algebra, then Definition 2.4.9 is the "right" one for $K_{2} A$.

We finish this section with a discussion of the functoriality of $K_{2} A$. A ring homomorphism $f: A \rightarrow B$ induces, in an obvious way, maps from $K_{2} A$, St $A$, and $E A$, to $K_{2} B, S t B$, and $E B$, respectively: given a generator $x_{i j}^{u}$ of St $A$, map it to $x_{i j}^{f(a)}$, and so on.

Lemma 2.4.14. Let $A$ be a ring such that $A^{2}=A$ and let

$$
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
$$

be a short exact sequence of rings. Then the sequence

$$
S t A \rightarrow S t B \rightarrow S t C
$$

is exact at StB.
Note that the map $S t B \rightarrow S t C$ is onto, since each generator $x_{i j}^{c}$ of $S t C$ has a pre-image in $S t B$.

Proof. (Compare [40, Lemma 6.1].) The kernel of the homomorphism $S t B \rightarrow S t C$ is equal to the normal subgroup generated by the image of $S t A$. This is because by adjoining the relations $x_{i j}^{a}=1(a \in A)$ to those for $\operatorname{StB}$, we obtain the relations for $S t C$. So we need only show that the image of St $A$ in $S t B$ is a normal subgroup, and for this, it suffices to show that if $x_{i j}^{a}$ is a generator for $S t A$ and $x_{k l}^{b}$ is a generator for $S t B$, then $x_{k l}^{b} x_{i j}^{a} s_{k l}^{-b}$ is an element of the image of $S t A$ in $S t B$. Unless $k=j$ and $i=l$, this is clear from the Steinberg relations. If $k=j$ and $i=l$, then choose an index $h$ distinct from $i, j, k$, and $l$, and using the fact that $A^{2}=A$, write $x_{i j}^{i}$ as a product of commutators [ $x_{i h}^{u_{1}}, x_{h j}^{u_{2}}$ ]. Since conjugation with $x_{k l}^{b}$ is an automorphism of StB, it follows from the fact that each $x_{k l}^{b} x_{i h}^{a_{1}} x_{k l}^{-b}$ and each $x_{k l}^{b} x_{h j}^{a_{2}} x_{k l}^{-b}$ is in the image of StA that the commutator

$$
\left[x_{k l}^{b} x_{i h}^{a_{1}} x_{k l}^{-b}, x_{k l}^{b} x_{h j}^{a_{2}} x_{k l}^{-b}\right]=x_{k l}^{b}\left[x_{i h}^{a_{1}}, x_{h j}^{a_{2}}\right] x_{k l}^{-b}
$$

is as well. From this we get that $x_{k l}^{b} x_{i j}^{a} x_{k l}^{-b}$ is an element of the image, as required.

Theorem 2.4.15. (Compare [40, Theorem 6.2].) $A$ short exact sequence of rings $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, all of which satisfy $R^{2}=R$, gives rise to a long exact sequence

$$
K_{2}(A) \rightarrow K_{2}(B) \rightarrow K_{2}(C) \xrightarrow{a} K_{1}(A) \rightarrow K_{1}(B) \rightarrow K_{1}(C) .
$$

Proof. In the diagram

the rows are exact by the definitions of $K_{1}$ and $K_{2}$. The rightmost column of maps is exact by 2.4.5; the next by definition; the next by Lemma 2.4.14; and the final one by the definition of $K_{2}$ and the exactness of the rest of the diagram. The boundary map $\delta: K_{2} C \rightarrow K_{1} A$ is defined in the usual fashion from a diagram of this sort: given $x \in K_{2} C$, lift it to $x^{\prime} \in S t B$, map it next to $x^{\prime \prime} \in G L B$; in fact, $x^{\prime \prime}$ is an element of $G L A \subset G L B$, and so it determines an element of the quotient $G L A / E A=K_{1} A$. The usual diagram chasing shows that $\partial$ is well defined and that the sequence of $K$-theory groups is exact.
Let us note that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of $C^{*}$-algebras and $*$-homomorphism then the diagram

commutes, where the maps $\alpha$ are defined prior to Theorem 2.4 .6 and in (2.4.11). In both the topological and the algebraic case, $\partial$ is defined by lifting a loop $\alpha$ in GLC to a path in GLB starting at the identity and then taking $\partial([\alpha])$ to be the class in $K_{1} A$ of the free endpoint of the path.

### 2.5. Some Results from Algebraic Topology

If $G$ is any topological group then we may associate to it a classifying space, $B G$ (see [27]), so called because for any space $X$, the set of homotopy classes of maps from $X$ to $B G$ is in one-to-one correspondence
with the set of principal $G$-bundles over $X$ (modulo technical conditions which come into effect if the space $X$ is not paracompact). If $G$ is discrete, then the classifying space $B G$ forms an important link between algebra and topology, and as such, it plays an important role in algebraic $K$-theory, and also in the comparison of algebraic and topological $K$-theory. It is clear from the definition of $B G$ as a space which classifies $G$-bundles that $B G$ is defined only up to homotopy type. This is sometimes convenient since it allows us to choose particular realizations of $B G$ for various particular purposes. It will, however, be useful to have a single standard model for $B G$; and so we review the "infinite join model" of $B G$, due to Milnor [38]. For full details the reader is referred to [27, Sect. 4.11].
Abstractly, if $Y$ is any space and $E$ is a locally trivial principal $G$-bundle over $Y$ which as an ordinary topological space is contractible, then $Y$ is a model for the classifying space of $G$. The correspondence between homotopy classes of maps from $X$ into $Y$ and bundles over $X$ takes a map $f: X \rightarrow Y$ to the bundle over $X$ pulled back from the bundle $E$ via the map $f$. (Strictly speaking, any model for $B G$ should come equipped with a distinguished $G$-bundle over it so that a particular isomorphism of $[X, B G]$ with $G$-bundles over $X$ is specified, and hence a particular homotopy equivalence with any other model is determined. When we construct different models for $B G$, they will be spaces obtained from a standard model, say the one below, by means of some geometric construction, and so this requirement is satisfied.) We begin then with a particular contractible $G$-space. This is $E G$, defined to be the set of all sequences

$$
x=\left\{t_{0} g_{0}, t_{1} g_{1}, t_{2} g_{2}, \ldots\right\}
$$

where $g_{i} \in G, t_{i} \in[0,1]$, subject to the conditions that $t_{i}$ be zero for almost all $i$, and $\sum_{i=0}^{\infty} t_{t}=1$. (If $t=0$ then we identify any two $t g$, $t h$, where $g$, $h \in G$.) We give $E G$ the weak topology from the coordinate maps $x \mapsto t_{i}$ and $x \mapsto g_{i}$ (see [27]), and let $G$ act on the left in the obvious manner:

$$
g \cdot x=\left\{t_{0} g g_{0}, t_{1} g g_{1}, \ldots\right\}
$$

It is not hard to show that $E G$ is contractible, and that the quotient mapping $E G \rightarrow G \backslash E G$ is a principal $G$-bundle. So we define $B G$ to be the quotient space $G \backslash E G$.

Let us note here one obvious advantage of this construction: it is functorial, in the sense that a continuous group homomorphism induces a natural map on classifying spaces.

Consider for a moment the case of a discrete group $G$. In this case, $E G$ is a covering space of $B G$, with covering group $G$. Since $E G$ is a contractible, it follows from the long exact homotopy sequence for the fibration
$G \rightarrow E G \rightarrow B G$ that the homotopy groups of $B G$ are zero in all dimensions except for $\pi_{1}$, and $\pi_{1}(B G)=G$. In the discrete case it is sometimes convenient to realize $E G$ as an infinite simplicial complex, as follows. The $p$-simplices of $E G$ are all the ordered ( $p+1$ )-tuples ( $g_{0}, \ldots, g_{p}$ ) of elements in $G$. This is a contractible space and the group $G$ acts properly discontinuously on $E G$ by left multiplication:

$$
g \cdot\left(g_{0}, \ldots, g_{p}\right)=\left(g g_{0}, \ldots, g g_{p}\right)
$$

One advantage of this is that it allows us to directly identify the homology of $B G, H_{*}(B G)$, with the Eilenberg-MacLane homology, $H_{*}(G)$, of the group $G$. For a treatment of the homology of groups, see, for example, [37, Chap. IV]. We do not need to say much about it here; we will frequently write $H_{*}(G)$, but if the reader likes he can view this simply as $H_{*}(B G)$ with $B G$ constructed say as above (since all models for $B G$ are homotopy equivalent they have the same homology). We mention it only to stress the purely algebraic nature of $H_{*}(B G)$, and because we will use the following two facts, which are perhaps best seen from this point of view.

Lemma 2.5.1. If $g \in G$ then the inner automorphism $\operatorname{Ad}(g)$ of $G$ induces the identity map on $H_{*}(G)$.

Lemma 2.5.2. If $G$ is the direct limit of a directed system of groups $\left\{G_{x}\right\}$ then

$$
H_{*}(G)=\underline{\varliminf} H_{*}\left(G_{z}\right) .
$$

For proofs the reader is referred to [37].
Let us make one or two remarks about the classifying space in the opposite case of $G L A$ considered as a topological group. First, there is a close relationship between the classifying space of a group and the loop space $\Omega G L A$. Indeed, from the short exact sequence of $C^{*}$-algebras

$$
0 \rightarrow C_{0}(\mathbf{R}) \otimes A \rightarrow C_{0}(0,1] \otimes A \rightarrow A \rightarrow 0
$$

we get a principal $G L C_{0}(\mathbf{R}) \otimes A$ bundle

$$
G L C_{0}(0,1] \otimes A \rightarrow G L^{0} A
$$

Since the total space is contractible, it follows that $G L^{0} A$ is a model for the classifying space of $G L C_{0}(\mathbf{R}) \otimes A$. But $G L C_{0}(\mathbf{R}) \otimes A=\Omega G L A$, and so we get that

$$
B \Omega G L A=G L^{0} A
$$

Similarly, it is not hard to show that

$$
\Omega B G \cong G
$$

and so $B$ is more or less inverse to $\Omega$. This is sometimes a useful point of view. Let us note, for example, that by virtue of the above equalities, $\pi_{n}(B G L A)=\pi_{n-1}(G L A)$, and so $K_{n}^{\prime}(A)=\pi_{n}(B G L A)$.

The second point concerns periodicity. It follows from the periodicity theorem that

$$
G L T_{0} \otimes A \rightarrow G L^{0} C_{0}(\mathbf{R}) \otimes A
$$

(where $T_{0}$ denotes the reduced Toeplitz algebra) is a principal $G L \mathscr{K} \otimes A$ bundle with contractible total space. Therefore, the base space is a model for the classifying space of $G L \mathscr{K} \otimes A$. Now, it follows from the stability of topological $K$-theory that the inclusion $G L A \rightarrow G L \mathscr{K} \otimes A$ is a homotopy equivalence; therefore $B G L A \rightarrow B G L \mathscr{K} \otimes A$ is a homotopy equivalence, and so

$$
B G L A \cong(\Omega G L A)^{0}
$$

where ( $)^{0}$ denotes the connected component of the base point. This shows that for $G L A$, the classifying space depends only on the structure of $G L A$ as a lopological space.

Let us move on to the next topic. We will encounter the following situation quite frequently: a map $f: X \rightarrow Y$ to which we wish to apply the long exact homotopy sequence, as if $f$ were a fibration. Of course, $f$ may not be a fibration, but there is a well-known way of making it one, up to homotopy (see [50, p. 99]): denote by $P_{f}$ the space of pairs $(x, \gamma)$, where $x \in X$ and $\gamma$ is a free path in $Y$ (i.e., $\gamma$ does not necessarily begin at the base point of $Y$ ) such that $\gamma(1)=f(x)$. There is a commutative diagram

where $s$ maps $x$ to the pair $(x, f(x))$ (here $f(x)$ denotes the path which is constantly equal to $f(x)$ ), and $p(x, \gamma)=\gamma(0)$. The map $s: X \rightarrow P_{f}$ is a homotopy equivalence-the projection $(x, \gamma) \mapsto x$ is a homotopy inver-se-and $p: P_{f} \rightarrow Y$ is a fibration. The fiber of $p$ will be denoted by $F_{f}$; it is called the homotopy fiber of $f: X \rightarrow Y$. Let us note some of the properties of this construction:
(2.5.2) It is obviously functorial, in the sense that from a commutative diagram

we obtain the commutative diagram

where $\tilde{g}(x, \gamma)=(g(x), h \gamma)$.
(2.5.3) Given a sequence of spaces and maps $Z \rightarrow{ }^{g} X \rightarrow{ }^{f} Y$, if the composite $f g: Z \rightarrow Y$ is homotopic to a constant map, then $g: Z \rightarrow X$ factors through the homotopy fibre of $f$. Indeed, choose a homotopy $H(z, t)$ connecting $f g$ to the constant map onto the base point of $Y$. Then we can define $Z \rightarrow F_{f}$ by $z \mapsto(g(z), H(z, \cdot))$. We obtain a diagram

which commutes up to homotopy. We will say that

$$
Z \xrightarrow{g} X \xrightarrow{\text { I }} Y
$$

is a fibration, up to homotopy, if $Z \rightarrow F_{f}$ is a homotopy equivalence for some choice of the homotopy $H$ (we note that $Z \rightarrow F_{f}$ may depend on this choice of $H$ ).

Next we must say a few words about the type of spaces we are going to be dealing with. All spaces will be assumed to be either $C W$-complexes (see [50]) or spaces with homotopy type of a CW-complex. The advantage of working in this category is due mostly to the following well-known, and useful fact: if a map $f: X \rightarrow Y$ between such spaces induces isomorphisms of all homotopy groups then $f$ is a homotopy equivalence. By a theorem of Milnor [39], if $X$ and $Y$ have the homotopy types of $C W$-complexes, and $f: X \rightarrow Y$ is any map, then $P_{f}$ and $F_{f}$ have the homotopy types of
$C W$-complexes as well. Also, if $X$ is an open paracompact subset of a locally convex topological vector space, then $X$ has the homotopy type of a $C W$-complex (see [39]). This applies to the spaces $G L A$, where $A$ is a $C^{*}$-algebra, and so justifies the remarks made above in connection with classifying spaces. For example, from the fact that $G L T_{0} \otimes A$ has trivial homotopy we can deduce that it is a contractible space.

Now, getting back to the construction (2.5.1), checking fundamental groups reveals immediately that if $N \rightarrow G \rightarrow G / N$ is an extension of (discrete) groups then

$$
\begin{equation*}
B N \rightarrow B G \rightarrow B G / N \tag{2.5.4}
\end{equation*}
$$

is a fibration, up to homotopy (indeed, $\pi_{1}$ is the only non-zero homotopy group of $B N$ and the homotopy fiber $F$, and since it is easily checked that $\pi_{1}(B N) \rightarrow \pi_{1}(F)$ is an isomorphism, $B N \rightarrow F$ is a homotopy equivalence).

Suppose that $F \rightarrow E \rightarrow B$ is a fibration. Recall (or see, e.g., [50, p. 476]) that $\pi_{1}(B)$ acts on $H_{*}(F)$, as follows: given a loop $\gamma:[0,1] \rightarrow B$ (i.e., $\gamma(0)=e=\gamma(1))$ we solve the homotopy lifting problem

obtaining a map $\Gamma: F \times[0,1] \rightarrow E$. Since $\tilde{\gamma}$ maps $F \times\{1\}$ to the base point of $B, \Gamma$ maps $F \times\{1\}$ into $F$. This map-call it $\Gamma_{1}: F \rightarrow F$-is well defined up to homotopy, and $\Gamma_{1 *}: H_{*}(F) \rightarrow H_{*}(F)$ defines the action of $\gamma$ on $H_{*}(F)$. Consider, for example, the fibration associated with the fiber squence (2.5.4). An extremely useful, and well-known fact (see, e.g., [54, p. 356]) is this:
(2.5.5) Let $B N$ denote the Milnor model of the classifiying space (we specify this so that $B N$ is obviously functorial). The action of $\pi_{1}(B G / N)=G / N$ on $H_{*}(B N)$ is given by the action of $G / N$ on $N$ (and hence on $B N$ ) by conjugation: $n \mapsto g n g^{-1}$. (It.is well defined on the level of homology.)

The proof is a straightforward direct verification. Here is a related result (also well known, also easy to prove directly):
(2.5.6) If $N$ is a normal subgroup of $G$ then a check of homotopy groups reveals that $B N$ is homotopy equivalent to the covering of $G$, with group $G / N$. The action of $G / N$ on $H_{*}(B N)$ by deck transformations is equal to the action by conjugation.

We will define the algebraic $K$-theory of a ring $R$ to be the homotopy of a certain space associated with $R$. It is preferable to work with homotopy (rather than homology) because decomposing $R$ tends to lead to fibrations of the associated space, and by virtue of the long exact sequence, fibrations are quite amenable to study by homotopy groups. However, the space associated with $R$ is in many ways more accessible through homology than through homotopy. So it is important to have at our disposal theorems which compare homotopy and homology. The following famous result is of this sort.

Theorem 2.5 .3 (Whitehead theorem, see [50, p. 399]). Let $X$ and $Y$ be simply connected spaces. If $f: X \rightarrow Y$ is a map which induces an isomorphism of homology groups then $f$ induces an isomorphism of homotopy groups (and is hence a homotopy equivalence).

Since most of the spaces we will be dealing with are not simply connected, a generalization of this theorem is needed.

Definition 2.5.4. (See [54]). A space $X$ will be called weakly simple if $\pi_{1}(X)$ acts trivially on the homology of the universal cover $\tilde{X}$ of $X$. (To be specific, $\pi_{1}(X)$ acts on $\tilde{X}$ by deck transformations, and it is the induced action on homology that we are talking about.)

Theorem 2.5.5. [9, Lemma 6.2]. Let $X$ and $Y$ be weakly simple spaces. If $f: X \rightarrow Y$ induces an isomorphism of homology groups, as well as of fundamental groups, then finduces an isomorphism of all homotopy groups (and so it is a homotopy equivalence).

Recall that an $H$-space is a space $X$, equipped with a (base point preserving) "multiplication" map

$$
\mu: \quad X \times X \rightarrow X
$$

such that the maps

$$
\mu: \quad X \times\{e\} \rightarrow X \quad \text { and } \quad \mu: \quad\{e\} \times X \rightarrow X
$$

are both homotopic to the identity (here $e$ denotes the base point of $X$ ). These are of interest to us because every connected $H$-space is weakly simple [9, Lemma 6.2]. Thus Theorem 2.5.5 applies to $H$-spaces. In fact we may simplify the hypotheses somewhat: the fundamental group of an $H$-space $X$ is abelian (see [50, p. 44]), and so $\pi_{1}(X)=H_{1}(X)$. Thus the condition that $f$ induce an isomorphism on fundamental groups is subsumed under the condition that it induce an isomorphism on homology.

Our final topic in this section comes from the theory of spectral sequences.

Definition 2.5.6. (See [50, p. 476].) A fibration $F \rightarrow E \rightarrow B$ is said to be orientable if $\pi_{1}(B)$ acts trivially on the homology of $F$.

Theorem 2.5.7. Let

be a commuting diagram, where the rows are orientable fibrations. If any two of the maps

$$
\begin{array}{ll}
\alpha_{*}: & H_{*}\left(F_{1}\right) \rightarrow H_{*}\left(F_{2}\right), \\
\beta_{*}: & H_{*}\left(E_{1}\right) \rightarrow H_{*}\left(E_{2}\right), \\
\gamma_{*}: & H_{*}\left(B_{1}\right) \rightarrow H_{*}\left(B_{2}\right)
\end{array}
$$

are isomorphisms then so is the third.
For the proof see [37, p.355]. We do not want to say anything about it here, except that the theorem follows from the existence of the Spectral Sequence for a fibration over a $C W$-complex - given the spectral sequence it is nothing more than an enormous diagram chasing argument.

Let us give a simple application, which will be made use of in the next section (another application is in Sect. 5.1).

Definition 2.5.8. A map $f: X \rightarrow Y$ is said to be acyclic if the homotopy fiber $F_{f}$ of $f$ is cyclic, that is, if $F_{f}$ has the same homology as a single point space.

Theorem 2.5.9. If $f: X \rightarrow Y$ is acyclic then $f_{*}: H_{*}(X) \rightarrow H_{*}(Y)$ is an isomorphism.

Proof. Apply Theorem 2.5 .8 to the diagram of fibrations


The fibrations are orientable because the fibers have trivial homology. The map $p t \rightarrow F_{f}$ induces an isomorphism on homology by hypothesis. Therefore, so does $f: X \rightarrow Y$.

### 2.6. Higher Algebraic $K$-Theory

We begin by stating a theorem, due to Quillen [45], which characterizes what is known as the plus construction. The goal, roughly speaking, is to pass from a space $X$ to a space $X^{+}$with more manageable homotopy, but without altering the homology.

Theorem 2.6.1. Let $X$ be a CW-complex. There exists a space $X^{+}$, which may be chosen to be a CW-complex as well, and an acyclic map $q: X \rightarrow X^{+}$, which induces an isomorphism $\pi_{1} X / P \pi_{1} X \cong \pi_{1} X^{+}$. Furthermore, if $Y$ is another $C W$-complex and $f: X \rightarrow Y$ is a map, then there is a map $f^{+}: X^{+} \rightarrow Y^{+}$, which is unique up to homotopy, such that the diagram

commutes up to homotopy. In particular, $X^{+}$and $q: X \rightarrow X^{+}$are unique up to homotopy.

There is no need for us to prove this theorem; the reader is referred to [7] for a good exposition. But let us at least sketch how $X^{+}$is constructed since it is not especially complicated. The idea is simply to adjoin enough 2-cells to $X$ to kill the maximal perfect subgroup of $\pi_{1} X$. Having done this, we get the right $\pi_{1}$, but the homology has been altered. To remedy this, we adjoin 3-cells sufficient to kill the new generators of $\mathrm{H}_{2}$ created from the 2 -cells. It follows from the van Kampen theorem that adding the 3 -cells does not alter $\pi_{1}$, and so we have obtained a space with the desired properties.

Definition 2.6.2. The algebraic $K$-theory of a ring $A, K_{*}(A)$, is the homotopy of the space $B G L A^{+}$, where $B G L A$ denotes the classifying space of the stable general linear group of $A$. Thus

$$
\begin{equation*}
K_{n}(A)=\pi_{n}\left(B G L A^{+}\right) \tag{2.6.1}
\end{equation*}
$$

For our purposes it is perhaps best to think of these groups as follows. Bearing in mind the situation in topological $K$-theory, we want to view $B G L A$, appropriately modified, as the "inverse" of some sort of "algebraic loop space" of $G$. Of course, $B G L A$ as it stands is not satisfactory for this;
but if we modify it is so that the elements of $P G L A$, which are supposed to be thought of as connected to the identity, do indeed give the trivial clement in homotopy, then we naturally obtain the space $B G L A^{+}$, which is suitable for our purposes.

Let us compare Definition 2.6 .2 with the low-dimensional definitions given in Section 2.4. First, if $A^{2}-A$ then the maximal perfect subgroup of $G L A$ is [GLA, GLA]; so if $A$ satisfies this condition, then according to Definition 2.6.2, $K_{1}(A)=G L A /[G L A, G L A]$, which agrees with Definition 2.4.3. A more significant result, which lends considerable weight to Definition 2.6.2, and which was, in fact, one of the main motivations behind the definition (see [45]), is

Theorem 2.6 .3 (see, e.g., [7, Chap. 8]). For any discrete group G, $\pi_{2}\left(B G^{+}\right)$is the kernel of the universal central extension of the maximal perfect subgroup of $G$.

Proof. We will borrow a result from Section V, namely that the extension of groups

$$
P G \rightarrow G \rightarrow G / P G
$$

gives rise to a fibration

$$
\begin{equation*}
B P G^{+} \rightarrow B G^{+} \rightarrow B(G / P G)^{+} \tag{2.6.2}
\end{equation*}
$$

(see Corollary 5.1.5). The maximal perfect subgroup of $G / P G$ is trivial, and so $B(G / P G)^{+}=B(G / P G)$. But $\pi_{n}(B(G / P G))$ is zero if $n>1$, and therefore from the long exact sequence for the fibration (2.6.2) we get that $\pi_{2}\left(B P G^{+}\right)=\pi_{2}\left(B G^{+}\right)$. Now, $P G$ is perfect, and so $\pi_{1}\left(B P G^{+}\right)=0$. Therefore, it follows from the Hurewicz theorem [50, 7.5.2) that $\pi_{2}\left(B P G^{+}\right)$ is isomorphic to $H_{2}\left(B P G^{+}\right)$. But by definition of the plus construction, and Theorem 2.5.9,

$$
H_{2}\left(B P G^{+}\right)=H_{2}(B P G)=H_{2}(P G)
$$

and so the result follows from the fact that the second homology group of a perfect group is the kernel of the universal extension (see, e.g., [40, Corollary 5.8]).

As usual, our interest in algebraic $K$-theory lies in comparisons with topological $K$-theory. Let $A$ be a $C^{*}$-algebra and denote by $G L^{t} A$ the general linear group considered as a topological group (the unadorned $G L A$ will from here on only refer to GLA as a discrete group). The classifying space $B G L^{t} A$ has an abelian fundamental group, namely
$\pi_{1}\left(B G L^{\prime} A\right)=K_{1}^{\prime}(A)$, and so the space $B G L^{\prime} A$ is unaffected by the plus construction

$$
\begin{equation*}
B G L^{\prime} A \cong B G L^{t} A^{+} . \tag{2.6.3}
\end{equation*}
$$

Also there is a natural map $B G L A \rightarrow B G L^{\prime} A$ (if we are working with, say, Milnor's infinite join model of $B$, then this is clear from the functorial nature of that construction: the map is the one induced from the "identity" map $G L A \rightarrow G L^{\prime} A$, which is, of course, continuous). By plussing this map and then using (2.6.3) we obtain a canonical (up to homotopy) map

$$
B G L A^{+} \rightarrow B G L^{\prime} A .
$$

Applying the homotopy group functors $\pi_{n}$ we obtain homomorphisms

$$
\begin{equation*}
\alpha_{n}: \quad K_{n}(A) \rightarrow K_{n}^{\prime}(A) \tag{2.6.4}
\end{equation*}
$$

comparing the algebraic and topological $K$-theory.
Let us describe now our main result concerning this comparison, which concerns the following $C^{*}$-algebras.

Definition 2.6.4. Let $B$ be a $C^{*}$-algebra. The Calkin algebra for $B$, denoted $\mathscr{Q}(B)$ is the quotient $C^{*}$-algebra $\mathscr{M}(\mathscr{K} \otimes B) / \mathscr{K} \otimes B$.

The terminology is of course derived from the case $B=\mathbf{C}: \mathscr{2}(\mathbf{C})$ is just the quotient $\mathscr{B} / \mathscr{K}$ of bounded operators by compact operators, or in other words, the ordinary Calkin algebra.

We are interested in the $K$-theory of stable $C^{*}$-algebras (by reason of, e.g., Theorem 2.4.6). However, it is very difficult to approach the $K$-theory of $\mathscr{K} \otimes B$-even say $K_{2}$, as we shall see in Section IV-and so we compromise and study $\mathscr{2}(B)$. There is a close relationship between $\mathscr{K} \otimes B$ and $2(B)$, as a result of the following important result.

Theorem 2.6.5. Compare [54, Proposition 2.1].) The algebraic and topological $K$-theory groups of $\mathscr{M}(\mathscr{K} \otimes B)$ are trivial.

We will prove this in a moment. Let us note now that as a result of the long exact sequence in topological $K$-theory, together with stability,

$$
\begin{equation*}
K_{n}^{\prime}(\mathscr{2}(B))=K_{n-1}^{\prime}(B) . \tag{2.6.5}
\end{equation*}
$$

Thus $\mathscr{2}(B)$ plays the role of a suspension of $B$. A similar result is true in the algebraic situation. If $B$ is a unital ring then denote by $\mathscr{2}^{a}(B)$ the quotient of the ring of all infinite matrices over $B$, whose rows and columns each
have only finitely many non-zero entries, by the ideal of finite matrices. Then

$$
K_{n}\left(\mathscr{Q}^{a}(B)\right)=K_{n-1}(B)
$$

(see [54]). However, it is not clear that the equality (2.6.5) holds with algebraic $K$-theory replacing topological $K$-theory, and the most we can say is that $K_{n}(\mathscr{2}(B))$ is a sort of relative $K$-theory group for $\mathscr{K} \otimes B$.

Our main result, proved in Section V is as follows:
Theorem. Let $A$ be a unital $C^{*}$-algebra and let $B$ be a $\sigma$-unital $C^{*}$-algebra. The homomorphism $\alpha: K *(A \otimes \mathscr{2}(B)) \rightarrow K_{*}^{\prime}(A \otimes \mathscr{2}(B))$ is an isomorphism.

In particular, by setting $B=\mathbf{C}$, we obtain a conjecture of Karoubi [30] that $K_{*}(A \otimes \mathbb{Q})$ equals $K_{*-1}^{t}(A)$.
As another consequence, we have, for example,

$$
\begin{equation*}
H_{*}(B G L 2(A))=H_{*}\left(B G L^{\prime} 2(A)\right) . \tag{2.6.6}
\end{equation*}
$$

This is quite remarkable (we think): the left-hand side of (2.6.6) is the same as the Eilenberg-MacLane homology $H_{*}(G L \mathscr{Q}(A))$, which is purely algebraic in character, and depends only on the structure of $G L 2(A)$ as a group. The right-hand side is the homology of $B G L^{t} \mathcal{Q}(A)$; but by Bott Periodicity, $B G L^{\prime} \mathscr{Q}(A)$ is equal to the connected component of the base point in $\Omega G L^{\prime} \mathscr{2}(A)$, and so the right-hand side depends solely on the structure of $\left.G L^{t} \mathcal{Q}^{2} A\right)$ as a topological space.

The remainder of this section is devoted to establishing the basic facts concerning algebraic $K$-theory that we will need.

Definition 2.6 .6 (cf. [54]). A discrete group $G$ will be called a stable group if it has the following properties.
(i) The commutator subgroup $[G, G] \subset G$ is perfect (and hence it is the maximal perfect subgroup).
(ii) There is a homomorphism $G \times G \rightarrow G$, called direct sum, and denoted by $\oplus$, with the property that for any finitely generated subgroup $F \subset G$, there exist elements $a$ and $b$ in $G$ with

$$
a(e \oplus f) a^{-1}=f=b(f \oplus e) b^{-1}
$$

for every $f \in F$, where $e$ denotes the identity element of $G$.
We will find that this is a very convenient class of groups to work with. A homomorphism between stable groups will be assumed to commute with
the direct sum maps. Let us note that any subgroup $G^{\prime}$ of $G$ containing $[G, G]$ is itself a stable group (with the same $\oplus$ ). To see that $G^{\prime}$ is invariant under $\oplus$, note first that since $[G, G] \subset G^{\prime}, G^{\prime}$ is a normal subgroup of $G$. It follows then from (ii) above that if $g^{\prime} \in G^{\prime}$ then $e \oplus g^{\prime}$ and $g^{\prime} \oplus e$ are in $G^{\prime}$, too. Thus if $g_{1}^{\prime}, g_{2}^{\prime} \in G^{\prime}$ then $g_{1}^{\prime} \oplus g_{2}^{\prime} \in G^{\prime}$.

Condition (ii) in the definition is designed with the following result in mind.

Lemma 2.6.7. Let $G$ be a stable group.
(i) The endomorphisms $g \mapsto g \oplus e$ and $g \mapsto e \oplus g$ of $G$ induce the identity map on $H_{*}(G)$.
(ii) If $G^{\prime}$ is a stable normal subgroup of $G$ (by which we mean that the normal subgroup $G^{\prime}$ is a stable group in its own right, and the direct sum on $G^{\prime}$ is the restriction of the direct sum on $G$ ), then $G$ acts trivially on $H_{*}\left(G^{\prime}\right)$.

In part (ii), the action is by conjugation, of course.
Proof. (i) Since the two cases are the same, we consider only the map $g \mapsto g \oplus e$. By Lemma 2.5.2 it suffices to show that for every finitely generated subgroup $F$ of $G$, the map $f \mapsto f \oplus e$ from $F$ to $G$ induces the same map on homology as the natural inclusion $F \subsetneq G$. But according to part (ii) of Definition 2.6.6, these two maps differ by an inner automorphism of $G$. Since by Lemma 2.5.1 inner automorphisms act trivially on homology, the result follows.
(ii) Let $g \in G$. By part two of Definition 2.6.6, there exists an element $a \in G$ such that $g=a(e \oplus g) a^{-1}$. Therefore it suffices to show that $e \oplus g$ acts trivially on $H_{*}\left(G^{\prime}\right)$. Consider the commutative diagram


By part (i) of this lemma, applied to $G^{\prime}$, the horizontal maps induce the identity map on homology. It follows that $\operatorname{Ad}(e \oplus g)_{*}=$ id as required

The proof of part (ii) illustrates a useful technique that we will use several times.
The reason for the term "stable group" is that the stable general linear group of any unital ring is a stable group. We will in fact need a slightly stronger result.

Lemma 2.6.8. Let $R$ be a $C^{*}$-algebra and suppose that for any finite subset $\left\{r_{1}, r_{2}, \ldots, r_{n}\right\}$ of $R$ there exists an element $u \in R$ such that $1 \geqslant u \geqslant 0$ and $u r_{i}=r_{i}=r_{i} u$ for every $i$. Then GLR is a stable group.

Proof. Condition (i) of Definition 2.6 .6 follows from the Whitehead lemma (Theorem 2.4.2). As for condition (ii), the direct sum is defined by mapping a pair of matrices $(X, Y) \in G L R \times G L R$ to the matrix

$$
X \oplus Y=\left(\begin{array}{cccccc}
x_{11} & 0 & x_{12} & 0 & x_{13} & 0
\end{array}\right] .
$$

(In other words, $X \oplus Y$ is simply a rearrangement of $\left(\begin{array}{ll}X & 0 \\ 0 & y\end{array}\right)$, desined so as to make sense of it as an element of GLR.) Now, any finitely generated subgroup $F$ of $G L R$ is contained in some $G L_{n} R$, and then for any $f \in F, e \oplus f$ is equal to $f$, modulo conjutation by some $2 n \times 2 n$ permutation matrix. This matrix is a product of single alternations $\left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right)$, so it suffices to show that conjugation by $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ has the same effect on $F$ as conjugation by some matrix in $G L R$. But $F$ being finitely generated, there is some $u \in R$ such that $1 \geqslant u \geqslant 0$ and $u f_{i j}=f_{i j}=f_{i j} u$ for any matrix element $f_{i j}$ of a matrix in $F$. Then $\left(e^{e^{\text {riu }}\left(1^{1,} u\right)^{1 / 2}}\left(0^{\text {riu }}{ }^{1 / 1 / 2}\right)\right.$ is the matrix we are seeking (it is invertible, in fact unitary).

The same sort of proof shows that if $R$ is any unital ring then $G L R$ is a stable group. It is convenient to give $C^{*}$-algebras which satisfy the hypotheses of the lemma a name: for want of a better one let us call them weakly unital.

Lemma 2.6.9. If $G_{1}$ and $G_{2}$ are discrete groups then the natural map

$$
B\left(G_{1} \times G_{2}\right)^{+} \rightarrow B G_{1}^{+} \times B G_{2}^{+}
$$

is a homotopy equivalence. In particular, the functor $K_{*}$ is additive.
For the definition, see 2.1.17.
Proof. Let us first make a remark about the plus construction in general. For any two spaces, $\pi_{1}\left(X_{1} \times X_{2}\right) \cong \pi_{1}\left(X_{1}\right) \times \pi_{1}\left(X_{2}\right)$, and also, $P\left(\pi_{1}\left(X_{1}\right) \times \pi_{1}\left(X_{2}\right)\right)=P \pi_{1}\left(X_{1}\right) \times P \pi_{1}\left(X_{2}\right)$, as is the case for any two groups.

Therefore, since the direct product of two acyclic maps is acyclic (the homotopy fibre of the product is the product of the homotopy fibres), it follows that $X_{1}^{+} \times X_{2}^{+}-\left(X_{1} \times X_{2}\right)^{+}$. Now, if $G_{1}$ and $G_{2}$ are discrete groups then the natural map $B\left(G_{1} \times G_{2}\right) \rightarrow B G_{1} \times B G_{2}$ is a homotopy equivalence, as a check on fundamental groups immediately reveals. It follows that the natural map $B\left(G_{1} \times G_{2}\right)^{+} \rightarrow B G_{1}^{+} \times B G_{2}^{+}$is indeed a homotopy equivalence, and since $G L\left(B_{1} \oplus B_{2}\right)=G L B_{1} \times G L B_{2}$, the result follows.

Theorem 2.6.10 [54, Proposition 1.2]. If $G$ is a stable group then $B G^{+}$ is an H-space.

Proof. We will repeat Wagoner's argument. The first step is to show that $B G^{+}$is a weakly simple space. Choose a model for $q: B G \rightarrow B G^{+}$ which is a fibration (see (2.5.1)). Let $\pi: \widehat{B G^{+}} \rightarrow B G^{+}$denote the universal cover and consider the commutative diagram

where $X$ is defined to be the pullback

$$
X-\left\{(a, b) \in B G \times \widetilde{B G^{+}} \mid q(a)=\pi(b)\right\}
$$

It is easy to see that $X$ is a covering space of $B G$ with $\pi_{1}(X)=P G$; hence $X$ is homotopy equivalent to $B P G$. Since $q: B G \rightarrow B G^{+}$is a fibration, so is $X \rightarrow \widetilde{B G^{+}}$(see $[50,2.8 .6]$ ), and furthermore, the fibers of these two fibrations are equal. Thus the map $X \rightarrow \widetilde{B G^{+}}$is acyclic and, in particular, by Theorem 2.5.9, it induces an isomorphism in homology. Now, by (2.5.6), the action of $\pi_{1}(B G)$ on $H_{*}(X)$ by deck transformations corresponds to the action of $G$ on $H_{*}(P G)$ by conjugation. This is trivial by Lemma 2.6.7, and since $H_{*}(X) \rightarrow H_{*}\left(\widetilde{B G^{+}}\right)$is an isomorphism, it follows that the action of $\pi_{1}\left(B G^{+}\right)$on $H_{*}\left(\overparen{B G^{+}}\right)$is trivial as well. Thus $B G^{+}$is indeed weakly simple. By Lemma 2.6.9, the natural map $B(G \times G)^{+} \rightarrow B G^{+} \times B G^{+}$is a homotopy equivalence. It follows from Theorem 2.5.5 and Lemma 2.6.7 that the maps

$$
l=(\cdot \oplus e)_{+}: \quad B G^{+} \rightarrow B G^{+}
$$

and

$$
r=(e \oplus \cdot)_{+}: \quad B G^{+} \rightarrow B G^{+}
$$

are homotopy equivalencess. If we denote by $\hat{r}$ and $\hat{l}$ (base-point preserving) homotopy inverses to these maps then we can construct an $H$-space multiplication

$$
B G^{+} \times B G^{+} \xrightarrow{\hat{i \times i}} B G^{+} \times B G^{+} \xrightarrow{\cong} B(G \times G)^{+} \xrightarrow{\oplus} B G^{+}
$$

Here is an important by-product of this reasoning.

THEOREM 2.6.11. If $A$ is a weakly unital $C^{*}$-algebra then the map $A \rightarrow M_{2} A, a \mapsto\left(\begin{array}{cc}a & 0 \\ 0 & 0\end{array}\right)$ induces an isomorphism $K_{*}(A) \cong K_{*}\left(M_{2} A\right)$.

Proof. The map $A \rightarrow M_{2} A$ induces a map $G L A \rightarrow G L M_{2} A$, which, after identifying $G L M_{2} A$ with $G L A$ in the obvious way, is exactly the map $(\cdot \oplus e): G L A \rightarrow G L A$. Passing to $B G L A^{+}$, this map is a homology isomorphism between connected $H$-spaces. Hence it is a homotopy equivalence.

We remark that this result will be used in Section IV in the special case of $K_{2}$ of a unital algebra. In this case, a direct proof using roughly the same technique, but avoiding the topology, is possible. The key fact is that the conjugation by an element of $G L A$ induces the identity map on $K_{2}$, which follows from the fact that $K_{2}$ is central.

Lemma 2.6.12. Let $A$ be any ring and let $v$ be an invertible element, or merely a left-invertible element, in a ring containing $A$ as an ideal. If $F$ is any functor and if the map $A \rightarrow M_{2} A$ given by $a \mapsto\left(\begin{array}{ll}a & 0 \\ 0 & 0\end{array}\right)$ induces an isomorphism $F(A) \cong F\left(M_{2} A\right)$, then $\operatorname{Ad}(v): F(A) \rightarrow F(A)$ is the identity map.

Proof. The diagram

commutes, and since the horizontal maps induce isomorphisms when $F$ is applied, it follows from applying $F$ to the diagram that $\operatorname{Ad}\left(\begin{array}{ll}1 & 0 \\ 0 & v \\ v\end{array}\right)_{*}$ is the identity. Since

$$
\operatorname{Ad}\left(\begin{array}{ll}
v & 0 \\
0 & 1
\end{array}\right)=\operatorname{Ad}\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) \operatorname{Ad}\left(\begin{array}{ll}
1 & 0 \\
0 & v
\end{array}\right) \operatorname{Ad}\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

it follows that $\operatorname{Ad}\left(\begin{array}{ll}n & 0 \\ 0 & 1\end{array}\right)_{*}$ is the identity. So from the commuting square

it follows that $\operatorname{Ad}(v)_{*}$ is the identity.
We can now complete some unfinished business.
Proof of Theorem 2.6.5. We consider only algebraic $K$-theory, the topological $K$-theory case being entirely similar. Let $P_{n} \otimes 1(n \in \mathbf{Z})$ be a family of pairwise orthogonal projections in $\mathscr{H}(\mathscr{K} \otimes B)$, each equivalent to 1 , via isometries $v_{n}(n \in \mathbf{Z})$. Define endomorphisms $r, r^{(x)}$ of $\mathscr{H}(\mathscr{K} \otimes B)$ by

$$
r(x)=\operatorname{Ad}\left(v_{0}\right) \cdot x, \quad r^{(x)}(x)=\sum_{n>0} \operatorname{Ad}\left(v_{n}\right)(x) .
$$

Note that $r_{*}=$ id, by Lemma 2.6.12; and also $\left(r+r^{(x)}\right)_{*}=r_{*}^{(x)}$, by the same lemma since $r+r^{(x)}$ and $r^{(x)}$ are unitarily equivalent via a bilateral shift. Finally, $\left(r+r^{(x)}\right)_{*}=r_{*}+r_{*}^{(x)}$ by Lemma 2.1.18. Thus

$$
\begin{aligned}
\mathrm{id}_{K * 1 / H(* \otimes B)} & =r_{*}=r_{*}+r_{*}^{(x)}-r_{*}^{(x)}=\left(r+r^{(x)}\right)_{*}-r_{*}^{(x)} \\
& =r_{*}^{(x)}-r_{*}^{(x)}=0 .
\end{aligned}
$$

The final result in this section is of a more technical nature. For the most part it is possible to be very lax about the choices of the space $X^{+}$and maps $f^{+}: X^{+} \rightarrow Y^{+}$, variation within a homotopy equivalence class being of no consequence. However, in Section $V$ we will want to compare long exact sequences in topological and algebraic $K$-theory, and in order to do this, we will have to produce diagrams of fibrations which commute exactly, not merely up to homotopy. Let

$$
1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1
$$

be an extension of topological groups: thus we asssume that $N$ is a closed subgroup of $G$, and that $H$ has the quotient topology. Denote by $B^{t} N, B^{t} G$, and $B^{t} H$ the Milnor classifying spaces of these groups, and denote by $B^{d} N$, $B^{d} G$, and $B^{d} H$ the Milnor classifying spaces of $N, G$, and $H$, considered as discrete groups. Suppose that for each of the groups, the maximal perfect subgroup is equal to the connected component of the identity (and that this is equal to the path-connected component of the identity). As a con-
sequence, $B^{t} N, B^{t} G$, and $B^{t} H$ are equal to $B^{t} N^{+}, B^{t} G^{+}$, and $B^{t} H^{+}$, respectively, since, for example,

$$
\pi_{1}\left(B^{t} N\right)=N / N^{0}=N / P N
$$

and so on. Thus we obtain a homotopy commutative diagram

where the top and bottom rows come from the functoriality of the classifying space, and the composition, $B^{d} N \rightarrow B^{t} N$ (and so on for $G$ and $H$ ), comes, up to homotopy, from the functoriality of $B$ as well.

Theorem 2.6.13. We may choose the plussed spaces and maps in (2.6.7) in such a way that:
(i) the diagram commutes exactly, and all maps between classifying spaces (including compositions) come from the functoriality of $B$;
(ii) there are homotopies contracting the compositions $B^{d} N \rightarrow B^{d} H$, $B^{d} N^{+} \rightarrow B^{d} H^{+}$and $B^{t} N \rightarrow B^{t} H$, to constant maps, which are compatible in the sense that the diagram

commutes.
We want to make a number of comments before starting on the proof. As we have already said, the theorem will be used in the comparison of long exact sequences in algebraic and topological $K$-theory. Suppose that $I$ is an ideal in a $C^{*}$-algebra $A$. Associated to the pair $(A, I)$ is an extension $N \rightarrow G \rightarrow H$ of topological groups, where:
(i) $N=G L I$,
(ii) $G=G L A$,
(iii) $H=$ Image of $G L A$ in $G L A / I$
(the group $H$ is an open and closed subgroup of $G L A / I)$. Now, choose maps and spaces as in Theorem 2.6 .13 so that the diagram (2.6.7) commutes. Passing to the associated fibrations, we have a commuting diagram

where $F_{d}^{(+)}$and $F_{t}$ are homotopy fibers, and $P_{d}^{(+)}$and $P_{t}$ are the total spaces as in (2.5.1). Furthermore, in view of part (ii) of the theorem we may choose maps

$$
\begin{equation*}
B^{d} N^{+} \rightarrow F_{d}^{(+)} \quad \text { and } \quad B^{\prime} N \rightarrow F_{i} \tag{2.6.8}
\end{equation*}
$$

as in (2.5.3), such that the diagram

commutes. Now suppose that the maps (2.6.8) induce isomorphisms on $\pi_{n}$ ( $n \geqslant 1$ ). Then from the commuting diagram

by identifying the homotopy of the fibers with $K_{*}(I)$ and $K_{*}^{\prime}(I)$ by means of (2.6.8), and using the elementary fact (to be proved in Sect. 5.1) that the maps

$$
B^{d} H^{+} \rightarrow B^{d} G L A / I^{+} \quad \text { and } \quad B^{t} H \rightarrow B^{t} G L A / I
$$

induce isomorphisms on $\pi_{n}(n \geqslant 1)$, we obtain the commuting diagram


The point is that without part (ii) of Theorem 2.6 .13 we would not know that diagram (2.6.9) commutes, even up to homotopy, and so we would not be able to conclude that the transformation $\alpha$ commutes with the boundary map $\partial$ in the above diagram.

The proof of Theorem 2.6.13 uses certain results concerning the plus construction which we have not bothered to mention, the reason being that they do not shed much more light on the nature of $X^{+}$. The basic fact is that if we consider maps which are cofibrations then the diagram in Theorem 2.6 .1 can be made to commute exactly, not merely up to homotopy. Rather than go into this and other results needed, we simply give references. Part (ii) of the theorem, and its proof, are similar to constructions of Wagoner in [54].

Proof of Theorem 2.6.13. We may choose $B^{d} N \rightarrow B^{d} N^{+}$to be a cofibration. Then by $[7,(5.2)]$, since the kernel of the map $\pi_{1}\left(B^{d} N\right) \rightarrow \pi_{1}\left(B^{t} N\right)$ contains (and is in fact equal to) the kernel of the map $\pi_{1}\left(B^{d} N\right) \rightarrow \pi_{1}\left(B^{d} N^{+}\right)$, there exists a map $B^{d} N^{+} \rightarrow B^{t} N$ such that the diagram

commutes. Now, choose $B^{d} G \rightarrow B^{d} G^{+}$to be a cofibration. We can similarly extend the map $B^{d} G \rightarrow B^{\prime} G$ to a map $B^{d} G^{+} \rightarrow B^{t} G$. However, $B^{d} G^{+}$will not be the space ultimately appearing in diagram (2.6.7). Rather, we use the space

$$
X_{1}=B^{d} G^{+} \cup_{B^{d} N} B^{d} N^{+}
$$

By $[7,(4.20)]$, the natural map from $B^{d} G^{+}$into $X_{1}$ is an acyclic cofibration. It follows from [7, (4.12)] that the composition

$$
B^{d} G \rightarrow B^{d} G^{+} \rightarrow X_{1}
$$

of two acyclic cofibrations is also an acyclic cofibration. A computation of the fundamental group of $X_{1}$ using the van Kampen theorem shows that $\pi_{1}\left(X_{1}\right) \cong G / P G$, and so $X_{1}$ serves as a model for $B^{d} G^{+}$(in other words, the $\operatorname{map} B^{d} G^{+} \rightarrow X_{1}$ is a homotopy equivalence). We have maps from $B^{d} N^{+}$
and $B^{d} G^{+}$into $B^{t} G$, which agree on $B^{d} N$, and so from these we obtain a map from $X_{1}$ into $B^{t} G$, and a commuting diagram


Next, it follows from [7, (5.11)] that since $\pi_{1} B G \rightarrow \pi_{1} B H$ maps a maximal perfect subgroup onto a maximal perfect subgroup, the space

$$
X_{2}=B^{d} G^{+} \cup_{B^{d} G} B^{d} H,
$$

together with the obvious map $B^{d} H \rightarrow X_{2}$, serves as a model for the plus construction $B^{d} H \rightarrow B^{d} H^{+}$. There is, of course, a natural map from $X_{2}$ to $B^{\prime} H$, extending $B^{d} H \rightarrow B^{t} H$, since there exist natural maps from $B^{d} G^{+}$and $B^{d} I I$ into $B^{d} H$ which agree over $B^{d} G$. We must, however, still construct the map $X_{1} \rightarrow X_{2}$, extending $B^{d} G \rightarrow B^{d} H$, such that the diagram

commutes. Recalling the definition of $X_{1}$, it suffices to construct a map $B^{d} N^{+} \rightarrow B^{d} H$ which agrees with the natural map on $B^{d} N$. However, with part (ii) of the theorem in mind, we should be a little more careful. The space $B^{d} N$ is mapped into the subspace $B^{d} E$ of $B^{d} H$, where $E$ denotes the trivial subgroup of $H$. By [7, (5.2)], the map $B^{d} N \rightarrow B^{d} E$ extends to $B^{d} N^{+} \rightarrow B^{d} E$ and, in this manner, we define $X_{1} \rightarrow X_{2}$. We have now constructed spaces and maps so that the diagram (2.6.7) commutes, and thus it remains to prove (ii). But $B^{d} N, B^{d} N^{+}$, and $B^{t} N$ map into the subspaces $B E$ of $B^{d} H$ and $B^{t} H$. Note that $B E$, the subspace of $B^{d} H$, is mapped homeomorphically onto $B E$, the subspace of $B^{t} H$. This space $B E$ is contractible, and using a contraction of $B E$ to a point we obtain compatible homotopies.

We note that if we look at only the top two rows of the diagram (2.6.7), then the statement of the theorem makes sense for any extension of groups. The proof will carry through if we make the assumption that the maximal perfect subgroup of $G$ maps onto the maximal perfect subgroup of $H$. This is the case, for example, with stable groups, since for these the maximal perfect subgroup is the commutator subgroup.

## III. A Homotopy Invariance Theorem

The main result of this section is a homotopy invariance theorem, proved in Section 3.1, which asserts that if a functor $E$ from $C^{*}$-algebras to abelian groups admits a suitable pairing with the set of Fredholm modules then $E$ is homotopy invariant. The proof is essentially an adaptation of Kasparov's proof of homotopy invariance for the extension groups $\operatorname{Ext}^{-1}(C, A)$.

In order to make the theorem applicable it must be modified a little. To begin with, we consider in Section 3.1 the business of replacing "Fredholm module" with "Fredholm pair." Having done this, we go on in Section 3.2 to prove a homotopy invariance theorem whose hypotheses are that $E$ satisfies a certain excision property (split exactness) and is stable. The machinery used to deduce this result from the former is derived from Cuntz's theory of quasi-homomorphisms.

It is sometimes useful to have versions of these results for functors of quasi-unital maps. The necessary changes are indicated in Section 3.3.

The final two Sections are applications of the homotopy invariance theorems to extension theory. In Section 3.4 we indicate how to construct pairings between extension groups and the set of Fredholm modules: this amounts to the construction of the Kasparov product from $K K$-theory. Our approach is perhaps a little more conceptual than that of Kasparov [33,35]. For example, our definition of the pairing does not require the Kasparov technical theorem (Theorem 1.1.11) or anything similar (however, we will use this to show that the pairing is well defined). Section 3.5 is devoted to a brief discussion of the excision properties of exten sion groups, followed by a discussion of the Brown-Douglas-Fillmore characterization of $\mathrm{Ext}^{-1}(A, \mathbf{C})$.

### 3.1. Pairings with Fredholm Modules

Throughout this section and Section $3.2, E$ will denote a functor from the category of $C^{*}$-algebras and *-homomorphisms to abelian groups. The particular class of $C^{*}$-algebras on which $E$ is defined-all $C^{*}$-algebras, separable $C^{*}$-algebras, $\sigma$-unital $C^{*}$-algebras, nuclear $C^{*}$-algebras, etc.-is not especially important. All we need in this section is that if $A$ is in the class, then so is $A \otimes C[0,1]$; in Section 3.2 we will require in addition that $A \otimes \mathscr{K}$ is in the class, and if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a degenerate extension, with $A$ and $C$ in the class, then so is $B$. All of the above mentioned families of $C^{*}$-algebras have these properties.

By a pairing of $E$ with the set of Fredholm modules we mean simply the association to each Fredholm $B$-module $\left(\varphi_{+}, \varphi_{-}, F\right)$ of a homomorphism

$$
\begin{equation*}
\times\left(\varphi_{+}, \varphi_{-}, F\right): \quad E(A \otimes B) \rightarrow E(A) \tag{3.1.1}
\end{equation*}
$$

for each $A$ (the notation is meant to suggest "multiplication with $\left(\varphi_{+}, \varphi_{-}, F\right) "$ ). Before going on we might mention that we are tacitly assuming that $E$ is covariant. The treatment for a contravariant $E$ is entirely similar (just reverse the arrows); it will be omitted, but made use of in Section 3.4 and 3.5.

We will only need to deal with Fredholm modules for which $\varphi_{+}=\varphi_{-}$; we will let $(\varphi, F)$ be an abbreviation for $(\varphi, \varphi, F)$.

Now, a general pairing is of course of no interest at all, and so we impose the following conditions.
(3.1.2a) Functoriality. If $(\varphi, F)$ is a Fredholm $B^{\prime}$-module, and if $f: B \rightarrow B^{\prime}$ is a *-homomorphism, then the diagram

commutes.
(3.1.2b) Additivity. If $(\varphi, F)$ and $(\varphi, G)$ are Fredholm $B$-modules then

$$
\times(\varphi, F)+\times(\varphi, G)=\times(\varphi, F G) .
$$

(3.1.2c) Stability. If $(\varphi, F)$ is a Fredholm $B$-module and $\varphi^{\prime}: B \rightarrow \mathscr{B}(\mathscr{H})$ is any *-homomorphism then

$$
\times(\varphi, F)=\times\left(\left(\begin{array}{cc}
\varphi & 0 \\
0 & \varphi^{\prime}
\end{array}\right),\left(\begin{array}{ll}
F & 0 \\
0 & 1
\end{array}\right)\right)
$$

(3.1.2d) Non-degeneracy. If $(\varphi, F)$ is a Fredholm C-module for which $\varphi$ maps $1 \in \mathbf{C}$ to $1 \in \mathscr{B}(\mathscr{H})$, and $F$ is an index one operator, then the map

$$
\times(\varphi, F): \quad E(A) \rightarrow E(A)
$$

is the identity.
Our main theorem is this:
Theorem 3.1.1. If the functor $E$ admits a pairing with the set of Fredholm modules $(\varphi, F)$ which satisfies conditions (3.1.2) above, then $E$ is a homotopy functor.

The first step in the proof is to establish operator homotopy invariance for the pairing:

Lemma 3.1.2. Let $\left(\varphi, F_{t}\right)$, for $t \in[0,1]$, be a family of Fredholm $B$-modules. If the map $t \mapsto F_{t}$ is norm continuous then $\times\left(\varphi, F_{0}\right)=\times\left(\varphi, F_{1}\right)$.
Proof. Choose a "parametrix" $\tilde{F}$ for $F_{1}$ such that $F_{1} \tilde{F}^{*}=1-P$, where $P$ is some finite rank projection. By additivity we have that

$$
\times(\varphi, 1-P)+\times(\varphi, 1-P)=\times(\varphi, 1-P),
$$

and so $\times(\varphi, 1-P)=0$. It follows from additivity that $\times\left(\varphi, F_{1}\right)=-\times\left(\varphi, \widetilde{F}^{*}\right)$, and therefore

$$
\times\left(\varphi, F_{0}\right)=\times\left(\varphi, F_{1}\right)+\times\left(\varphi, \tilde{F}^{*} F_{0}\right) .
$$

Thus it suffices to show that $\times\left(\varphi, \tilde{F}^{*} F_{0}\right)=0$. Now, $\tilde{F}^{*} F_{0}$ is an index zero operator and so we may find an invertible operator $G$, unitary modulo the compacts, and a finite rank projection $P^{\prime}$ such that $G\left(1-P^{\prime}\right)=$ $\tilde{F}^{*} F_{0}\left(1-P^{\prime}\right)$. By additivity once again, it suffices to show that $\times(\varphi, G)=0$. Denote by $C$ the $C^{*}$-algebra of all operators in $\mathscr{B}(\mathscr{H})$ which commute with $\varphi[B]$, modulo compact operators. Since the image of $G$ in the quotient $C / \mathscr{K}$ is connected by a path of unitaries to the identity (by the path $t \mapsto \dot{F}_{t}^{*} \dot{F}_{0}$ ), for example), it follows that $G$ may be written as a product of exponentials $e^{i 4}$, where $A \in C$ is self-adjoint, modulo $\mathscr{K}$; it suffices to show that each $\times\left(\varphi, e^{i A}\right)$ is zero. By the stability of the pairing, this will follow if we show that $\times(1 \otimes \varphi, H)=0$, where $(1 \otimes \varphi)(b)=1 \otimes \varphi(b) \in \mathscr{B}(\mathscr{H} \otimes \mathscr{H})$, and $H \in \mathscr{B}(\mathscr{H} \otimes \mathscr{H})$ is the operator $p \otimes e^{i A}+(1-p) \otimes 1, p$ being a rankone projection: in matrix form this is

$$
H=\left(\begin{array}{cc}
e^{i A} & 0 \\
0 & 1
\end{array}\right) .
$$

At this point, we appeal to the construction used in Theorem 2.4.7. Decomposing $(1-p) \otimes 1$ into a sequence of pairwisc disjoint projections $p_{n} \otimes 1$, each $p_{n}$ of rank one, we may write $H$ as a product $X_{1} X_{2}$ of two infinite diagonal matrices as in (2.4.4) and (2.4.5):

$$
\begin{aligned}
& \operatorname{diag}\left(X_{1}\right)=(x, x^{-1 / 2}, x^{-1 / 2}, \underbrace{x^{1 / 4}, \ldots, x^{1 / 4}}_{4 \text { times }}, x^{-1 / 8}, \ldots) \\
& \operatorname{diag}\left(X_{2}\right)=(1, x^{1 / 2}, x^{1 / 2}, \underbrace{x^{1 / 4}, \ldots, x^{-1 / 4}}_{4 \text { times }}, x^{1 / 8}, \ldots)
\end{aligned}
$$

(Here, $x$ denotes $e^{i 4}$.) Note that $X_{1}$ and $X_{2}$ are in $1+\mathscr{K} \otimes C$, and so $\left(1 \otimes \varphi, X_{1}\right)$ and $\left(1 \otimes \varphi, X_{2}\right)$ are Fredholm modules. We may write each $X_{i}$ as a product of two matrices of the form

$$
\left(\begin{array}{ccc}
W & 0 & 0 \\
0 & W^{-1} & 0 \\
0 & 0 & 1
\end{array}\right)
$$

where these $3 \times 3$ matrix decompositions of $\mathscr{H} \otimes \mathscr{H}$ are given by projections of the form $Q \otimes 1 \in \mathscr{B}(\mathscr{H} \otimes \mathscr{H})$. Then since

$$
\left(\begin{array}{ccc}
W & 0 & 0 \\
0 & W^{-1} & 0 \\
0 & 0 & 1
\end{array}\right)=\left[\left(\begin{array}{ccc}
W & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right),\left(\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right)\right]
$$

it follows that $H$ may be written as a product of four commutators [ $H_{1}^{(i)}, H_{2}^{(i)}$ ], where all the operators are essentially unitary, and commute with $1 \otimes \varphi[B]$ modulo compacts (note that the matrix

$$
\left(\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

commutes exactly with $1 \otimes \varphi[B]$ ). But by additivity,

$$
\begin{aligned}
\times & \left(1 \otimes \varphi,\left[H_{1}^{(i)}, H_{2}^{(i)}\right]\right) \\
& =\times\left(1 \otimes \varphi, H_{1}^{(i)}\right)+\times\left(1 \otimes \varphi, H_{2}^{(i)}\right)+\times\left(1 \otimes \varphi, H_{1}^{(i)-1}\right)+\times\left(1 \otimes \varphi, H_{2}^{(i)-1}\right) \\
& =\times\left(1 \otimes \varphi, H_{1}^{(i)} H_{1}^{(i)-1}\right)+\times\left(1 \otimes \varphi, H_{2}^{(i)} H_{2}^{(i)-1}\right) \\
& =0 .
\end{aligned}
$$

Hence $\times(1 \otimes \varphi, H)=0$.
Corollary 3.1.3. If $F$ and $G$ are equal, modulo the compacts, then

$$
\times(\varphi, F)=x(\varphi, G) .
$$

Proof. We may connct $F$ to $G$ by a homotopy, namely the straight line from $F$ to $G$, and then apply Lemma 3.1.2.

The remainder of the proof of Theorem 3.1.1 involves the construction of a Fredholm $C[0,1]$-module $\left(\varphi, F_{0}\right)$, for which the map $\times\left(\varphi, F_{0}\right): E(A \otimes C[0,1]) \rightarrow E(A)$ is equal to the homomorphism induced from evaluation of a function $f \in C[0,1]$ at 0 ; and which is operator
homotopic to a Fredholm module $\left(\varphi, F_{1}\right)$, for which $\times\left(\varphi, F_{1}\right)$ is the map induced from evaluation at 1 . We will follow Kasparov [33, Sect. 5].

Proof of Theorem 3.1.1. Construct a Fredholm $C[0,1]$-module ( $\varphi, F_{0}$ ) as follows. The Hilbert space on which $C[0,1]$ acts is $L^{2}(-\pi, \pi)$. With respect to the decomposition

$$
L^{2}(-\pi, \pi)=L^{2}(-\pi, 0) \oplus L^{2}(0,1) \oplus L^{2}(1, \pi)
$$

a function $f \in C[0,1]$ acts on the first summand by multiplication by the scalar $f(0)$; it acts on the second summand by pointwise multiplication; and it acts on the third summand by multiplication by the scalar $f(1)$. Denote by $P$ the projection of $L^{2}(-\pi, \pi)$ onto the Hardy space (the closed linear span of the functions $e^{i n \theta}$ with $n \geqslant 0$ ), and let $S$ be the corresponding symmetry: $S=2 P-1$. It is well known that $S$ commutes, modulo compacts, with multiplication by any continuous function $g$ on $[-\pi, \pi]$ such that $g(-\pi)=g(\pi)$. Now let $h$ be any real-valued continuous function on $[-\pi, \pi]$ such that
(i) $1 \geqslant h \geqslant-1$;
(ii) $h(-\pi)=-1$; and
(iii) $h(\pi)=1$;
and denote by $U_{h}$ the unitary operator

$$
U_{h}=h+i \sqrt{1-h^{2}} S \in \mathscr{B}\left(L^{2}(-\pi, \pi)\right) .
$$

Since $\sqrt{1-h^{2}}$, evaluated at both $-\pi$ and $\pi$, is zero, it follows that $\sqrt{1-h^{2}}$ commutes with $S$, modulo compacts, and furthermore, $\sqrt{1-h^{2}} S$ commutes with any continuous function on $[-\pi, \pi]$, modulo compact operators. Thus $U_{h}$ is unitary modulo the compact operators and $U_{h}$ commutes with the action $\varphi$ of $C[0,1]$ on $L^{2}(-\pi, \pi)$. Any two continuous functions $h_{0}, h_{1}$ with the properties (i), (ii), and (iii) above are connected by a norm-continuous path of such functions, and so the corresponding Fredholm modules $\left(\varphi, U_{h_{i}}\right)$ are operator homotopic, in the sense of Lemma 3.1.2. A particular consequence is that all $U_{h}$ have the same Fredholm index, and by direct computation in the special case $h(t)=\sin t / 2$, we find that this index is 1 (cf. [33, p. 760]). We define $F_{0}$ to be $U_{h_{0}}$, where in addition to (i), (ii), and (iii), we require that $h_{0}$ is equal to 1 on $[0, \pi]$. It follows that $\sqrt{1-h^{2}}$ is equal to zero on $[0, \pi]$, and from this it follows easily that $U_{h_{0}}$ commutes, modulo compacts, with the projection $Q$ onto $L^{2}(-\pi, 0)$. By Corollary 3.1.3, we have

$$
\times\left(\varphi, F_{0}\right)=\times\left(\varphi, Q F_{0} Q+(1-Q) F_{0}(1-Q)\right) .
$$

In matrix form, the right-hand side is

$$
\times\left(\left(\begin{array}{cc}
\varphi_{Q} & 0 \\
0 & \varphi_{1-Q}
\end{array}\right),\left(\begin{array}{cc}
Q F_{0} Q & 0 \\
0 & 1
\end{array}\right)\right)
$$

By the stability property of the pairing, this is equal to $\times\left(\varphi_{Q}, Q F_{0} Q\right)$; but $Q F_{0} Q$ is an index one operator, while $\varphi_{Q}: C[0,1] \rightarrow \mathscr{B}\left(L^{2}(-\pi, 0)\right)$ is obtained from the unital map $\mathbf{C} \rightarrow \mathscr{B}$ by composition with evaluation at zero, $C[0,1] \rightarrow \mathbf{C}$; so by the non-degeneracy and functoriality properties of the pairing, $\times\left(\varphi_{Q}, Q F_{0} Q\right)$ is equal to the map $E(A \otimes C[0,1]) \rightarrow E(A)$ induced from evaluation at 0 . Now, $\left(\varphi, F_{0}\right)$ is operator homotopic to ( $\varphi, F_{1}$ ), where $F_{1}=U_{h_{1}}$, and $h$ is a continuous function satisfying (i), (ii), (iii) and which is identically equal to -1 on $[-\pi, 1]$. Just as above, we see that $\times\left(\varphi, F_{0}\right)$ is equal to the map induced by evaluation at 1 . (There is one additional point. We get that $\times\left(\varphi, F_{0}\right)$ is equal to $\times\left(\left(\begin{array}{cc}\varphi_{R} & 0 \\ 0 & \varphi_{1-R}\end{array}\right)\left(\begin{array}{cc}R F_{0} R & 0 \\ 0 & -1\end{array}\right)\right)$, where $R$ is the projection onto $L^{2}(1, \pi)$, and in order to apply the stability property we need $\left(\begin{array}{cc}R F_{0} R & 0 \\ 0 & 1\end{array}\right)$, not $\left(\begin{array}{cc}R F_{0} R & 0 \\ 0 & -1\end{array}\right)$; but these two operators are homotopic, so by Lemma 3.1.2, the latter can be replaced by the former.) An application of Lemma 3.1.2 shows that $\times\left(\varphi, F_{0}\right)=\times\left(\varphi, F_{1}\right)$, so we are done.

In practice, it is easier to work with Fredholm pairs rather than Fredholm modules. Suppose then that there is a pairing between the functor $E$ and the set of all Fredholm pairs, with the following properties.
(3.1.3a) Functoriality. If $\left(\varphi_{+}, \varphi_{-}\right)$is a Fredholm pair for $B^{\prime}$ and if $f: B \rightarrow B^{\prime}$ is a $*$-homomorphism then the diagram

commutes.
(3.1.3b) Additivity. If $\left(\varphi_{1}, \varphi_{2}\right)$ and $\left(\varphi_{2}, \varphi_{3}\right)$ are Fredholm pairs then

$$
\times\left(\varphi_{1}, \varphi_{2}\right)+\times\left(\varphi_{2}, \varphi_{3}\right)=\times\left(\varphi_{1}, \varphi_{3}\right) .
$$

(3.1.3c) Stability. If $\varphi: B \rightarrow \mathscr{B}(\mathscr{H})$ is any $*$-homomorphism then

$$
\times\left(\varphi_{+}, \varphi_{-}\right)=\times\left(\left(\begin{array}{cc}
\varphi_{+} & 0 \\
0 & \varphi
\end{array}\right),\left(\begin{array}{cc}
\varphi_{-} & 0 \\
0 & \varphi
\end{array}\right)\right) .
$$

(3.1.3d) Non-degeneracy. If $e: \mathbf{C} \rightarrow \mathscr{B}(\mathscr{H})$ maps $1 \in \mathbf{C}$ to a rank one projection then $\times(e, 0): E(A) \rightarrow E(A)$ is the identity.
(3.1.3e) Unitary equivalence. If $U \in \mathscr{B}$ is a unitary then

$$
\times\left(\varphi_{+}, \varphi_{-}\right)=\times\left(\operatorname{Ad}(U) \varphi_{+}, \operatorname{Ad}(U) \varphi\right)
$$

(3.1.3f) Stability under compact perturbations. If $U \in \mathscr{B}$ is a unitary which is equal to the identity, modulo compact operators, then

$$
\times(\varphi, \operatorname{Ad}(U) \varphi)=0 .
$$

Let us note two useful consequences of these properties. First, if $U_{1}$ and $U_{2}$ are unitaries then

$$
\begin{equation*}
\times\left(\varphi, \operatorname{Ad}\left(U_{1}^{*}\right) \varphi\right)+\times\left(\varphi, \operatorname{Ad}\left(U_{2}^{*}\right) \varphi\right)=\times\left(\varphi, \operatorname{Ad}\left(U_{1}^{*} U_{2}^{*}\right) \varphi\right) \tag{3.1.4}
\end{equation*}
$$

This follows from (3.1.3b) and (3.1.3e). Second, if $U$ is equal to $V$, modulo compacts ( $U$ and $V$ being unitaries), then

$$
\begin{equation*}
\left.\times\left(\varphi, \operatorname{Ad}\left(U^{*}\right) \varphi\right)=\times\left(\varphi, \operatorname{Ad}\left(V^{*}\right) \varphi\right)\right) \tag{3.1.5}
\end{equation*}
$$

This follows from (3.1.4) and (3.1.3f).
TheOrem 3.1.4. If E admits a pairing with Fredholm pairs which satisfies the conditions (3.1.3), then $E$ is homotopy invariant.

Proof. We will construct a pairing of $E$ with the set of Fredholm modules ( $\varphi, F$ ), and then apply Theorem 3.1.1. The first step is to associate with $F$ an operator matrix

$$
\bar{F}=\left(\begin{array}{ll}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{array}\right)
$$

such that $\bar{F}$ is unitary, $F_{12}$ and $F_{21}$ are compact, and $F_{11}$ is equal to $F$, modulo compact operators. For example, we could take $F_{11}$ to be the partial isometry part of $F$ in its polar decomposition, and then dilate to a unitary as in Section 2.3. Then define $\times(\varphi, F): E(A \otimes B) \rightarrow E(A)$ by

$$
\times(\varphi, F)=\times\left(\left(\begin{array}{ll}
\varphi & 0  \tag{3.1.6}\\
0 & 0
\end{array}\right), \operatorname{Ad}\left(\bar{F}^{*}\right)\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right)\right)
$$

Our first observation is that this does not depend on the choice of $\bar{F}$. Indeed, suppose that $\bar{F}^{\prime}$ is another choice; then by (3.1.4),

$$
\begin{aligned}
& \times\left(\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right), \operatorname{Ad}\left(\bar{F}^{*}\right)\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right)\right)-\times\left(\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right), \operatorname{Ad}\left(\bar{F}^{\prime} *\right)\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right)\right) \\
& \quad=\times\left(\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right), \operatorname{Ad}\left(\bar{F}^{\prime} \bar{F}^{*}\right)\left(\begin{array}{cc}
\varphi & 0 \\
0 & 0
\end{array}\right)\right)
\end{aligned}
$$

But $\bar{F} \bar{F}^{*}$ is equal to ( $\left.\begin{array}{ll}1 & 0 \\ 0 & U\end{array}\right)$, modulo compacts, where $U$ is some unitary. Thus it follows from (3.1.5) that

$$
\times\left(\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right), \operatorname{Ad}\left(\bar{F}^{\prime} \bar{F}^{*}\right)\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right)\right)=\left(\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right), \operatorname{Ad}\left(\begin{array}{ll}
1 & 0 \\
0 & U
\end{array}\right)\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right)\right) .
$$

However, $\left(\begin{array}{cc}1 & 0 \\ 0 & U\end{array}\right)$ commutes with $\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right)$, and so the right side of (3.1.7) is $\times\left(\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right),\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right)\right.$ ), which is zero (by (3.1.3f), for example). Hence $\times(\varphi, F)$ is well defined. Let us show that the conditions (3.1.2) are satisfied. Property (3.1.2a) functoriality-follows immediately from the functoriality of the pairing with Fredholm pairs. For property (3.1.2b) additivity-given Fredholm modules $(\varphi, F)$ and ( $\varphi, G$ ) (with $F$ and $G$ unitary modulo compact operators) and having chosen $\bar{F}$ and $\bar{G}$ as "dilations" for $F$ and $G$, we may choose $\bar{F} \bar{G}$ as a dilation for $F G$. Then by (3.1.4),

$$
\begin{aligned}
\times(\varphi, F)+\times(\varphi, G)= & \times\left(\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right), \operatorname{Ad}\left(\bar{G}^{*}\right)\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right)\right) \\
& +\times\left(\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right), \operatorname{Ad}\left(\bar{F}^{*}\right)\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right)\right) \\
= & \times\left(\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right), \operatorname{Ad}\left(\bar{G}^{*} \bar{F}^{*}\right)\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right)\right) \\
= & \times(\varphi, F G) .
\end{aligned}
$$

Property (3.1.2c) follows easily from the corresponding property (3.1.3c). Finally, we consider the non-degeneracy condition (3.1.2d). Let $V$ be an index one coisometry and let $\bar{V}=\left(1-v_{*}^{v} v_{v^{*}}^{0}\right)$. If $\varphi: \mathbf{C} \rightarrow \mathscr{B}(\mathscr{H})$ is unital then the $*$-homomorphism $\operatorname{Ad}\left(\bar{V}^{*}\right)\binom{\left(\begin{array}{l}0 \\ 0\end{array}\right.}{0}$ takes $1 \in \mathbf{C}$ to $\left(\begin{array}{cc}1-e & 0 \\ 0\end{array}\right)$, where $e$ is a rank-one projection. Applying the stability condition (3.1.3c) we get that

$$
\begin{aligned}
\times(\varphi, V) & =\times\left(\left(\begin{array}{ll}
\varphi & 0 \\
0 & 0
\end{array}\right),\left(\begin{array}{cc}
\varphi_{(1-e)} & 0 \\
0 & 0
\end{array}\right)\right) \\
& =\times\left(\left(\begin{array}{cc}
\varphi_{e} & 0 \\
0 & 0
\end{array}\right), 0\right),
\end{aligned}
$$

and this last map is the identity on $E(A)$ by the non-degeneracy condition (3.1.3d). It follows that the pairing with Fredholm modules we have constructed satisfies the hypotheses of Theorem 3.1.1, and so $E$ is a homotopy functor.

### 3.2. Excision Properties

In order to apply the previous results, it is of course necessary to construct pairings between a given functor $E$ and the set of Fredholm modules, or Fredholm pairs. Sometimes (as with extension groups-see Sect. 3.4), it is possible to do this directly from the definition of the groups $E(A)$. Often though, as with algebraic $K$-theory, it is much easier to deduce the existence of a pairing from properties of $E$ as a functor.

Definition 3.2.1. We shall call a functor $E$ split exact if for every split, short exact sequence of $C^{*}$-algebras

$$
0 \rightarrow J \xrightarrow{j} D \underset{p}{\stackrel{s}{\leftrightarrows}} D / J \rightarrow 0,
$$

the sequence of abelian groups

$$
0 \rightarrow E(J) \xrightarrow{j_{*}} E(D) \underset{p_{*}}{\stackrel{s_{*}}{\leftrightarrows}} E(D / J) \rightarrow 0
$$

is also split exact.
Our main result is

Theorem 3.2.2. If $E$ is split exact, and if in addition $E$ is stable (see Definition 2.1.8), then $E$ is homotopy invariant.

The proof relies on the construction of split exact sequences associated with Fredholm pairs: then by means of these, and the split exactness of $E$, we construct a pairing of $E$ with the set of Fredholm pairs. The whole procedure is essentially due to Cuntz [19, 20], who associates with any Fredholm module, or more generally, any $K K$-element, a "quasihomomorphism" (compare (3.2.2) below), and then constructs pairings between quasihomomorphisms and functors of the same general character as $E$.

Definition 3.2.3. If $\varphi: B \rightarrow \mathscr{B}(\mathscr{H})$ is a $*$-homomorphism then let $B_{\varphi}$ be the $C^{*}$-algebra

$$
\begin{equation*}
B_{\varphi}=\{b \oplus x \in B \oplus \mathscr{B} \mid \varphi(b)=x, \text { modulo } \mathscr{K}\} \tag{3.2.1}
\end{equation*}
$$

Note that $B_{\varphi}$ fits into a short exact sequence

$$
0 \rightarrow \mathscr{K} \xrightarrow{j} B_{\varphi} \xrightarrow{p} B \rightarrow 0,
$$

where $p: B_{\varphi} \rightarrow B$ is the obvious projection onto the $B$-summand of $B \oplus \mathscr{B}$, and $j: \mathscr{K} \rightarrow B_{\varphi}$ maps $\mathscr{K}$ into the $\mathscr{B}$ summand of $B_{\varphi}$ in the natural way:
$j(x)=0 \oplus x$. Furthermore, (3.2.1) is split by the $*$-homomorphism $\hat{\varphi}: B \rightarrow B_{\varphi}$ defined by $\hat{\varphi}(b)=b \oplus \varphi(b)$.

We make the important observation that if $\varphi^{\prime}: B \rightarrow \mathscr{B}$ is equal to $\varphi: B \rightarrow \mathscr{B}$, modulo $\mathscr{K}$, then $B_{\varphi}=B_{\varphi^{\prime}}$, and furthermore, the exact sequence (3.2.1) remains unchanged with $B_{\varphi}$, replacing $B_{\varphi}$. However, the section $\hat{\varphi}^{\prime}: B \rightarrow B_{\varphi}$ is different. Now, let $E$ be a stable and split exact functor, let $\left(\varphi_{+}, \varphi_{-}\right)$be a Fredholm pair, and let $A$ be a $C^{*}$-algebra. The short exact sequence

$$
0 \rightarrow A \otimes \mathscr{K} \xrightarrow{j} A \otimes B_{\varphi_{+}} \xrightarrow{\rho} A \otimes B \rightarrow 0
$$

(where, for simplicity, we write $j$ and $p$ instead of $1 \otimes j$ and $1 \otimes p$ ) is split by either of the $*$-homomorphisms $\hat{\varphi}_{+}, \hat{\varphi}_{-}: A \otimes B \rightarrow A \otimes B_{\varphi_{+}}$. Since $p \hat{\varphi}_{+}=\mathrm{id}_{A \otimes B}=p \hat{\varphi}_{-}$, the homomorphism $p_{*}\left(\hat{\varphi}_{+*}-\hat{\varphi}_{-*}\right): E(A \otimes B) \rightarrow$ $E(A \otimes B)$ is equal to zero or, in other words $\hat{\varphi}_{+*}-\hat{\varphi}_{-*} \operatorname{maps} E(A \otimes B)$ into the kernel of $p_{*}: E\left(A \otimes B_{\varphi_{+}}\right) \rightarrow E(A \otimes B)$. But by split exactness, this kernel is exactly $E(A \otimes \mathscr{K})$ or, to be precise, the isomorphic image of $E(A \otimes \mathscr{K})$ in $E\left(A \otimes B_{\varphi_{+}}\right)$. Consequently, from the Fredholm pair $\left(\varphi_{+}, \varphi_{-}\right)$we obtain a map $\times\left(\varphi_{+}, \varphi_{-}\right): E(A \otimes B) \rightarrow E(A)$,

$$
\begin{equation*}
E(A \otimes B) \xrightarrow{\dot{\varphi}_{+*}-\dot{\varphi}_{-*}} \operatorname{kernel}\left(p_{*}\right) \xrightarrow{\cong} E(A \otimes \mathscr{H}) \xrightarrow{\cong} E(A) . \tag{3.2.2}
\end{equation*}
$$

Proof of Theorem 3.2.2. It suffices to show that the pairing defined by (3.2.2) satisfies the hypotheses (3.1.3) of Theorem 3.1.4. The first two-additivity and functoriality-are straightforward, and are left to the reader. As for the stability property ( 3.1 .3 c ), consider the commuting diagram
where $f(b \oplus x)=b \oplus\left(\begin{array}{cc}x \\ 0 \\ 0 & 0 \\ \hline\end{array}\right)$. . Note only does (3.2.3) commute, but also, the sections $\hat{\varphi}_{ \pm}$of the top short exact sequence correspond via $f$ to the sections $\left(\begin{array}{l}\varphi_{+} \\ 0 \\ 0 \\ 0\end{array}\right)$ (or the bottom sequence. It follows then from the definition (3.2.2) of the pairing that

$$
\times\left(\varphi_{+}, \varphi_{-}\right)=\times\left(\left(\begin{array}{cc}
\varphi_{+} & 0 \\
0 & \varphi
\end{array}\right),\left(\begin{array}{cc}
\varphi- & 0 \\
0 & \varphi
\end{array}\right)\right) .
$$

The non-degeneracy condition (3.1.3d) is satisfied because for the Fredholm C-pair ( $e, 0$ ), we have $\mathbf{C}_{e}=\mathbf{C} \oplus \mathscr{K}$; and $\hat{e}: \mathbf{C} \rightarrow \mathbf{C} \oplus \mathscr{K}$ is given by $\hat{e}(\lambda)=\lambda \oplus \lambda e$, while $\hat{0}(\lambda)=\lambda \oplus 0$. It follows from Lemma 2.1.18 that
$\hat{e}_{*}-\hat{0}_{*}=(\hat{e}-\hat{0})_{*}$, and since $\hat{e}-\hat{0}: \mathbf{C} \rightarrow \mathscr{K}$ is precisely the canonical map, it follows from the definition (3.2.2) that $\times(e, 0)=\mathrm{id}$. Condition (3.1.3e) follows from the commuting diagram

because by Lemma $2.6 .12, \operatorname{Ad}(1 \otimes U)_{*}: E(A \otimes \mathscr{K}) \rightarrow E(A \otimes \mathscr{K})$ is equal to the identity. Similarly, for condition (3.1.3f), if $U$ is equal to the identity, modulo compacts, then $1 \otimes U$ is an element of the multiplier algebra of $A \otimes B_{\varphi}$, and so $\hat{\varphi}_{*}-\operatorname{Ad}(1 \otimes U)_{*} \hat{\varphi}_{*}=0$.

Remark 3.2.4. We point out that in order to prove the above theorem, it of course sufficed to know that $E$ was split exact with respect to the split exact sequences

$$
0 \rightarrow \mathscr{K} \rightarrow B_{\varphi} \rightarrow B \rightarrow 0
$$

where $B$ is commutative. We will need the following rather technical observation in Section V: the short exact sequence

$$
0 \rightarrow \operatorname{Ann}(\mathscr{K}) \rightarrow B_{\varphi} \rightarrow B_{\varphi} / \operatorname{Ann}(\mathscr{K}) \rightarrow 0
$$

has a completely positive section. Indeed, the annihilator ideal of $\mathscr{K}$ in $B_{\varphi}$ is of the form $J \oplus 0$, where $J$ is some ideal in $B$. Since $B$ is commutative, the projection map $B \rightarrow B / J$ has a completely positive section $s: B / J \rightarrow B$; then $\dot{b} \oplus x \mapsto s(\dot{b}) \oplus x$ is a completely positive section for the above short exact sequence. (Another way of dealing with this point: it is easy enough to arrange things so that we need only consider short exact sequences where $\mathscr{K}$ is an essential ideal in $B_{\varphi}$.)

### 3.3. The Quasi-Unital Case

As a cursory inspection of the proof of Theorem 3.1.1 reveals, in order to establish homotopy invariance of a functor $E$ we need only construct a pairing of $E$ with the sets of Fredholm $C[0,1]$-modules and Fredholm C-modules, which satisfies the conditions (3.1.2). Furthermore, in the functoriality condition (3.1.2a), we need only consider surjections (indeed, in the proof, functoriality is used only with respect to the maps $\varepsilon_{0}, \varepsilon_{1}$ : $C[0,1] \rightarrow \mathbf{C}$ given by evaluation at zero and one). Thus the analogs of Theorems 3.11 and 3.1 .4 for functors from $C^{*}$-algebras and quasi-unital *-homomorphisms to abelian groups are easily proved.

The theorem below is an analog of Theorem 3.2.2 for functors $E$ from
$C^{*}$-algebras and quasi-unital *-homomorphisms to abelian groups. Since we do not need the result in this work, we will not spell out the proof in detail.

THEOREM 3.3.1. Let $E$ be a stable functor from $C^{*}$-algebras and quasiunital *-homomorphisms to abelian groups. Suppose that $E$ is split exact in the weakened sense that associated to every short exact sequence of $C^{*}$-algebras

$$
0 \rightarrow J \rightarrow B \xrightarrow{p} B / J \rightarrow 0
$$

which is split by some quasi-unital *-homomorphism, there is a homomorphism $\pi$ : $\operatorname{kernel}\left(p_{*}\right) \rightarrow E(J)$ such that:
(i) The map $\pi$ is natural, in the sense that a commutative diagram

where the vertical maps are quasi-unital, gives rise to a commuting square

(ii) In the case of an exact sequence of the form

$$
0 \rightarrow J \rightarrow J \oplus B / J \rightarrow B / J \rightarrow 0
$$

$\operatorname{kernel}\left(p_{*}\right)$ is equal to the image of $E(J)$ in $E(J \oplus B / J)$, and furthermore, $\pi$ : $\operatorname{kernel}\left(p_{*}\right) \rightarrow E(J)$ is induced from the projection of $J \oplus B / J$ onto $J$.

Then $E$ is homotopy invariant, in the sense that the maps $\varepsilon_{0 *}, \varepsilon_{1 *}$ : $E(A \otimes C[0,1]) \rightarrow E(A)$ induced from evaluation at zero and one are equal.

Proof. We construct a pairing with Fredholm $B$-pairs ( $\varphi_{+}, \varphi_{-}$), where $B$ is unital, by means of the following analog of (3.2.2):

$$
\begin{equation*}
E(A \otimes B) \xrightarrow{\stackrel{\varphi_{+}-\hat{\varphi}_{-}}{*}} E\left(A \otimes B_{\varphi}\right) \xrightarrow{\pi} E(A \otimes \mathscr{K}) \xrightarrow{\cong} E(A) . \tag{3.3.1}
\end{equation*}
$$

Here, $\pi$ is the projection corresponding to the split exact sequence

$$
0 \rightarrow A \otimes \mathscr{K} \rightarrow A \otimes B_{\varphi} \rightarrow A \otimes B \rightarrow 0
$$

It is a straightforward matter to check that the conditions (3.13) are satisfied (with the necessary restrictions to quasi-unital maps: for example, we need only verify that functoriality-condition (3.1.3a)-holds for maps between unital algebras). The result then follows from the quasi-unital analog of Theorem 3.1.4.

### 3.4. Application to Extension Groups

Our goal in this section is to show how $\operatorname{Ext}^{-1}(C, A)$ pairs with Fredholm modules. There are a number of possibilities here, for example,

$$
\begin{align*}
\mathrm{Ext}^{-1}(C, A) \times\{\text { Fredholm } B \text {-modules }\} & \rightarrow \mathrm{Ext}^{-1}(C \otimes B, A),  \tag{3.4.1}\\
\mathrm{Ext}^{-1}(C, A \otimes B) \times\{\text { Fredholm } B \text {-modules }\} & \rightarrow \operatorname{Ext}^{-1}(C, A),  \tag{3.4.2}\\
\mathrm{Ext}^{-1}(C, B) \times\{\text { Fredholm } B \text {-modules }\} & \rightarrow \mathrm{Ext}^{-1}(C, \mathrm{C}), \tag{3.4.3}
\end{align*}
$$

and furthermore, we may generalize and consider "Fredholm ( $B_{1}, B_{2}$ )modules," or in other words, Kasparov bimodules (see [49], from where the terminology originates). All of these are particular cases of the Kasparov product - see [35]-and our aim is to indicate that the product is quite a natural thing in the context of extension theory. We will consider only the pairing (3.4.1) in detail, leaving it to the reader to make the simple adaptations to the other cases.

In the following, $A$ will be assumed to be $\sigma$-unital and $B$ and $C$ will be assumed to be separable.

It is a little more convenient to work with Fredholm pairs than with Fredholm modules. Let us begin by constructing the map Ext ${ }^{1}(C, A) \rightarrow \operatorname{Ext}^{1}(C \otimes B, A)$ corresponding to a Fredholm pair $\left(\varphi_{+}, \varphi_{-}\right)$of the simplest sort, where $\varphi_{+}$and $\varphi_{-}$are $*$-homomorphisms from $B$ to $\mathscr{K}$. In fact, let us consider first the even simpler Fredholm pair $\left(\varphi_{+}, 0\right)$. From $\varphi_{+}$and an extension $\psi: C \rightarrow \mathscr{2}(A)$ we obtain an extension of $C \otimes B$ by $\mathscr{K} \otimes A \otimes \mathscr{K}(\cong \mathscr{K} \otimes A)$,

$$
C \otimes B \xrightarrow{\mu \otimes \leftrightarrow}(\mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A) \otimes \mathscr{K} \leftrightarrows \mathscr{M}(\mathscr{K} \otimes A \otimes \mathscr{K}) / \mathscr{K} \otimes A \otimes \mathscr{K} .
$$

To obtain the pairing with $\left(0, \varphi_{-}\right)$we do a natural enough thing: we define it to be the negative of the pairing obtained from $\left(\varphi_{-}, 0\right)$. Putting the two together, we define

$$
\begin{align*}
\times\left(\varphi_{+}, \varphi_{-}\right) & =\times\left(\varphi_{+}, 0\right)+\times\left(0, \varphi_{-}\right) \\
& =\times\left(\varphi_{+}, 0\right)-\times\left(\varphi_{-}, 0\right) \tag{3.4.4}
\end{align*}
$$

Now, for a general pair ( $\varphi_{+}, \varphi_{-}$), where $\varphi_{ \pm}: B \rightarrow \mathscr{B}$ do not necessarily map into $\mathscr{K}$, it is not possible to define $\times\left(\varphi_{+}, \varphi_{-}\right)$in this manner. But a
closer examination of (3.4.4) shows us what to do. Assuming that $\psi$ is an invertible extension, we may choose a *-homomorphism

$$
\left(\begin{array}{ll}
\psi_{11} & \psi_{12}  \tag{3.4.5}\\
\psi_{21} & \psi_{22}
\end{array}\right): \quad C \rightarrow M_{2}(. \mathscr{M}(\mathscr{K} \otimes A))
$$

such that $\psi_{11}: C \rightarrow \mathscr{M}(\mathscr{K} \otimes A)$ is a lifting of $\psi: C \rightarrow \mathscr{M}(\mathscr{K} \otimes A) / \mathscr{K} \otimes A$. The extension obtained by applying the pairing (3.4.4) is, by the definition of the inverse of an extension, simply the extension determined by the map

$$
\left(\begin{array}{l}
\psi_{11} \otimes \varphi_{+}  \tag{3.4.6}\\
\psi_{21} \otimes \psi_{12} \otimes \varphi_{+} \\
\psi_{22} \otimes \varphi_{-}
\end{array}\right): \quad C \otimes B \rightarrow M_{2}(\mathscr{M}(\mathscr{K} \otimes A) \otimes \mathscr{M}(\mathscr{K}))
$$

by following with the inclusion

$$
M_{2}(\mathscr{M}(\mathscr{K} \otimes A) \otimes \mathscr{M}(\mathscr{K})) \subseteq M_{2}(\mathscr{M}(\mathscr{K} \otimes A \otimes \mathscr{K})),
$$

and then the projection onto the quotient

$$
M_{2}(\mathscr{M}(\mathscr{K} \otimes A \otimes \mathscr{K})) \rightarrow M_{2}(\mathscr{M}(\mathscr{K} \otimes A \otimes \mathscr{K}) / \mathscr{K} \otimes A \otimes \mathscr{K}) .
$$

Now, the off-diagonal terms of the map (3.4.6) are of no particular relevance in the case of a Fredholm module ( $\varphi_{+}, \varphi_{\ldots}$ ), where $\varphi_{ \pm}$map $B$ into $\mathscr{K}$ (anything that maps $C$ into $\mathscr{K} \otimes A \otimes \mathscr{K}$ would do). However:

Lemma 3.4.1. If $\left(\varphi_{+}, \varphi_{-}\right)$is any Fredholm pair then the map (3.4.6) is a *-homomorphism, modulo $\mathscr{K} \otimes A \otimes \mathscr{K}$, and hence determines an extension of $C \otimes B$ by $\mathscr{K} \otimes A \otimes \mathscr{K}$.

Proof. This is a simple, direct computation on elementary tensors $c \otimes b \in C \otimes B$, using the observation that $\psi_{12} \otimes \varphi_{+}$and $\psi_{21} \otimes \varphi_{+}$are equal to $\psi_{12} \otimes \varphi_{-}$and $\psi_{21} \otimes \varphi_{-}$, modulo $\mathscr{K} \otimes A \otimes \mathscr{K}$ (which, in turn, follows from the fact that $\psi_{12}$ and $\psi_{21}$ map $C$ into $\mathscr{K} \otimes A$ ). We extend from $C \odot B$ to $C \otimes B$ by using the continuity of the map (3.4.6).

We define the pairing between ( $\varphi_{+}, \varphi_{-}$) and an extension $\psi$ to be the extension obtained from (3.4.6). It must be shown that this passes to a map $\mathrm{Ext}^{-1}(C, A) \rightarrow \mathrm{Ext}^{-1}(C \otimes B, A)$; the following lemma is the technical result needed to establish this.

Lemma 3.4.2. If the extension $\psi$ determines the zero element of $\operatorname{Ext}^{-1}(C, A)$ (that is, if $\psi$ is stably split), then the extension given by (3.4.6) determines the zero element of $\operatorname{Ext}^{-1}(C \otimes B, A)$.

Proof. By hypothesis, there exists a *-homomorphism $\theta: C \rightarrow \mathscr{M}(\mathscr{K} \otimes A)$ such that the extension $\dot{\theta} \oplus \psi: C \rightarrow M_{2}(\mathscr{2}(A))$ is degenerate. Since the map

$$
\left(\begin{array}{ll}
\psi_{11} \otimes \varphi_{+} & \psi_{12} \otimes \varphi_{+} \\
\psi_{21} \otimes \varphi_{+} & \psi_{22} \otimes \varphi_{-}
\end{array}\right)
$$

passes to a stably split extension if and only if the map

$$
\left(\begin{array}{ccc}
\theta \otimes \varphi_{+} & 0 & 0 \\
0 & \psi_{11} \otimes \varphi_{+} & \psi_{12} \otimes \varphi_{+} \\
0 & \psi_{21} \otimes \varphi_{+} & \psi_{22} \otimes \varphi_{-}
\end{array}\right)
$$

does, by replacing $\psi_{11}$ with $\left(\begin{array}{cc}\theta & 0 \\ 0 & \psi_{11}\end{array}\right), \psi_{12}$ with $\binom{0}{\psi_{12}}$, and $\psi_{21}$ with $\left(0, \psi_{21}\right)$, we may assume that the extension $\psi$ is actually split. Thus we may assume that there exists a $*$-homomorphism $\bar{\Psi}: \rightarrow \mathscr{M}(\mathscr{K} \otimes A)$ which lifts the extension $\psi$, and which is therefore equal to $\psi_{11}$, modulo $\mathscr{K} \otimes A$. Now, let $\widetilde{E}_{1}$, $E_{2}$, and $\mathscr{F}$ be $C^{*}$-subalgebras of $M_{2}(\mathscr{M}(\mathscr{K} \otimes A \otimes \mathscr{K}))$, generated by the elements (where $b, c$, and $k$ denote arbitrary elements of $B, C$, and $\mathscr{K}$, respectively):

$$
\begin{aligned}
& \tilde{E}_{1}:\left(\begin{array}{cc}
\psi_{11}(c) \otimes k & 0 \\
0 & 0
\end{array}\right)\left(\begin{array}{cc}
\tilde{\psi}(c) \otimes k & 0 \\
0 & 0
\end{array}\right), \\
& E_{2}:\left(\begin{array}{cc}
0 & \psi_{12}(c) \otimes \varphi_{ \pm}(b) \\
\psi_{21}(c) \otimes \varphi_{ \pm}(b) & \psi_{22}(c) \otimes \varphi_{-}(b)
\end{array}\right)\left(\begin{array}{cc}
\left(\tilde{\psi}(c)-\psi_{11}(c)\right) \otimes \varphi_{ \pm}(b) & 0 \\
0 & 0
\end{array}\right), \\
& \mathscr{F}:\left(\begin{array}{cc}
\psi_{11}(c) \otimes \varphi_{ \pm}(b) & \psi_{12}(c) \otimes \varphi_{ \pm}(b) \\
\psi_{21}(c) \otimes \varphi_{ \pm}(b) & \psi_{22}(c) \otimes \varphi_{-}(b)
\end{array}\right)\left(\begin{array}{cc}
\tilde{\psi}(c) \otimes \varphi_{-}(b) & 0 \\
0 & 0
\end{array}\right) .
\end{aligned}
$$

Then $\tilde{E}_{1} \cdot E_{2} \subset M_{2}(\mathscr{K} \otimes A \otimes \mathscr{K})$ (it is helpful to remember that $\psi_{12}$ and $\psi_{21}$ map into $\mathscr{K} \otimes A)$ and also $\widetilde{F}_{1} \cdot \mathscr{F} \cdot E_{2} \subset M_{2}(\mathscr{K} \otimes A \otimes \mathscr{K})$. So if we let $E_{1}$ be the $C^{*}$-algebra generated by $\widetilde{E}_{1}$ and $\widetilde{E}_{1} \cdot \mathscr{F}$, then

$$
E_{1} \cdot E_{2} \subset M_{2}(\mathscr{K} \otimes A \otimes \mathscr{K}) \quad \text { and } \quad \mathscr{F} \cdot E_{1} \subset E_{1}
$$

By Theorem 1.1.11, there exists an operator $M \in M_{2}(\mathscr{M}(\mathscr{K} \otimes A \otimes \mathscr{K}))$ such that $1 \geqslant M \geqslant 0, \quad M \cdot E_{1} \subset \dot{M}_{2}(\mathscr{K} \otimes A \otimes \mathscr{K}), \quad(1-M) \cdot E_{2} \subset$ $M_{2}(\mathscr{K} \otimes A \otimes \mathscr{K})$, and $M$ commutes with $\mathscr{F}$, modulo $M_{2}(\mathscr{K} \otimes A \otimes \mathscr{K})$. Let $N=1-M$. Then using the computation

$$
\begin{align*}
& \left(\begin{array}{cc}
-\dot{M}^{1 / 2} & \dot{N}^{1 / 2} \\
\dot{N}^{1 / 2} & \dot{M}^{1 / 2}
\end{array}\right)\left(\begin{array}{cc}
\alpha & 0 \\
0 & \beta
\end{array}\right)\left(\begin{array}{cc}
-\dot{M}^{1 / 2} & \dot{N}^{1 / 2} \\
\dot{N}^{1 / 2} & \dot{M}^{1 / 2}
\end{array}\right) \\
& \quad=\left(\begin{array}{cc}
\dot{M} \alpha+\dot{N} \beta & \dot{M}^{1 / 2} \dot{N}^{1 / 2}(\beta \\
\dot{M}^{1 / 2} \dot{N}^{1 / 2}(\beta-\alpha) & \dot{M} \beta+\dot{N} \alpha
\end{array}\right) \tag{3.4.7}
\end{align*}
$$

which is valid for $\alpha$ and $\beta$ which commute with $\dot{M}$, we see that the direct sum

$$
\left(\begin{array}{ll}
\psi_{11} \otimes \varphi_{+} & \psi_{12} \otimes \varphi_{+} \\
\psi_{21} \otimes \varphi_{+} & \psi_{22} \otimes \varphi_{-}
\end{array}\right) \oplus\left(\begin{array}{cc}
\tilde{\psi} \otimes \varphi_{-} & 0 \\
0 & 0
\end{array}\right)
$$

is unitarily equivalent to the direct sum

$$
\left(\begin{array}{ll}
\psi_{11} \otimes \varphi_{-} & \psi_{21} \otimes \varphi_{-} \\
\psi_{21} \otimes \varphi_{-} & \psi_{22} \otimes \varphi_{-}
\end{array}\right) \oplus\left(\begin{array}{cc}
\tilde{\psi} \otimes \varphi_{+} & 0 \\
0 & 0
\end{array}\right),
$$

modulo compacts, via the unitary $\left(\begin{array}{c}-M_{N^{1 / 2}}^{1 / 2}\end{array} M_{M^{1 / 2}}^{N_{2}}\right.$. Since the former is stably equal to the extension (3.4.6) we started with, while the latter is a split extension, the lemma is proved.

Theorem 3.4.3. For any Fredholm pair $\Phi=\left(\varphi_{+}, \varphi_{-}\right)$, the construction (3.4.6) passes to a homomorphism

$$
\Phi^{*}: \quad \operatorname{Ext}^{-1}(C, A) \rightarrow \operatorname{Ext}^{-1}(C \otimes B, A) .
$$

Proof. The construction (3.4.6) of the pairing involves, for a given $x \in \operatorname{Ext}^{-1}(C, A)$, choosing an extension $\psi$ such that $[\psi]=x$, and then choosing a $*$-homomorphism $\left(\begin{array}{l}\psi_{21} \\ \psi_{21}\end{array} \psi_{22}\right)$ such that $\psi_{11}$ lifts $\psi$. We must show that $\Phi^{*} x$ depends on neither of these choices. Note, first, that the construction is in an obvious sense additive: if we choose for $y \in \operatorname{Ext}^{-1}(C, A)$ an extension $\sigma$ and $*$-homomorphism $\left(\begin{array}{c}\sigma_{11} \\ \sigma_{2} \\ \sigma_{12} \\ \sigma_{2}\end{array}\right)$, then we may choose the extension $\psi \oplus \sigma$ and the $*$-homomorphism $\left(\psi_{i j}\right) \oplus\left(\sigma_{i j}\right)$ for the element $x+y \in \operatorname{Ext}^{-1}(C, A)$. Having done so, if we define $\Phi^{*} x, \Phi^{*} y$, and $\Phi^{*}(x+y)$ by means of these choices, then

$$
\begin{equation*}
\Phi^{*} x+\Phi^{*} y=\Phi^{*}(x+y) . \tag{3.4.8}
\end{equation*}
$$

This observation is of importance because we can now bring the lemma to bear: choosing $y$ to be $-x$, we see from the lemma that $\Phi^{*}(x+y)$ is stably split, and therefore $\Phi^{*} x$ is invertible by (3.4.8). Furthermore, if we choose another construction for $\Phi^{*} x$ by means of some $\psi^{\prime}$ and some $\left(\psi_{i j}^{\prime}\right)$-then since $\Phi^{*}(-x)$, as constructed from $\sigma$ and ( $\sigma_{i j}$ ), is an inverse in Ext ${ }^{-1}(C \otimes B, A)$ for both constructions of $\Phi^{*} x$, these constructions must give the same elements of $\operatorname{Ext}^{-1}(C \otimes B, A)$.

Let us go on to show that the pairing so defined statisfies the hypotheses (3.1.3) of the homotopy invariance Theorem 3.1.4. We will revert to the notation $\times\left(\varphi_{+}, \varphi_{-}\right)$for the map $\Phi^{*}$ associated with the Fredholm pair $\left(\varphi_{+}, \varphi_{-}\right)$.

Lemma 3.4.4. If $\left(\varphi_{1}, \varphi_{2}\right)$ and $\left(\varphi_{2}, \varphi_{3}\right)$ are Fredholm pairs then

$$
\times\left(\varphi_{1}, \varphi_{2}\right)+\times\left(\varphi_{2}, \varphi_{3}\right)=\times\left(\varphi_{1}, \varphi_{3}\right)
$$

Proof. We will show that the direct sum

$$
\left(\begin{array}{ll}
\psi_{11} \otimes \varphi_{1} & \psi_{12} \otimes \varphi_{1} \\
\psi_{21} \otimes \varphi_{1} & \psi_{22} \otimes \varphi_{2}
\end{array}\right) \oplus\left(\begin{array}{ll}
\psi_{11} \otimes \varphi_{2} & \psi_{12} \otimes \varphi_{2} \\
\psi_{21} \otimes \varphi_{2} & \psi_{22} \otimes \varphi_{3}
\end{array}\right)
$$

of the extensions obtained by applying ( $\varphi_{1}, \varphi_{2}$ ) and ( $\varphi_{2}, \varphi_{3}$ ), is unitarily equivalent, modulo compacts, to

$$
\left(\begin{array}{ll}
\psi_{11} \otimes \varphi_{1} & \psi_{12} \otimes \varphi_{1} \\
\psi_{21} \otimes \varphi_{1} & \psi_{22} \otimes \varphi_{3}
\end{array}\right) \otimes\left(\begin{array}{ll}
\psi_{11} \otimes \varphi_{2} & \psi_{12} \otimes \varphi_{2} \\
\psi_{21} \otimes \varphi_{2} & \psi_{22} \otimes \varphi_{2}
\end{array}\right)
$$

which is the direct sum of the extension obtained by applying ( $\varphi_{1}, \varphi_{3}$ ) and a trivial extension. This will, of coursc, prove the lemma. Using Theorem 1.1.11, we obtain an element $M$ of $M_{2}(\mathscr{H}(\mathscr{K} \otimes A \otimes \mathscr{K}))$ such that:
(i) $1 \geqslant M \geqslant 0$.
(ii) $(1-M) \cdot\left(\begin{array}{lc}\psi_{11}(c) \otimes \varphi_{1}(b) & \psi_{12}(c) \otimes \varphi_{1}(b) \\ \psi_{21}(c) \otimes \varphi_{1}(b) & 0\end{array}\right)$ is compact for all $b \in B$ and all $c \in C$.
(iii) $M \cdot\left(\begin{array}{lc}0 & 0 \\ 0 & \psi_{22}(c) \otimes\left(\varphi_{2}(b)-\varphi_{3}(b)\right)\end{array}\right)$ is compact for every $b \in B$ and $c \in C$.
(iv) $M$ commutes, modulo compacts, with all elements of the form $\left(\begin{array}{ll}\psi_{11}(c) \otimes \varphi_{1}(b) & \psi_{12}(c) \otimes \varphi_{1}(b) \\ \psi_{21}(c) \otimes \varphi_{1}(b) & \psi_{22}(c) \otimes \varphi_{2}(b)\end{array}\right) \quad$ or $\quad\left(\begin{array}{ll}\psi_{11}(c) \otimes \varphi_{2}(b) & \psi_{12}(c) \otimes \varphi_{2}(b) \\ \psi_{21}(c) \otimes \varphi_{2}(b) & \psi_{22}(c) \otimes \varphi_{3}(b)\end{array}\right)$.

Then it follows from the properties of $M$, together with the computation (3.4.7), that the unitary $\left(\begin{array}{cl}-M^{1 / 2} & N^{1 / 2}\end{array}\right)$ implements the desired essential unitary equivalence.

This establishes the additivity property of the pairing. The remaining hypotheses of Theorem 3.1.4 are easily verified. Let us, for example, consider one more: stability under compact perturbations. We must show that if $U \in \mathscr{B}$ is a unitary which is equal to the identity, modulo the compact operators, then the pairing with the Fredholm pair $(\varphi, \operatorname{Ad}(U) \varphi)$ is trivial. Thus we must show that the extension

$$
\left(\begin{array}{cc}
\psi_{11} \otimes \varphi & \psi_{12} \otimes \varphi \\
\psi_{21} \otimes \varphi & \psi_{22} \otimes \operatorname{Ad}(U) \varphi
\end{array}\right)
$$

is trivial. But conjugation with the unitary $\left(\begin{array}{cc}1 & 0 \\ 0 & 1 \\ \otimes\end{array}\right)$ shows that this extension is equivalent to the extension

$$
\left(\begin{array}{cc}
\psi_{11} \otimes \varphi & \psi_{12} \otimes \varphi U^{*} \\
\psi_{21} \otimes U \varphi & \psi_{22} \otimes \varphi
\end{array}\right),
$$

and since $U$ is equal to the identity, modulo the compacts, this is equal, modulo the compacts, to the map

$$
\left(\begin{array}{ll}
\psi_{11} \otimes \varphi & \psi_{12} \otimes \varphi \\
\psi_{21} \otimes \varphi & \psi_{22} \otimes \varphi
\end{array}\right),
$$

which certainly determines the trivial extension. Thus from 3.1.4 we obtain the following well-known result (due to Kasparov [34] in this generality, and to Brown, Douglas, and Fillmore [11,12], in the case $A=\mathbf{C}$ ).

Theorem 3.4.5. The functor $\operatorname{Ext}^{-1}(C, A)$ is homotopy invariant in the first variahle.

Now, we do not want to go into the details, but let us mention that we may modify the above arguments and constructions so as to obtain a pairing between $\operatorname{Ext}^{-1}(C, A)$, in the second variable, and the class of quasi-unital Fredholm pairs. We then obtain

Theorem 3.4.6. The functor $\operatorname{Ext}^{-1}\left(A, C^{\prime}\right)$ is homotopy invariant in the second variable.

### 3.5. Excision Properties of Extension Groups

We have two goals in this section. First, we want to point out how excision properties for the extension groups may be easily obtained from the separation theorem of Kasparov (Theorem 1.1.11). Second, we want to remind the reader how the excision properties, together with the homotopy invariance results of this section, and the Bott periodicity theorem, may be put together to obtain the beautiful result of Brown, Douglas, and Fillmore $[11,12]$ that $\operatorname{Ext}^{-1}(C(X), \mathbf{C})$ is equal to the $K$-homology of $X$. The reason for doing this is that the main results of this paper are of a similar type-the equality of an algebraically defined object and a topologically defined one-and so it seems worthwhile to present the most outstanding result of this kind, since it is easily within our reach.

Because it simplifies a number of points, we will consider only the Brown-Douglas-Filmore group $\mathrm{Ext}^{-1}(A, \mathrm{C})$, and not the more general Kasparov extension groups $\operatorname{Ext}^{-1}(A, B)$. We will use the abbreviated notation $\operatorname{Ext}^{-1}(A)=\operatorname{Ext}^{-1}(A, \mathbf{C})$. All arguments of the functor $\mathrm{Ext}^{-1}$ will be separable $C^{*}$-algebras.

Theorem 3.5.1. The functor Ext $^{-1}$ is split exact.
Proof. Suppose that

$$
0 \rightarrow A \xrightarrow{j} B \underset{\rho}{\stackrel{s}{\leftrightarrows}} C \rightarrow 0
$$

is a split exact sequence of $C^{*}$-algebras and $*$-homomorphisms. We must show that the sequence

$$
\begin{equation*}
0 \rightarrow \operatorname{Ext}^{-1}(C) \stackrel{p^{*}}{\stackrel{s^{*}}{ }} \operatorname{Ext}^{-1}(B) \xrightarrow{j^{*}} \operatorname{Ext}^{-1}(A) \rightarrow 0 \tag{3.5.1}
\end{equation*}
$$

is a split exact sequence of abelian groups. Obviously, $j^{*} p^{*}=0$. If $[\varphi] \in \operatorname{Ext}^{-1}(B)$ and $j^{*}[\varphi]=0$, then by definition, there exists a *-homomorphism $\psi: A \rightarrow \mathscr{B}(\mathscr{H})$ such that $\varphi j \oplus \dot{\psi}$ lifts to a *-homomorphism from $A$ into $\mathscr{B}\left(\mathscr{H}^{(2)}\right)$. By extending $\psi$ to a *-homomorphism from $B$ into $\mathscr{B}(\mathscr{H})$, and then replacing $\varphi$ with $\varphi \oplus \psi$, we may assume that $\varphi j$ lifts to a $*$-homomorphism $\theta: A \rightarrow \mathscr{B}(\mathscr{H})$. Having made this simplification, we proceed as follows. Extend $\theta$ to a *-homomorphism from $B$ to $\mathscr{B}(\mathscr{H})$. We will show that the extensions

$$
\left(\begin{array}{cc}
\dot{\theta} & 0 \\
0 & \varphi s p
\end{array}\right): \quad B \rightarrow M_{2}(\mathscr{2}(\mathscr{H}))
$$

and

$$
\left(\begin{array}{cc}
\dot{\theta} s p & 0 \\
0 & \varphi
\end{array}\right): \quad B \rightarrow M_{2}(\mathscr{2}(\mathscr{H}))
$$

are unitarily equivalent, and since the former gives an element of the image of $p^{*}$ in $\operatorname{Ext}^{-1}(B)$, while the latter is stably equal to the extension $\varphi$ in the kernel of $j^{*}$ that we started with, this will show that the sequence (3.5.1) is exact at $\operatorname{Ext}^{-1}(B)$. In fact, the two extensions are unitarily equivalent, via the unitary $\left(\begin{array}{cc}-M^{1 / 2} & N^{1 / 2} \\ N^{1 / 2}\end{array}\right)$, where $M \in \mathscr{B}(\mathscr{H})$ satisfies:
(i) $1 \geqslant M \geqslant 0$.
(ii) For every element $b$ of $B,(1-\dot{M}) \cdot(\varphi(b)-\dot{\theta}(b))=0$.
(iii) For every $a$ in $A, \dot{M} \cdot \varphi j(a)=0$.
(iv) $\dot{M}$ commutes with every element of $\varphi[B]$.

Such an element exists by Theorem 1.1.11; the verification that this unitary does the job is a computation using the formula (3.4.7). So it remains to show that $j^{*}$ is onto. For this, let $\varphi: A \rightarrow \mathscr{2}$ be any invertible extension of $A$ by the compact operators, and let $\left(\varphi_{i j}\right): A \rightarrow \mathscr{B}\left(\mathscr{H}^{(2)}\right)$ be a
*-homomorphism such that $\dot{\varphi}_{11}=\varphi$. Extend $\left(\varphi_{i j}\right)$ to a *-homomorphism from $B$ to $\mathscr{B}\left(\mathscr{H}^{(2)}\right)$, and then define $\theta: B \rightarrow \mathscr{B}\left(\mathscr{H}^{(2)}\right)$ by the formula

$$
\theta(b)=\left(\begin{array}{cc}
\varphi_{11}(b) & \varphi_{12}(b) \\
\varphi_{21}(b) & \varphi_{22}(s p(b))
\end{array}\right) .
$$

Now $\varphi_{11} j$ and $\varphi_{22} j$ are $*$-homomorphisms, modulo the compact operators, and so it follows that if $a \in A$ then $\varphi_{21} j(a)$ and $\varphi_{21} j(a)$ are elements of $\mathscr{K}$. Therefore, $\varphi_{12}(b)$ and $\varphi_{21}(b)$ are equal to $\varphi_{12}(s p(b))$ and $\varphi_{21}(s p(b))$, respectively, modulo compact operators, and from this it follows easily that $\theta$ is a *-homomorphism, modulo compacts. Thus $\theta$ determines an extension of $B$ by the compact operators, and since it is clear that $\theta j$ determines an extension equivalent to $\varphi$ (because $s p$ kills $A$ ), it remains to show that $\theta$ determines an invertible extension. This is so because, modulo compact operators, $\theta$ is equal to the completely positive map

$$
M^{1 / 2}\left(\varphi_{i j}\right) M^{1 / 2}+N^{1 / 2}\left(\varphi_{i j} s p\right) N^{1 / 2},
$$

where $N=(1-M)$, and $M \in \mathscr{B}\left(\mathscr{H}^{(2)}\right)$ satisfies:
(1) $1 \geqslant M \geqslant 0$.
(2) $(1-M) \cdot\left(\varphi_{\varphi_{21}(t)}(b){ }_{\left(\varphi_{2}(b)\right.}^{\varphi_{0}}\right) \in M_{2}(\mathscr{K})$ for every $b \in B$.
(3) $M \cdot\left(\begin{array}{c}0 \\ 0 \\ \varphi_{22} j(\alpha)\end{array}\right) \in M_{2}(\mathscr{K})$ for every $a \in A$.
(4) $M$ commutes, modulo compact operators, with $\left(\begin{array}{l}\varphi_{211}(b) \\ \varphi_{21}(b) \\ \varphi_{p_{21}(b)}(b)\end{array}\right)$ for every $b \in B$.

As usual, $M$ exists thanks to Theorem 1.1.11.
Now, it is easy to show that $\operatorname{Ext}^{-1}(A)$ is a stable functor (see, e.g., [47]), and so from Theorems 3.5 .1 and 3.2.2, we obtain another proof that $\mathrm{Ext}^{-1}$ is homotopy invariant.

By using the techniques of the above proof, together with Stinespring's theorem, we can prove half-exactness for Ext ${ }^{-1}$, at least with respect to invertible extensions. See also [35, 21].

Theorem 3.5.2. If $0 \rightarrow A \rightarrow^{j} B \rightarrow{ }^{p} C \rightarrow 0$ is a short exact of $C^{*}$-algebras for which there exists a completely positive contractive section $s: C \rightarrow B$, then the sequence of abelian groups

$$
\operatorname{Ext}^{-1}(C) \xrightarrow{p^{*}} \operatorname{Ext}^{-1}(B) \xrightarrow{j^{*}} \operatorname{Ext}^{-1}(A)
$$

is exact at $\operatorname{Ext}^{-1}(B)$.
Proof. Suppose that $[\varphi] \in \operatorname{Ext}^{-1}(B)$ and $j^{*}([\varphi])=0$. Then as in the proof of Theorem 3.5.1, we may assume that $\varphi j$ lifts to a $*$-homomorphism
into $\mathscr{B}(\mathscr{H})$, and we can choose a $*$-homomorphism $\theta: B \rightarrow \mathscr{B}(\mathscr{H})$ such that $\dot{\theta} j=\varphi j$. Also, as in the proof of Theorem 3.5.1, there is a unitary $U \in \mathscr{B}\left(\mathscr{H}^{(2)}\right)$ such that

$$
\operatorname{Ad}(\dot{U})(\varphi \oplus \dot{\theta} s p)=(\dot{\theta} \oplus \varphi s p)
$$

The goal of the rest of the proof is to dilate $\theta s p$ and $\varphi s p$ to *-homomorphisms in such a way that above equation, so modified, asserts that the sum of $\varphi$ and a degenerate extension is unitarily equivalent to the sum of a degenerate extension and an extension in the image $p^{*}$. Let

$$
\Psi=\left(\begin{array}{ll}
\psi_{11} & \psi_{12} \\
\psi_{21} & \psi_{22}
\end{array}\right): \quad C \rightarrow \mathscr{B}\left(\mathscr{H}^{(2)}\right)
$$

be a *-homomorphism such that $\psi_{11}=\theta s$. (The existence of $\Psi$ follows from Stinespring's theorem.) If $W$ denotes the unitary

$$
\left(\begin{array}{ccc}
U_{11} & U_{12} & 0 \\
U_{21} & U_{22} & 0 \\
0 & 0 & 1
\end{array}\right)
$$

where ( $U_{i j}$ ) denotes the unitary $U$, then

$$
\operatorname{Ad}(\dot{W})(\varphi \oplus \dot{\Psi} p)=\left(\begin{array}{ccc}
\dot{\theta} & 0 & \dot{U}_{21}^{*} \psi_{12} p  \tag{3.5.2}\\
0 & \varphi s p & \dot{U}_{22}^{*} \psi_{12} p \\
\psi_{21} p \dot{U}_{21} & \psi_{21} p \dot{U}_{22} & \gamma_{22} p
\end{array}\right)
$$

Since $\dot{\theta}$ is a *-homomorphism, it follows that the "off-diagonal" terms $\psi_{21} p \dot{U}_{21}$ and $\dot{U}_{21}^{*} \psi_{21} p$ in (3.5.2) are zero. Hence we may rewrite (3.5.2) as

$$
\operatorname{Ad}(\dot{W})\left(\begin{array}{cc}
\varphi & 0 \\
0 & \dot{\Psi} p
\end{array}\right)=\left(\begin{array}{cc}
\dot{\theta} & 0 \\
0 & X p
\end{array}\right)
$$

where $X$ is some extension of $C$ by the compacts. Note that this equation implies that $X$ is invertible: since the extension on the left-hand side of the equation is invertible, so must be the "summand" $X p$ of the right-hand side. Thus $X p$ has a completely positive lifting, and composing with the completely positive section $s$, we obtain from this lifting a completely positive lifting for $X$. It follows that $[\varphi]$ is in the image of $p^{*}$, as required.

Finally, we turn to the Brown-Douglas-Fillmore characterization of the groups $\operatorname{Ext}^{-1}(C(X))$.
$\operatorname{Ext}^{-1}(C(X))$ is naturally isomorphic to the odd-dimensional $K$-homology group of $X$.

We only want to sketch the proof. So far we know that Ext ${ }^{-1}$ is a halfexact, stable, homotopy functor. A natural transformation is defined from Ext ${ }^{-1}$ to the $K$-homology group $K^{1}$ as follows. (The topologists would write $K_{1}$ instead of $K^{1}$, the former suggesting a covariant functor, the latter suggesting a contravariant one. However, we are dealing with algebras-the dual of spaces-and so for us the superscript is more suggestive.) Correspond to an element $\alpha$ of $\operatorname{Ext}^{-1}(C(X))$ and extension

$$
0 \rightarrow \mathscr{K} \rightarrow E \rightarrow C(X) \rightarrow 0
$$

Then if $Y$ is any space, the element $\alpha$ determines a map

$$
\Gamma_{x}: \quad K_{*}(C(X \times Y)) \rightarrow K_{*-1}(C(X)),
$$

namely, the boundary map associated with the short exact sequence

$$
0 \rightarrow \mathscr{K} \otimes C(Y) \rightarrow E \otimes C(Y) \rightarrow C(X) \otimes C(Y) \rightarrow 0
$$

Now there is associated to any such map an element of $K^{1}(C(X))$, by Spanier-Whitehead duality (see [5] for a discussion of this) and this is our natural transformation. By applying the Bott periodicity theorem, one shows that this map is an isomorphism if $X$ happens to be a sphere (the point about using the periodicity theorem is that this amounts to checking only the case of $S^{1}$ ), and then it follows that it is an isomorphism for arbitrary polyhedra by decomposing into spheres.

## IV. Algebraic $K_{2}$ of a Stable $C^{*}$-Algebra ${ }^{1}$

We prove in Section 4.1 the analog of the Steinberg-Kervaire theorem (Theorem 2.4.13) for $C^{*}$-algebras, namely that if $A$ is a $C^{*}$-algebra then the Steinberg extension for $A$ is the universal central extension of the perfect group $E A$. The whole section is of a very technical nature. We apologize for this, and offer the following two-sentence overview. Apart from the techniques used in the proof of the ordinary Steinberg-Kervaire theorem (for which we follow Milnor's exposition [40]) the main ingredients in the proof are some lemmas on the factorization of elements in a $C^{*}$-algebra. Roughly speaking, the condition $A^{2}=A$ (of importance in the theory of $K_{1}$ ) which states that every element can be factored, must be strengthened to the condition that there be, up to a suitable notion of equivalence, a unique factorization for every element of $A$.

[^0]In Section 4.2 we prove that the algebraic and topological $K_{2}$-groups of a stable $C^{*}$-algebra are isomorphic. The method is to verify that the functor $A \mapsto K_{2}(\mathscr{K} \otimes A)$ satisfies the hypotheses of our homotopy invariance theorem (Theorem 3.2.2). From this and the excision properties of the lowdimensional $K$-theory groups (see Theorem 2.4.14), we deduce the result by reduction of dimensions from $K_{2}$ to $K_{1}$. The technique developed here will be applied in the next section to $K_{*}(2(A))$.

### 4.1. The Steinberg-Kervaire Theorem

We begin by stating the main theorem.
Theorem 4.1.1. If $A$ is a $C^{*}$-algebra then the Steinberg extension

$$
1 \rightarrow K_{2}(A) \rightarrow S t A \xrightarrow{\pi} E A \rightarrow 1
$$

is the universal central extension of the perfect group $E A$.
The proof has two parts: the bulk of it is simply a modification of the proof of the Steinberg-Kervaire theorem as presented in say Milnor's book [40]; the remainder is a pair of factorization results for $C^{*}$-algebras, needed to make the arguments of the first part work in the absence of a unit for $A$. We tackle the functional analysis first.

For a result related to the following lemmas, see [43, Proposition 1.4.5].

Lemma 4.1.2. Let $A$ be $a C^{*}$-algebra. If $a, b \in A$ then there exist elements $c, a_{1}, b_{1}$ in $A$ such that $a=c a_{1}$ and $b=c b_{1}$.

Proof. Define $c, a_{1}$, and $b_{1}$ as

$$
\begin{aligned}
c & =\left(a a^{*}+b b^{*}\right)^{1 / 4} \\
a_{1} & =\lim _{t \rightarrow 0+}(c+t)^{-1} a \\
b_{1} & =\lim _{t \rightarrow 0+}(c+t)^{-1} b
\end{aligned}
$$

We will check that the limits exist, and of course that $a=c a_{1}$ and $b=c b_{1}$. Since the cases of $a$ and $b$ are the same, we will consider only that of say $a$. The derivative of $(c+t)^{-1} a$ with respect to $t \in(0,1]$ is $-(c+t)^{-2} a$, and this is bounded independently of $t$ :

$$
\begin{aligned}
\left\|(c+t)^{-2} a\right\|^{2} & =\left\|(c+t)^{-2} a a^{*}(c+t)^{-2}\right\| & & \\
& \leqslant\left\|(c+t)^{-2} c^{4}(c+t)^{-2}\right\| & & \text { (since } \left.a a^{*} \leqslant c^{4}\right) \\
& \leqslant 1 & & \text { (by the functional calculus). }
\end{aligned}
$$

It follows that $(c+t)^{-1} a$ converges, as $t \rightarrow 0+$, because we can bound the difference $\left(c+t_{1}\right)^{-1} a-\left(c+t_{2}\right)^{-1} a$ by $\left|t_{1}-t_{2}\right|$ times a bound for the derivative. To see that $a=c a_{1}$ note that by continuity of multiplication,

$$
\begin{aligned}
a-c a_{1} & =\lim _{t \rightarrow 0+}\left(1-\frac{c}{c+t}\right) a \\
& =\lim _{t \rightarrow 0+} \frac{t}{c+t} a
\end{aligned}
$$

and since $(c+t)^{-1} a$ converges to $a_{1}$, the expression $t(c+t)^{-1} a$ converges to zero as $t \rightarrow 0+$.

Lemma 4.1.3. Let $A$ be a $C^{*}$-algebra, let $a \in A$, and let $S$ be the set of pairs $(b, c) \in A \times A$ such that $a=b c$. All elements of $S$ are equivalent with respect to the equivalence relation generated by the relation $\left(a_{1} a_{2}, a_{3}\right) \sim\left(a_{1}, a_{2} a_{3}\right)$.

Proof. We will show that any factorization $a=b c$ is equivalent to a "standard" one. Let $b_{1}=\lim _{t \rightarrow 0+}\left(\left|b^{*}\right|^{1 / 2}+t\right)^{-1} b$. The limits exists by exactly the computation given in the above proof (in the particular case where the element $a$ of Lemma 4.1.2 is zero). As above, $b=\left|b^{*}\right|^{1 / 2} b_{1}$. Next, let us show that the limit

$$
\begin{equation*}
c_{1}=\lim _{s \rightarrow 0+} b_{1} c\left(|a|^{1 / 4}+s\right)^{-1} \tag{4.1.1}
\end{equation*}
$$

exists. We begin by computing the derivative with respect to $s \in(0,1]$, obtaining $-b_{1} c\left(|a|^{1 / 4}+s\right)^{-2}$, which, by the definition of $b_{1}$, is equal to the expression

$$
\begin{equation*}
-\lim _{t \rightarrow 0+}\left(\left|b^{*}\right|^{1 / 2}+t\right)^{-1} a\left(|a|^{1 / 4}+s\right)^{-2} \tag{4.1.2}
\end{equation*}
$$

In order to show that this is bounded we will need the following facts:
(i) If $x, y \in A$ then $\|x y\|=\left\|x\left|y^{*}\right|\right\|$. Indeed,

$$
\|x y\|^{2}=\left\|x y y^{*} x^{*}\right\|=\left\|x\left|y^{*}\right|^{2} x^{*}\right\|=\left\|x\left|y^{*}\right|\right\|^{2}
$$

(ii) If $f$ is a continuous function then $x f(|x|)=f\left(\left|x^{*}\right|\right) x$, for any $x \in A$. (To prove this, approximate $f$ and the square root function by polynomials.)
(iii) If $x, y \in A$ and $0 \leqslant x \leqslant y$, then $x^{1 / 2} \leqslant y^{1 / 2}$. For a proof, see [43, Proposition 1.3.8]. Our application: because $a=b c$ we have the inequality $\left|a^{*}\right|^{2} \leqslant\|c\|^{2}\left|b^{*}\right|^{2}$, and so $\left|a^{*}\right| \leqslant\|c\|\left|b^{*}\right|$.

Putting all this together we have

$$
\begin{aligned}
\|\left(\mid b^{*}\right. & \left.\left.\right|^{1 / 2}+t\right)^{-1} a\left(|a|^{1 / 4}+s\right)^{-2} \| \\
& =\left\|\left(\left|b^{*}\right|^{1 / 2}+t\right)^{-1}\left(a\left(|a|^{1 / 4}+s\right)^{-4} a^{*}\right)^{1 / 2}\right\| \\
& =\left\|\left(\left|b^{*}\right|^{1 / 2}+t\right)^{-1}\left(a a^{*}\right)^{1 / 2}\left(\left|a^{*}\right|^{1 / 4}+s\right)^{-2}\right\| \\
& \leqslant\left\|\left(\left|b^{*}\right|^{1 / 2}+t\right)^{-1}\left|a^{*}\right|^{1 / 2}\right\|\left\|\left|a^{*}\right|^{1 / 2}\left(\left|a^{*}\right|^{1 / 4}+s\right)^{-2}\right\| .
\end{aligned}
$$

The first equality is an application of (i); the second is an application of (ii). The first factor in the bottom expression is bounded independently of $t$ because

$$
\begin{aligned}
& \left\|\left(\left|b^{*}\right|^{1 / 2}+t\right)^{-1}\left|a^{*}\right|^{1 / 2}\right\|^{2} \\
& \quad=\left\|\left(\left|b^{*}\right|^{1 / 2}+t\right)^{-1}\left|a^{*}\right|\left(\left|b^{*}\right|^{1 / 2}+t\right)^{-1}\right\| \\
& \quad \leqslant\|c\|\left\|\left(\left|b^{*}\right|^{1 / 2}+t\right)^{-1}\left|b^{*}\right|\left(\left|b^{*}\right|^{1 / 2}+t\right)^{-1}\right\| \quad \text { (by (iii)) } \\
& \quad \leqslant\|c\|
\end{aligned}
$$

By the functional calculus, the second factor is bounded independently of $s$. Hence the derivative (4.1.2) is bounded, and so the limit $c_{1}$ exists. Furthermore,

$$
\begin{equation*}
c_{1}|a|^{1 / 4}=b_{1} c \tag{4.1.3}
\end{equation*}
$$

because by the definition (4.1.1) of $c_{1}$,

$$
\left(c_{1}|a|^{1 / 4}-b_{1} c\right)=\lim _{s \rightarrow 0+} b_{1} c \frac{-s}{|a|^{1 / 4}+s}
$$

and since $b_{1} c\left(|a|^{1 / 4}+s\right)^{-1}$ converges, $s b_{1} c\left(|a|^{1 / 4}+s\right)^{-1}$ converges to zero, as $s \rightarrow 0+$. Finally,

$$
\begin{equation*}
\left|b^{*}\right|^{1 / 2} c_{1}=\lim _{s \rightarrow 0+} a\left(|a|^{1 / 4}+s\right)^{-1}, \tag{4.1.4}
\end{equation*}
$$

by the definition of $c_{1}$, and because $\left|b^{*}\right|^{1 / 2} b_{1}=b$. Therefore,

$$
\begin{align*}
(b, c) & \sim\left(\left|b^{*}\right|^{1 / 2}, b_{1} c\right) \\
& =\left(\left|b^{*}\right|^{1 / 2}, c_{1}|a|^{1 / 4}\right)  \tag{4.1.3}\\
& \sim\left(\left|b^{*}\right|^{1 / 2} c_{1},|a|^{1 / 4}\right) \\
& =\left(\lim _{s \rightarrow 0+} a\left(|a|^{1 / 4}+s\right)^{-1},|a|^{1 / 4}\right) \tag{4.1.4}
\end{align*}
$$

and the last factorization is independent of the pair $(b, c)$.

Before going on, let us note that the two lemmas above amount to the following assertion: if $A$ is a $C^{*}$-algebra then

$$
\begin{equation*}
A \otimes_{A} A=A, \tag{4.1.5}
\end{equation*}
$$

where $A \otimes_{A} A$ denotes the quotient of $A \otimes_{\mathrm{Z}} A$ by the subgroup generated by elements of the form $a \otimes b c-a b \otimes c$. (The map from $A \otimes{ }_{A} A$ to $A$ is the natural one, namely $a \otimes b \mapsto a b$.) We mention this so that the reader may compare Theorems 4.1.1 and 4.2.1 to an obviously related result of van der Kallen [29]. He proves that if $A$ is any ring satisfying (4.1.5) and if $B$ is any unital ring which contains $A$ as an ideal, then the relative group $K_{2}(B, A)$ (see [29] and the references cited there for the definition) is independent of $B: K_{2}(B, A)=K_{2}(\tilde{A}, A)$. Theorem 4.2.2 asserts that the sequence

$$
0 \rightarrow K_{2}(J) \rightarrow K_{2}(B) \rightarrow K_{2}(B / J) \rightarrow 0
$$

associated with a split exact sequence of $C^{*}$-algebras is split exact. Van der Kallen's result asserts that this sequence is exact if $K_{2}(J)$ is replaced by $K_{2}(\widetilde{J}, J)$.
The proof of Theorem 4.1.1 relies on the following characterization of the universal central extension of a perfect group. A central extension $P: X \rightarrow G$ is said to split if there exists a homomorphism $s: G \rightarrow X$ such that the composition $p s: G \rightarrow G$ is the identity.

Theorem 4.1.1 [40, Theorem 5.3]. A central extension $\pi: U, G$ is universal if and only if every central extension $p: X \rightarrow U$ splits.

If $A$ is a $C^{*}$-algebra, then it is clear from the Steinberg relations that St $A$ is a perfect group. So it suffices to show that every central extension $p: X \rightarrow$ StA is split. Define a map $s$ from the generators of StA to $X$ by the formula

$$
\begin{equation*}
s\left(x_{i j}^{a}\right)=\left[p^{-1} x_{i k}^{h}, p^{-1} x_{k i}^{c}\right], \quad k \neq i, j, \quad a=b c . \tag{4.1.6}
\end{equation*}
$$

The notation in (4.1.6), which we will use throughout the proof, means the following: choose some $\alpha \in p^{1}\left\{x_{i k}^{b}\right\}$ and some $\beta \in p^{1}\left\{x_{k j}^{c}\right\}$, and let $s\left(x_{i j}^{c}\right)$ be the commutator $[\alpha, \beta]$. Because $p: X \rightarrow S t A$ is a central extension, the elements $\alpha$ and $\beta$ can vary only up to elements of the center of $X$, and it follows from this that the commutator $[\alpha, \beta]$ does not depend at all on the choice of $\alpha$ and $\beta$. However, it is not immediately clear that $s\left(x_{i j}^{a}\right)$ is independent of the choice of factorization $a=b c$ in (4.1.6), or on the choice of the index $k$. In fact, most of the proof of the theorem is devoted to showing this.

Lemma 4.1.5. If $u, v$, and $w$ are elements of a group $G$ then:
(i) $[u, v][u, w]=[u, v w][v,[w, u]]$.
(ii) $[u,[v, w]][v,[w, u]][w,[u, v]]$ is equal to the identity, modulo the second commutator subgroup $G^{\prime \prime}=[[G, G],[G, G]]$.

Proof. See [40, p. 49].
Lemma 4.1.6 [40, Lemma 5.1]. If $j \neq k$ and $l \neq i$ then the commutator $\left[p^{-1} x_{i j}^{a}, p^{-1} x_{k l}^{b}\right]$ is equal to the identity.

Proof. The proof is a minor modification of Muilnor's (where the existence of a unit for $A$ is assumed). Write $a=a_{1} a_{2}$, and let $h$ be an index distinct from $i, j, k, l$. Then

$$
\left[p^{-1} x_{i j}^{u}, p^{-1} x_{k l}^{b}\right]=\left[\left[p^{-1} x_{i h}^{u_{1}}, p^{-1} x_{h i}^{u_{j}}\right], p^{-1} x_{k l}^{k}\right]
$$

and using the fact that $p^{-1} x_{i h}^{u_{1}}$ and $p^{-1} x_{h j}^{\alpha_{2}}$ both commute with $p^{-1} x_{k l}^{h}$ modulo central elements, it follows from part (ii) of the above lemma that $\left[p^{-1} x_{i j}^{u}, p^{-1} x_{k l}^{b}\right]$ is equal to the identity, modulo the second commutator subgroup of the group generated by $p^{-1} x_{i h}^{a_{1}}, p^{-1} x_{h j}^{a_{2}}$, and $p^{-1} x_{k l}^{b}$. Using the fact that the commutator subgroup of a group is generated, as a normal subgroup, by the commutators of the generators, it is easy to check that the commutator subgroup of this group is generated, as a group, by central elements and elements in $p^{-1} x_{i j}^{a}$. It follows that the second commutator subgroup is trivial.

Lemma 4.1.7. Let $a, b, c \in A$ and let $h, i, j, k$ be distinct indices. Then

$$
\begin{equation*}
\left[p^{-1} x_{i k}^{a}, p^{-1} x_{k i}^{b c}\right]=\left[p^{-1} x_{i h}^{a b}, p^{-1} x_{h j}^{c}\right] . \tag{4.1.7}
\end{equation*}
$$

Proof. We have that

$$
\begin{aligned}
& {\left[p^{-1} x_{i k}^{d}, p^{-1} x_{k j}^{h c}\right]=\left[p^{-1} x_{i k}^{a},\left[p^{-1} x_{k h}^{b}, p^{-1} x_{h j}^{c}\right]\right]} \\
& {\left[p^{-1} x_{i h}^{a h}, p^{-1} x_{h j}^{c}\right]=\left[\left[p^{-1} x_{i k}^{c}, p^{-1} x_{k h}^{h}\right], p^{-1} x_{h j}^{c}\right]}
\end{aligned}
$$

and by Lemma 4.1.5(ii) and Lemma 4.1:6, the two expressions on the righthand sides of these equalities are equal, modulo the second commutator subgroup of the group generated by $p^{-1} x_{i k}^{a}, p^{-1} x_{k h}^{b}$, and $p^{-1} x_{h j}^{c}$. However, it is easily checked that the first commutator subgroup is generated by elements lying in $p^{-1} x_{i h}^{a b}, p^{-1} x_{k j}^{b c}$, and $p^{-1} x_{i j}^{a b c}$; so the second commutator subgroup is trivial.

Now, it follows immediately from Eq. (4.1.7) that the definition (4.1.6) is independent of the choice of the index $k$. Also, (4.1.7), together with

Lemma 4.1.3, shows that the definition is independent of the factorization $a=b c$. Thus the definition of $s\left(x_{i j}^{d}\right)$ is independent of all choices. Let us go on then and show that $s$ so defined on generators determines a homomorphism from StA to $X$ or, in other words, that the $s\left(x_{i j}^{a}\right)$ satisfy the Steinberg relations. This will complete the proof of Theorem 4.1.1, since the $s\left(x_{i j}^{u}\right)$ certainly satisfy $p\left(s\left(x_{i j}^{u}\right)\right)=x_{i j}^{u}$, and so $s$ will be a section for the extension $p: X \rightarrow$ StA.

It follows from the definition (4.1.6) of $s$, and the fact that the definition is independent of all choices, that the $s\left(x_{i j}^{a}\right)$ satisfy the second Steinberg relation (2.4.8), and it follows from Lemma 4.1.6 that the third relation (2.4.9) is satisfied. This leaves the first relation: $s\left(x_{i j}^{a}\right) s\left(-x_{i j}^{b}\right)=s\left(x_{i j}^{a+b}\right)$. Using Lemma 4.1.2, find $a_{1}, b_{1}$, and $c$, such that $a=c a_{1}$ and $b=c b_{1}$. If we let $u=s\left(x_{i k}^{c}\right), v=s\left(x_{k j}^{d_{i}}\right)$, and $w=s\left(x_{k j}^{p_{1}}\right)$ then part (i) of Lemma 4.1.5, together with Lemma 4.1.6, gives the result we want.

### 4.2. Comparison with Topological $K_{2}$

In this section we prove that the homomorphisin $\alpha: K_{2}(\mathscr{K} \otimes A) \rightarrow$ $K_{2}^{\prime}(\mathscr{K} \otimes A)$ defined in (2.4.10) is an isomorphism. The most important step in the proof is establishing the homotopy invariance of the functor $A \mapsto K_{2}(\mathscr{K} \otimes A)$. For this we will use Theorem 3.2.2. Thus we will begin by showing that the hypotheses of this theorem are satisfied.

THEOREM 4.2.1. If $0 \rightarrow J \rightarrow^{\prime} B \leftrightarrows{ }_{p}^{\prime} B / J \rightarrow 0$ is a split short exact sequence of $C^{*}$-algebras and $*$-homomorphisms, then the sequence of $K_{2}$-groups

$$
\begin{equation*}
0 \rightarrow K_{2}(J) \xrightarrow{i_{*}} K_{2}(B) \underset{p_{*}}{\stackrel{s_{*}}{\longrightarrow}} K_{2}(B / J) \rightarrow 0 \tag{4.2.1}
\end{equation*}
$$

is also split exact.
In view of Theorem 2.4.15, all of (4.2.1) is exact, except possibly for the injectivity of the homomorphism $j_{*}: K_{2}(J) \rightarrow K_{2}(B)$. We will show that (4.2.1) is exact even in this last respect by finding a left inverse $\sigma: K_{2}(B) \rightarrow K_{2}(J)$ for $j_{*}$. The construction is conceptually very simple, but unfortunately the details are a bit long winded. Let us begin with a couple of preliminary results.

Lemma 4.2.2. Any automorphism of the group EA lifts uniquely to an automorphism of StA.

Proof. This follows immediately from the fact that $\pi: S t A \rightarrow E A$ is the universal central extension of $E A$, together with the definition of universal central extension.

Definition 4.2.3. Let $A$ be an ideal in $B$ and let $\beta$ be an element of St $B$. The corresponding element $\pi(\beta)$ of $E B$ acts on $E A$ by conjugation: $\alpha \mapsto \pi(\beta) \alpha \pi(\beta)^{-1}$, for $\alpha \in E A$. Denote by $\operatorname{Ad}(\beta)$ the automorphism of StA which lifts this automorphism of $E A$.

Lemma 4.2.4. The automorphism $\gamma=\operatorname{Ad}\left(x_{i j}^{b}\right)$ of $\operatorname{StJ}$ acts on generators of St J as follow's:
(i) $\gamma\left(x_{i j}^{d}\right)=x_{i j}^{i}$.
(ii) $\gamma\left(x_{k l}^{a}\right)=x_{k l}^{a}$ if $i \neq l$ and $j \neq k$.
(iii) $\gamma\left(x_{k i}^{u}\right)=x_{k i}^{u} x_{k i}^{-u h}$ if $j \neq k$.
(iv) $\gamma\left(x_{j k}^{a}\right)=x_{j k}^{a} x_{i k}^{b a}$ if $i \neq k$.

Proof. Let us prove, say, (iv) to illustrate the method: the rest are done in the same way and are no harder. Write $x_{i k}^{u}=\left[x_{i l}^{u_{1}}, x_{l k}^{a_{2}}\right]$, where $l$ is some unused index. We have

$$
\begin{aligned}
\gamma\left(x_{j k}^{a_{k}}\right) & =\left[\gamma\left(x_{i l}^{a_{1}}\right), \gamma\left(x_{l k}^{a_{2}}\right)\right] \\
& =\left[x_{j l}^{a_{1}} x_{i l}^{b a_{1}}, x_{l k}^{a_{2}}\right],
\end{aligned}
$$

where we obtain the second inequality because by definition of $\gamma$, $\gamma\left(x_{j l}^{a_{1}}\right)=x_{j l}^{a_{1}} x_{i l}^{h a_{1}}$ and $\gamma\left(x_{i k}^{a_{2}}\right)=x_{l k}^{d_{2}}$, modulo central elements; but the commutator of two elements is insensitive to perturbations by central elements. Computing the commutator by using the Steinberg relations gives the result.

Now, we will construct the map $\sigma: K_{2}(B) \rightarrow K_{2}(J)$ in several stages. Let $S$ denote the free semigroup generated by the symbols $\left(x_{i j}^{h}\right)^{\varepsilon}$, where $x_{i j}^{b}$ is a generator for $S t B$ and $\varepsilon= \pm 1$. In other words, $S$ is the free semigroup generated by the Steinberg symbols and their formal inverses. Let $F$ denote the free group on the Steinberg symbols. Recall that this is obtained as the quotient of $S$ by the equivalence relation generated by the relation
$x_{1}^{\xi_{1}} \cdots x_{n}^{\varepsilon_{n}} \sim x_{1}^{\varepsilon_{1}} \cdots x_{j}^{\varepsilon_{j}-1} x_{j+2}^{\varepsilon_{j}} \cdots x_{n}^{\varepsilon_{n}} \quad$ if $\quad x_{j}=x_{j+1}$ and $\varepsilon_{j}=-\varepsilon_{j+1}$
(where the $x_{i}$ denote Steinberg symbols). We will define a map $\sigma: S \rightarrow S t J$, show that it passes to maps $\sigma: F \rightarrow S t J$ and then $\sigma: S t B \rightarrow S t J$, and finally show that $\sigma$ restricts to a homomorphism from $K_{2} B$ to $K_{2} J$ left inverse to $j_{*}$.

The following notation will be used: if $b \in B$ then denote by $\bar{b}$ the element $s(p(b)) \in B$. Similarly, the homomorphism $s p: B \rightarrow B$ induces endomorphisms of $S, F$, and $S t B$, and if $x$ is an element of one of these, then we will denote by $\bar{x}$ the image of $x$ under the endomorphism. Also, we will expand the notation of Definition 4.2 .3 a little: if $x \in S$, then of course $x$
maps to some element $\tilde{x} \in S t B$; we will define the automorphism $\operatorname{Ad}(x)$ of St. $J$ to be $\operatorname{Ad}(\tilde{x})$. Here, then, is the definition of $\sigma$, which is inductive:

$$
\begin{align*}
\sigma\left(\left(x_{i j}^{b}\right)^{\varepsilon}\right) & =x_{i j}^{\varepsilon_{i j}^{\left(b-b_{1}\right.},}  \tag{4.2.3}\\
\sigma\left(x_{1}^{c_{1}} \cdots x_{n}^{\varepsilon_{n}}\right) & =\sigma\left(x_{1}^{t_{1}} \cdots x_{n-1}^{t_{n}-1}\right) \operatorname{Ad}\left(\bar{x}_{1}^{\varepsilon_{1}} \cdots \bar{x}_{n-1}^{\varepsilon_{n}-1}\right)\left(\sigma\left(x_{n}^{\varepsilon_{n}}\right)\right) . \tag{4.2.4}
\end{align*}
$$

Note that $b-\bar{b} \in J$, so that $x_{i j}^{\left(t\left(b-b_{1}\right.\right.}$ is an element of $S t J$ : thus $\sigma$ does indeed map $S$ into $S t J$. The motivation behind this is straightforward: informally, if $y \in S t B$ then we want to define $\sigma(y)$ by $\sigma(y)=y \bar{y}^{-1}$. In order to make sense of $\sigma(y)$ as an element of StJ some rearrangement of terms is necessary. Taking a simple example, where $y$ is a product of two Steinberg symbols $x_{i 1 / 1}^{b_{1}} x_{i / 2}^{p_{2}}$, we have

$$
\begin{aligned}
& \sigma(y)=x_{i, 11}^{h_{1}} x_{i 2 k}^{h_{2}}, x_{i / 22}^{k_{2}} x_{i, 1 / 1}^{-h_{1}} \\
& =\left(x_{i, 1}^{b_{1}} x_{1 / 1}^{-b_{1}}\right) \operatorname{Ad}\left(x_{i, 2}^{b_{1}}\right)\left(x_{i / 2}^{b_{2}-b_{2}}\right) \\
& =\sigma\left(x_{i 111}^{b_{1}}\right) \operatorname{Ad}\left(x_{i!1}^{b_{1}}\right)\left(\sigma\left(x_{i 2 / 2}^{b_{2}}\right)\right) .
\end{aligned}
$$

Note the following very useful multiplicative property of $\sigma$ :

$$
\begin{equation*}
\sigma\left(w_{1} w_{2}\right)=\sigma\left(w_{1}\right) \operatorname{Ad}\left(\bar{w}_{1}\right)\left(\sigma\left(w_{2}\right)\right) . \tag{4.2.5}
\end{equation*}
$$

Lemma 4.2.5. The function $\sigma: S \rightarrow S t J$ is well defined on $F$.
Proof. Applying $\sigma$ to the left-hand side of (4.2.2) and using (4.2.5) we get

$$
\sigma\left(x_{1}^{t_{1}} \cdots x_{n}^{t_{n}}\right)=\sigma\left(w_{1}\right) \operatorname{Ad}\left(\bar{w}_{1}\right)\left(\sigma\left(w_{2}\right)\right) \operatorname{Ad}\left(\bar{w}_{1}\right) \operatorname{Ad}\left(\bar{w}_{2}\right)\left(\sigma w_{3}\right),
$$

 computation using Lemma 4.2 .4 shows that $\sigma\left(w_{2}\right)=1$, while $\operatorname{Ad}\left(\bar{w}_{2}\right)$ is the identity, since $w_{2}$ determines the identity of $E B$. It follows that

$$
\begin{aligned}
\sigma\left(w_{1} w_{2} w_{3}\right) & =\sigma\left(w_{1}\right) \operatorname{Ad}\left(\bar{w}_{1}\right)\left(\sigma\left(w_{3}\right)\right) \\
& =\sigma\left(w_{1} w_{3}\right),
\end{aligned}
$$

so that $\sigma$ is well defined on $F$.
Lemma 4.2.6. The function $\sigma: F \rightarrow S t J$ is well defined as a function on St $B$.

Proof. What we must show is that if $w_{1}$ and $w_{2}$ are elements of $F$ which are in the same coset of the normal subgroup $N$ of $F$ generated by the

Steinberg relations then $\sigma\left(w_{1}\right)=\sigma\left(w_{2}\right)$. So suppose that $w_{1}=w_{2} n$, where $n \in N$. From the multiplicative property (4.2.5) we get

$$
\sigma\left(w_{1}\right)=\sigma\left(w_{2}\right) \operatorname{Ad}\left(\bar{w}_{2}\right)(\sigma(n))
$$

so it suffices to show that $\sigma(n)=1$. However, $n$ is the product of elements in $F$ of the form $v r v^{-1}$, where $v \in F$ is arbitrary, but $r \in F$ is one of the Steinberg relations. Now, if we show that $\sigma(r)=1$ for every such $r$, then it will follow that $\sigma\left(v r v^{-1}\right)=1$ :

$$
\begin{aligned}
\sigma\left(v r v^{-1}\right) & =\sigma(v) \times \operatorname{Ad}(\bar{v})(\sigma(r)) \times \operatorname{Ad}(\bar{v} \bar{r})\left(\sigma\left(v^{-1}\right)\right) \\
& =\sigma(v) \times \operatorname{Ad}(\bar{v}) \operatorname{Ad}(\bar{r})\left(\left(\sigma\left(v v^{1}\right)\right)\right. \\
& \left.=\sigma(v) \times \operatorname{Ad}(\bar{v}) \sigma\left(v^{-1}\right)\right) \\
& =\sigma\left(v v^{-1}\right) \\
& =\sigma(1) \\
& =1
\end{aligned}
$$

(For the third equality above we use the fact that $\operatorname{Ad}(\bar{r})=1$ which is true because $\bar{r}$ maps to the identity in $E B$ (and in $S t B$ even)). It will then follow from another application of the multiplicative property (4.2.5) of $\sigma$, that $\sigma$ maps each product $n$ of elements of the form $v r v^{-1}$ to the identity. So we have reduced the proof to showing that $\sigma\left(r_{i}\right)=1$, for $i=1,2,3$, where

$$
\begin{array}{ll}
r_{1}=\left[x_{i j}^{a}, x_{i k}^{b}\right]\left(x_{i k}^{a b}\right)^{-1} & (i \neq k) \\
r_{2}=\left[x_{i j}^{a}, x_{k l}^{h}\right] & (j \neq k, i \neq l) \\
r_{3}=x_{i j}^{a} x_{i j}^{h}\left(x_{i j}^{a+h}\right)^{-1}
\end{array}
$$

We will consider only the case of $r_{1}$, which is much the most tedious. Note first that $\operatorname{Ad}\left(\left[x_{i j}^{a}, x_{j k}^{b}\right]\right)=\operatorname{Ad}\left(x_{i k}^{a b}\right)$, and so by Lemma 4.2.4,

$$
\begin{align*}
\sigma\left(r_{1}\right) & =\sigma\left(\left[x_{i j}^{a}, x_{j k}^{b}\right]\right) \times \operatorname{Ad}\left(\left[x_{i j}^{a}, x_{j k}^{b}\right]\right)\left(\sigma\left(\left(x_{i k}^{a b}\right)^{-1}\right)\right) \\
& =\sigma\left(\left[x_{i j}^{a}, x_{j k}^{b}\right]\right) \times \operatorname{Ad}\left(x_{i k}^{a b}\right)\left(x_{i k}^{a \bar{h}-a b}\right) \\
& =\sigma\left(\left[x_{i j}^{a}, x_{j k}^{b}\right]\right) x_{i k}^{a b b-a b} . \tag{4.2.6}
\end{align*}
$$

Now, by expanding, using the inductive definition of $\sigma$ and Lemma 4.2.4, we get

$$
\begin{aligned}
\sigma\left(\left[x_{i j}^{u}, x_{j k}^{b}\right]\right) & =x_{i j}^{a--\bar{a}} \times \operatorname{Ad}\left(x_{i j}^{\bar{i}}\right)\left(x_{j k}^{b-\bar{b}}\right) \times \alpha \times \beta \\
& =x_{i j}^{a-\bar{a}} x_{j k}^{b-\bar{b}} x_{i k}^{a b-a}-\overline{a b} \alpha \beta
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha & =\operatorname{Ad}\left(x_{i j}^{\bar{a}}\right) \operatorname{Ad}\left(x_{j k}^{\bar{b}}\right)\left(x_{i j}^{\bar{a}-a}\right) \\
& =\operatorname{Ad}\left(x_{i j}^{\bar{a}}\right)\left(x_{i j}^{\bar{a}-a} x_{i k}^{a \bar{b}-\bar{a}}\right) \\
& =x_{i j}^{\bar{u}-a} x_{i k}^{a b} \cdots a \bar{b}
\end{aligned}
$$

and

$$
\begin{aligned}
& \beta=\operatorname{Ad}\left(x_{i j}^{\bar{u}}\right) \operatorname{Ad}\left(x_{j k}^{\bar{\zeta}}\right) \operatorname{Ad}\left(\left(x_{i j}^{\bar{i}}\right)^{-1}\right)\left(x_{j k}^{\zeta-h}\right) \\
& =\operatorname{Ad}\left(x_{i j}^{\bar{a}}\right) \operatorname{Ad}\left(x_{j k}^{\bar{b}}\right)\left(x_{j k}^{\bar{b}} \cdot{ }^{h} x_{i k}^{a b-\bar{a}}\right) \\
& =\operatorname{Ad}\left(x_{i j}^{\bar{a}}\right)\left(x_{j k}^{\bar{b}-b} x_{i k}^{\bar{a} h}-\bar{u} \sigma\right) \\
& =x_{j k}^{\bar{b}-b} x_{i k}^{\bar{a} \bar{a}-\dot{a} h} x_{i k}^{\bar{a} h-\alpha h} \\
& =x_{k}^{h-h} .
\end{aligned}
$$

Multiplying everything together, and noting that the $x_{i k}$ commute with the $x_{j k}$, we get $\sigma\left(\left[x_{i j}^{a} x_{j k}^{b}\right]\right)=x_{i k}^{a b-a ̈ b}$, which, in view of (4.2.6), completes the proof.

Having dealt with all the unsightly computations in the lemmas, the remainder of the proof is very easy.

Proof of Theorem 4.2.1. Restrict the function $\sigma: S t B \rightarrow S t J$ to $K_{2}(B)$. Because of the fact that $\operatorname{Ad}(\alpha)=1$ if $\alpha \in K_{2}(B)$, it follows from the multiplicative property of $\sigma$ that the restriction is a homomorphism of groups. Furthermore, if each $x_{k}$ is a Steinberg symbol $x_{i j}^{b}$ for which $b \in J$, then it is easy to see (by induction, say) that $\sigma\left(x_{1} \cdots x_{n}\right)=x_{1} \cdots x_{n}$. Therefore $\sigma$ is a left inverse of the homomorphism $j_{*}: K_{2}(J) \rightarrow K_{2}(B)$. It follows that $j_{*}$ is injective, and as we have remarked earlier, this is all we need to prove.

Lemma 4.2.6. The functor $A \mapsto K_{2}(\mathscr{K} \otimes A)$ is stable.
Proof. By identifying $\mathscr{K} \otimes \mathscr{K} \otimes A$ with $M_{2}(\mathscr{K} \otimes A)$ appropriately, the natural map $\mathscr{K} \otimes A \rightarrow \mathscr{K} \otimes \mathscr{K} \otimes A$ can be identified with the map $\mathscr{K} \otimes A \rightarrow M_{2}(\mathscr{K} \otimes A)$ which embeds $\mathscr{K} \otimes A$ in the top-left-hand corner of the $2 \times 2$ matrices over $\mathscr{K} \otimes A$. However, for any $C^{*}$-algebra $D$ the map $D \rightarrow M_{2} D$ gives an isomorphism $K_{2}(D) \rightarrow K_{2}\left(M_{2} D\right)$. Indeed, by Theorem 2.6.11, this is true if $D$ is unital, while in the non-unital case, by adjoining a unit to $D$ and applying $K_{2}$ we obtain the diagram


The rows are exact by Theorem 4.2.1 (they are split exact, in fact), while by Theorem 2.6.11, the two rightmost vertical maps are isomorphisms; it follows that the other is an isomorphism too.

We are now in a position to prove the main result of the section.

Theorem 4.2.7. For every $C^{*}$-algebra $A$ the homomorphism

$$
\alpha: \quad K_{2}(\mathscr{K} \otimes A) \rightarrow K_{2}^{\prime}(\mathscr{K} \otimes A)
$$

of (2.4.10) is an isomorphism.
Proof. The functor $A \mapsto K_{2}(\mathscr{K} \otimes A)$ is split exact by Theorem 4.2.1, and it is stable by Lemma 4.2.6. It follows from Theorem 3.2.2 that it is homotopy invariant. Consider now the "path fibration" short sequence

$$
\begin{equation*}
0 \rightarrow \mathscr{K} \otimes A \otimes C_{0}(\mathbf{R}) \rightarrow \mathscr{K} \otimes A \otimes C_{0}[0,1) \rightarrow \mathscr{K} \otimes A \rightarrow 0 . \tag{4.2.7}
\end{equation*}
$$

Apply to this the following diagram, which compares the long exact sequences in topological and algebraic $K$-theory associated with a short exact sequence of $C^{*}$-algebras and $*$-homomorphisms $0 \rightarrow J \rightarrow B \rightarrow B / J \rightarrow 0$ :


From homotopy invariance it follows that the end terms in the diagram are zero (since $B=\mathscr{K} \otimes A \otimes C_{0}[0,1)$ is contractible). Therefore the maps $\partial$ are isomorphisms, and so the fact that $\alpha: K_{2}(\mathscr{K} \otimes A) \rightarrow K_{2}^{\prime}(\mathscr{K} \otimes A)$ is an isomorphism follows from the corresponding fact for $\alpha: K_{1}(\mathscr{K} \otimes A) \rightarrow$ $K_{1}^{\prime}(\mathscr{K} \otimes A)$, which is Theorem 2.4.6.

## V. Algebraic $K$-Theory of Calkin Algebras

The purpose of this section is to prove that if $A$ is a unital $C^{*}$-algebra and $B$ is a $\sigma$-unital $C^{*}$-algebra, then the algebraic $K$-theory of $A \otimes \mathscr{2}(B)$ is isomorphic to its topological $K$-theory. The plan of the proof is broadly the same as that of the last section. However, two difficulties present themselves. First, the higher algebraic $K$-theory groups do not, in general, have good excision properties; and second, $\mathscr{2}(B)$ is functorial only with respect to quasi-unital *-homomorphisms between $C^{*}$-algebras. Section 5.1 is devoted to an exposition of some excision results in algebraic $K$-theory. These results require quite stringent hypotheses, but by applying the
separation theorem of Kasparov (Theorem 1.1.11) we are able to show in Section 5.2 that they are satisfied in the situations we need. By means of this we are also able to get around the problem of the non-functoriality of $2(B)$. In the next section we apply the results of Section III to obtain homotopy invariance for the algebraic $K$-theory groups, and then the proof of the main theorem is completed by reduction of dimensions as in Section IV. In Sections 5.2 and 5.3, for simplicity, we consider only $2(B)$ and not $A \otimes \mathscr{2}(B)$; we remedy this in the last section by showing how to modify our theorems so as to get the general result.

### 5.1. Excision in Algebraic $K$-Theory

Our goal in this section is to prove that if

$$
N \rightarrow G \rightarrow H
$$

is an extension of groups, then under suitable hypotheses (see Theorem 5.1.4), the sequence

$$
\begin{equation*}
\mathrm{BN}^{+} \rightarrow \mathrm{BG}^{+} \rightarrow \mathrm{BH}^{+} \tag{5.1.1}
\end{equation*}
$$

obtained by applying the classifying space functor, and then the plus construction, is a fibration, up to homotopy. We will then use this theorem to prove an excision result in algebraic $K$-theory.

As with the material on topology presented in Section II, the results of this section are all well known (they can all be found in [8]; see also [54]). However, although we make no claim about the originality of the results, our proofs arc, we hope, a little simpler than those in the literature, due mainly to the fact that we are considering only special cases of more general results.

Let us begin by writing down the mapping path fibration associated with the map $p_{+}: B G^{+} \rightarrow B H^{+}$. Recall from (2.5.1) that this is the fibration

$$
F^{(1)} \rightarrow P^{(1)} \xrightarrow{p_{t+1}} B H^{+},
$$

where $P^{(+)}$is the space of all pairs ( $x, \gamma$ ), with $x \in B G^{+}$, and $\gamma$ a path in $B H^{+}$for which $\gamma(1)=p_{+}(x)$. The map $p_{(+)}$is given by $p_{(+)}(x, \gamma)=\gamma(0)$. (We enclose the + 's in parentheses because it is not clear a priori that all the spaces so labelled arise from the plus construction: part of our job is to prove that they do.)

Theorem 5.1.1. Let $p: G \rightarrow H$ be a map between stable groups.
(i) There exists an $H$-space structure on $P^{(+)}$,

$$
P^{(+)} \times P^{(+)} \rightarrow P^{(+)}
$$

such that the diagram

commutes (exactly, not merely up to homotopy), where $\mu$ denotes the H -space multiplication on $\mathrm{BH}^{+}$given by Theorem 2.6.10.
(ii) $F^{(+1}$ is an $H$-space. In fact, there exist homotopy equivalences

$$
\hat{l}: F^{(+)} \rightarrow F^{(+)} \quad \text { and } \quad \hat{r}: F^{+1} \rightarrow F^{(+)}
$$

such that the map $(x, y) \mapsto \hat{l}(x) \times \hat{r}(y)$ is an $H$-space multiplication for $F^{(+)}$, where " $\times$ " denotes the $H$-space multiplication on $P^{(+)}$.

Proof. (i) If $\mu: B G^{+} \times B G^{+} \rightarrow B G^{+}$denotes the $H$-space multiplication on $B G^{+}$given by Theorem 2.6.10 then the diagram

commutes up to homotopy. Let

$$
\Gamma_{i}: B G^{+} \times B G^{+} \rightarrow B H^{+} \quad(t \in[0,1])
$$

be a homotopy such that

$$
\Gamma_{0}(x, y)=\mu\left(p_{+}(x), p_{+}(y)\right) \quad \text { and } \quad \Gamma_{1}(x, y)=p_{+}(\mu(x, y))
$$

If $(x, \gamma) \in P^{(+)}$and $(y, v) \in P^{(+)}$then define

$$
(x, \gamma) \times(y, v)=(\mu(x, y), \xi)
$$

where

$$
\xi(t)= \begin{cases}\mu(\gamma(2 t), v(2 t)) & \text { if } \quad 0 \leqslant t \leqslant \frac{1}{2} \\ \Gamma_{2 t-1}(x, y) & \text { if } \quad \frac{1}{2} \leqslant t \leqslant 1\end{cases}
$$

Clearly, this multiplication corresponds to the multiplication $\mu$ on $B H^{+}$ under the map $p_{(+)}$(i.e., the diagram (5.1.2) commutes), and so it remains to see that $P^{(+)}$is an $H$-space. In other words, we must show that the maps
$x \mapsto x \times e$ and $x \mapsto e \times x$, where $e$ denotes the base point, are homotopic to the identity. But consider the commuting diagram

where $\pi(x, \gamma)=x$. Beause $\pi$ is a homotopy equivalence, it follows from the fact that the bottom map is homotopic to the identity that the top map is too. Similarly, $e \times \cdot$ is homotopic to the identity.
(ii) Of course, since the diagram (5.1.2) commutes, the multiplication on $P^{(+)}$restricts to one on $F^{(+)}$. But it is not clear that this is an $H$-space multiplication on $F^{(+)}$. However, from the commuting diagram of fibrations

it follows from the long exact homotopy sequence that $\cdot \times e: F^{(+)} \rightarrow F^{(+)}$is a homotopy equivalence, since the other two maps are. Therefore there exists a map $\hat{r}: F^{(+)} \rightarrow F^{(+)}$so that the map $x \mapsto e \times \hat{r}(x)$ is homotopic to the identity. Similarly, there exists a map $\hat{l}: F^{(+)} \rightarrow F^{+\prime}$ such that the map $x \mapsto \hat{l}(x) \times e$ is homotopic to the identity. It follows that the map $(x, y) \mapsto \hat{l}(x) \times \hat{r}(y)$ is an $H$-space multiplication on $F^{(+1}$.

Theorem 5.1.2. If $p: G \rightarrow H$ is a surjection between stable groups then the mapping path fibration

$$
F^{(+)} \rightarrow P^{(+)} \xrightarrow{p_{1+1}} B H^{+}
$$

is orientable.
Proof. We must show that the group $\pi_{1}\left(B H^{+}\right)$acts trivially on the homology of the fibre $F^{+1}$, where the action is as defined in Section II. Since $p_{1+1 *}: \pi_{1}\left(P^{(+)}\right) \rightarrow \pi_{1}\left(B H^{+}\right)$is surjective (by virtue of $p: G \rightarrow H$ being surjective), and since the map $z \mapsto \mu(z, e)$ from $B H^{+}$to itself is homotopic to the identity, it suffices to show that loops in $B H^{+}$of the form $t \mapsto p_{(+)}(\gamma(t) \times e)$, where $\gamma$ is a loop in $P^{(+)}$, act trivially on $H_{*}\left(F^{+\prime}\right)$, since any loop is homotopic to one of these. We may suppose further that the
loop $\gamma$ is stationary near the endpoints of $[0,1]: \gamma(t)=e$, if $0 \leqslant t \leqslant \frac{1}{3}$ or $\frac{2}{3} \leqslant t \leqslant 1$. Recall how the action is constructed: we solve the homotopy lifting problem

where $\tilde{\gamma}$ in our particular case is the map $(x, t) \mapsto p_{(+)}(\gamma(t) \times e)$. The map $x \mapsto \Gamma(x, 1)$ then maps $F^{(+)}$into itself, and the induced map on homology is the action of the loop. In our case we can explicitly write down a suitable map $\Gamma$. It is

$$
\Gamma(x, t)= \begin{cases}R(x, 3 t) & \text { if } \quad 0 \leqslant t \leqslant \frac{1}{3} \\ \gamma(t) \times \hat{r}(x) & \text { if } \frac{1}{3} \leqslant t \leqslant \frac{2}{3} \\ R(x, 3-3 t) & \text { if } \quad \frac{2}{3} \leqslant t \leqslant 1,\end{cases}
$$

where $R: F^{(+)} \times[0,1] \rightarrow F^{+1}$ is a homotopy such that $R(x, 1)=e \times \hat{r}(x)$ and $R(x, 0)=x$ (and $\hat{r}$ is the map in part (ii) of Theorem 5.1.1). It follows immediately that this solves the homotopy extension problem above. Since $\Gamma(x, 1)=x$, we see that in fact the loop in $B H^{+}$acts trivially on $F^{+1}$, up to homotopy, and so it certainly acts trivially on the homology.

Lemma 5.1.3. The space $F^{(+1}$ is connected.
Proof. This follows immediately from the long exact homotopy sequence for the mapping path fibration, and the fact that the map $\pi_{1}\left(P^{++}\right) \rightarrow \pi_{1}\left(B H^{+}\right)$is onto.

Now, we are in a position to prove the main results. We will consider first extensions of groups, and then apply this to short exact sequences of rings. Let

$$
1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1
$$

be an extension of stable groups. Recall from Theorem 2.6.13 that we may choose plussed spaces and maps so that the diagram

commutes, and so that there are compatible homotopies


We obtain a commutative diagram

where $F$ is the homotopy fiber of the map $p: B G \rightarrow B H, \tilde{q}$ is the map induced by the maps $q_{+}$in diagram (5.1.3), and $j$ and $j_{(+)}$are the maps induced by the homotopies above, as in (2.5.3).

Theorem 5.1.4. The map $j_{(+1}: B N^{+} \rightarrow F^{+1}$ is a homotopy equivalence.
Proof. Since both $B N^{+}$and $F^{+1}$ are connected $H$-spaces (by Theorems 2.6.10 and 5.1.1, respectively), in view of Theorem 2.5.5 it suffices to show that $j_{1+1}$ induces an isomorphism on homology. For this, it suffices to show that the other three maps in diagram (5.1.4) induce isomorphisms on homology. The map $j: B N \rightarrow F$ is a homotopy equivalence, and so certainly it is a homology isomorphism; the map $q: B N \rightarrow B N^{+}$is a homology isomorphism by definition of the plus construction. So it remains to consider the map $\tilde{q}$. Consider the commuting diagram of fibrations


Since $H$ acts trivially on $H_{*}(N)$ (by Theorem 2.6.7), it follows that the top fibration is orientable; while by Theorem 5.1.3, the bottom fibration is orientable. Since the maps $q_{+}$are homology isomorphisms, by definition of the plus construction, it follows from the comparison theorem (Theorem 2.5.7) that $\tilde{q}: F \rightarrow F^{+1}$ induces an isomorphism on homology.

We need one more simple result which is actually a consequence of the above theorem.

Corollary 5.1.5. If

$$
N \xrightarrow{i} G \xrightarrow{p} A
$$

is an extension of stable groups and $A$ is abelian then the map

$$
i_{+}: \pi_{n}\left(B N^{+}\right) \rightarrow \pi_{n}\left(B G^{+}\right)
$$

is an isomorphism for all $n>1$, and an injection for $n=1$.
Proof. Since $A$ is abelian, its maximal perfect subgroup is trivial, and therefore $q: B A \rightarrow B A^{+}$is a homotopy equivalence. It follows that $\pi_{n}\left(B A^{+}\right)=0$ if $n>1$, and so by the long exact homotopy sequence, the map from homotopy fiber $F^{+1}$ of $p_{+}: B G^{+} \rightarrow B A^{+}$into $B G^{+}$induces an isomorphism on $\pi_{n}$ for $n>1$, and an injection if $n=1$. The corollary therefore follows from the homotopy commutative diagram

and the fact proved in the last theorem that the map $i_{(+)}$is a homotopy equivalence.

Remark. In fact Corollary 5.15 is much easier than we have made it out to be: it is very easy to see directly that if the quotient $A$ is abelian then the extension of groups passes to a fibration, up to homotopy, of plussed spaces-see $[7,(6.4)]$. It is not necessary to assume that the groups involved are stable.

Now, let

$$
0 \rightarrow I \rightarrow A \rightarrow A / I \rightarrow 0
$$

by a short exact sequence of weakly unital $C^{*}$-algebras (see Section II). We obtain from it an extension of stable groups

$$
N \rightarrow G \rightarrow H,
$$

where $N=G L I, G=G L A$, and $H$ is the image in $G L A / I$ of $G$. Note that $H$ is an open and closed subgroup of $G L A / I$ which contains the commutator subgroup (because the commutator subgroup is contained in the connected component of the identity). Therefore the quotient ( $G L A / I) / H$ is abelian, and Corollary 5.15 applies: $\pi_{n}\left(B H^{+}\right) \rightarrow \pi_{n}\left(B G L A / I^{+}\right)$is an isomorphism
for $n>1$ and an injection for $n=1$. Identifying $B N^{+}$with the fiber $F^{(+)}$by means of the map $j_{(+)}$, we obtain long exact $K$-theory sequence

$$
\begin{equation*}
\rightarrow K_{n+1}(A) \rightarrow K_{n+1}(A / I) \rightarrow K_{n}(I) \rightarrow K_{n}(A) \rightarrow \tag{5.1.5}
\end{equation*}
$$

(where $n \geqslant 1$ ) which by Theorem 2.6.13, and the discussion following it, is compatible with the long exact sequence in topological $K$-theory via the transformation $\alpha: K_{*} \rightarrow K_{*}^{\prime}$.

### 5.2. Application to Calkin Algebras

Throughout the rest of this section, $B$ will be a $\sigma$-unital $C^{*}$-algebra and $J$ will be a $\sigma$-unital ideal in $B$. Let us recall from Section I that the kernel $\mathscr{X}$ of the *-homomorphism $p: \mathscr{Q}(B) \rightarrow \mathscr{Z}(B / J)$ induced from the projection $B \rightarrow B / J$ is

$$
\begin{equation*}
\mathscr{X}=\mathscr{H}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J) / \mathscr{K} \otimes J \tag{5.2.1}
\end{equation*}
$$

(see Definition 1.3.8). Recall also that by Theorem 1.1.10, the map $\mathscr{M}(\mathscr{K} \otimes B) \rightarrow \mathscr{M}(\mathscr{K} \otimes B / J)$ is surjective, and so since the same is true of the $*$-homomorphism $p: \mathscr{Z}(B) \rightarrow \mathscr{Q}(B / J)$, we obtain the following short exact sequence

$$
\begin{equation*}
0 \rightarrow X \rightarrow \mathcal{Z}(B) \xrightarrow{D} \mathcal{Q}(B / J) \rightarrow 0 . \tag{5.2.2}
\end{equation*}
$$

Theorem 5.2.1. If $\mathscr{A}$ is a separable $C^{*}$-subalgebra of $\mathscr{X}$ then there exists an element $u \in \mathscr{X}$ such that $1 \geqslant u \geqslant 0$ and $u a=a=a u$ for every $a \in \mathscr{A}$.

Proof. Let $E_{1}=\mathscr{K} \otimes B$; let $E=\mathscr{K} \otimes J$; and let $E_{2}$ be a separable $C^{*}$-subalgebra of $\mathscr{M}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J)$ which maps onto $\mathscr{A} \subset \mathscr{X}$. It follows from the definition of $\mathscr{M}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J)$ that $E_{1} \cdot E_{2} \subset E$, and so, by Theorem 1.1.11, there exists an element $N \in \mathscr{H}(\mathscr{K} \otimes B)$ such that $1 \geqslant N \geqslant 0$, and
(i) $N \cdot \mathscr{K} \otimes B \subset \mathscr{K} \otimes J$.
(ii) $(1-N) \cdot E_{2} \subset \mathscr{K} \otimes J$.

Condition (i) implies that $N \in \mathscr{M}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J)$, while condition (ii) states that $N$ acts as a unit for $E_{2}$, modulo $\mathscr{K} \otimes J$. Therefore, if we let $u$ denote the image of $N$ in the quotient $\mathscr{X}=\mathscr{M}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J) / \mathscr{K} \otimes J$, then $u a=a=a u$ for every $a \in \mathscr{A}$.

In particular, $\mathscr{X}$ is weakly unital, and so from the long exact sequence (5.1.5), we obtain the following result.

Theorem 5.2.2. If $\mathscr{R}$ is a unital $C^{*}$-algebra which contains $\mathscr{X}$ as an ideal then from the exact sequence

$$
0 \rightarrow \mathscr{X} \xrightarrow{j} \mathscr{R} \xrightarrow{\rho} \mathscr{R} / \mathscr{X} \rightarrow 0
$$

we obtain a long exact sequence

$$
\xrightarrow{j_{*}} K_{n}(\mathscr{R}) \xrightarrow{p_{*}} K_{n}(\mathscr{R} \mid \mathscr{X}) \xrightarrow{\partial} K_{n-1}(\mathscr{X}) \xrightarrow{j_{*}} K_{n-1}(\mathscr{R}) \xrightarrow{p_{*}}
$$

Now, the restriction map $\rho: \mathscr{M}(\mathscr{K} \otimes B) \rightarrow \mathscr{M}(\mathscr{K} \otimes J)$ maps $\mathscr{K} \otimes J$ into itself, and so passes to a map $\rho: \mathscr{M}(\mathscr{K} \otimes B) / \mathscr{K} \otimes J \rightarrow \mathscr{Q}(J)$.

Theorem 5.2.3. If the annihilator ideal of $J \subset B$ is $\sigma$-unital, then the restriction map $\rho: \mathscr{X} \rightarrow 2(J)$ passes to an isomorphism in $K$-theory.

Proof. Let us consider first the case in which $J$ is an essential ideal of $B$, so that the map $\rho$ is an inclusion. By Theorem 2.6.10, both BGL. ${ }^{+}{ }^{+}$and $B G L \mathscr{Q}(J)^{+}$are connected $H$-spaces, and so by Theorem 2.5.5 it suffices to show that the mapping

$$
\begin{equation*}
\rho_{*}: \quad H_{*}(G L \mathscr{X}) \rightarrow H_{*}(G L \mathscr{Q}(J)) \tag{5.2.3}
\end{equation*}
$$

is an isomorphism. According to Theorem 1.3.13, there exists an isometry $v \in \mathscr{M}(\mathscr{K} \otimes J)$ such that $\operatorname{Ad}(v)$ maps $\mathscr{2}(J)$ into $\mathscr{X}$. It follows that (5.2.3) is surjective, because the composition

$$
\rho \circ \operatorname{Ad}(v): \mathscr{2}(J) \rightarrow \mathscr{2}(J)
$$

is equal to $\operatorname{Ad}(v)$, since $\rho$ is merely an inclusion. Therefore, passing to homology we get $\operatorname{Ad}(v)_{*} \circ \rho_{*}=\operatorname{Ad}(v)_{*}$, and $\operatorname{Ad}(v)_{*}=$ id, by Lemma 2.6.12. So it remains to prove that the map (5.2.3) is injective. By the continuity of homology (Lemma 2.5.2),

$$
H_{*}(G L \mathscr{X})=\underline{\lim } H_{*}(G L \mathscr{A}),
$$

where the direct limit is taken over all separable $C^{*}$-subalgebras $\mathscr{A}$ of $\mathscr{X}$. So it suffices to show that for each such $\mathscr{A}$, the map $\rho_{*}$ is an injection on the image of $H_{*}(G L \mathscr{A})$ in $H_{*}(G L \mathscr{X})$. However, by Theorem 1.3.14, there exist isometries $v_{1} \in \mathscr{M}(\mathscr{K} \otimes J)$ and $v_{2} \in \mathscr{M}(\mathscr{K} \otimes B)$ such that $\operatorname{Ad}\left(v_{1}\right)$ maps $\mathscr{M}(\mathscr{K} \otimes J)$ into $\mathscr{X}$, and also $\operatorname{Ad}\left(v_{1}\right)$ is equal to $\operatorname{Ad}\left(v_{2}\right)$ on $\mathscr{A}$. Thus the composition

$$
H_{*}(G L \mathscr{A}) \rightarrow H_{*}(G L \mathscr{X}) \xrightarrow{\rho_{*}} H_{*}(G L \mathscr{Q}(J)) \xrightarrow{\operatorname{Ad}\left(v_{1}\right)_{*}} H_{*}(G L \mathscr{X})
$$

is equal to

$$
H_{*}(G L \mathscr{A}) \rightarrow H_{*}(G L \mathscr{X}) \xrightarrow{\operatorname{Ad}\left(v_{2}\right)_{*}} H_{*}(G L \mathscr{X}),
$$

which is in turn equal to simply the inclusion $H_{*}(G L \mathscr{A}) \rightarrow H_{*}(G L \mathscr{X})$, because by Lemma 2.6.12 again, $\operatorname{Ad}\left(v_{2}\right)_{*}=$ id. Therefore $\rho_{*}$ is injective on the image of $H_{*}(G L \mathscr{A})$ in $H_{*}(G L X)$ and so the theorem is proved if $J$ is an essential ideal of $B$. We can reduce the general case to this case by introducing the annihilator ideal of $J$ into the picture, as follows. To make the notation a little less cumbersome, let $I=\operatorname{Ann}(J)$, and for any ideal $L$ of $B$ let

$$
\mathscr{X}(L)=\mathscr{H}(\mathscr{K} \otimes B ; \mathscr{K} \otimes L) / \mathscr{K} \otimes L
$$

Consider the following commutative square (cf. (1.3.8))


By what we have already shown, since $I \oplus J$ is an essential, $\sigma$-unital ideal in $B$, the right-hand vertical map is an isomorphism in $K$-theory. It follows that the left-hand vertical map is an isomorphism in $K$-theory too, and since $K$-theory is additive (see Theorem 2.6.9), it follows that the maps $\mathscr{X}(I) \rightarrow \mathscr{Q}(I)$ and $\mathscr{X}(J) \rightarrow \mathscr{Q}(J)$ both induce isomorphisms. The latter one gives us the theorem in the general case.

An important consequence of Theorem 5.2 .3 is that we are able to make the groups $K_{*}(\mathscr{Q}(B))$ functorial for arbitrary $*$-homomorphisms, and not simply quasi-unital ones. As in Section II, if $B$ is a $C^{*}$-algebra then denote by $\tilde{B}$ the $C^{*}$-algebra obtained by adjoining a unit to $B$ (and if $B$ is unital already then $\widetilde{B}=B \oplus \mathbf{C}$ ). Any $*$-homomorphism $f: B_{1} \rightarrow B_{2}$ induces a unital $\operatorname{map} f: \widetilde{B}_{1} \rightarrow \widetilde{B}_{2}$, and so a quasi-unital map $1 \otimes f: \mathscr{K} \otimes \widetilde{B}_{1} \rightarrow \mathscr{K} \otimes \widetilde{B}_{2}$. Thus $f$ induces a $*$-homomorphism from $\mathscr{M}\left(\mathscr{K} \otimes \widetilde{B}_{1}\right)$ to $\mathscr{M}\left(\mathscr{K} \otimes \widetilde{B}_{2}\right)$, and so a map from $K_{*}\left(2\left(B_{1}\right)\right)$ to $K_{*}\left(2\left(B_{2}\right)\right)$ by means of the diagram


The vertical maps, which are given by restriction, are isomorphisms for the following reason: from Theorem 5.2.2 and the exact sequence

$$
0 \rightarrow \mathscr{X} \rightarrow \mathscr{M}(\mathscr{K} \otimes B) / \mathscr{K} \otimes J \rightarrow \mathscr{M}(\mathscr{K} \otimes B / J) \rightarrow 0,
$$

it follows that the inclusion of $\mathscr{X}$ into $\mathscr{M}(\mathscr{K} \otimes B ; \mathscr{K} \otimes J)$ induces an isomorphism

$$
\begin{equation*}
K_{*}(\mathscr{X}) \xrightarrow{\cong} K_{*}(\mathscr{M}(\mathscr{K} \otimes B) / \mathscr{K} \otimes J), \tag{5.2.5}
\end{equation*}
$$

since the quotient $\mathscr{M}(\mathscr{K} \otimes B / J)$ has trivial $K$-theory (by Theorem 2.6.5). Therefore by Theorem 5.2.3, the restriction map from $\mathscr{M}(\mathscr{K} \otimes B) / \mathscr{K} \otimes J$ to $\mathscr{2}(J)$ induces an isomorphism in $K$-theory. We note that the annihilator ideal of $B$ in $\widetilde{B}$ is equal to zero if $B$ is not unital, and equal to $C$ if $B$ is unital: in either case it is $\sigma$-unital.

Lemma 5.2.4. If $f$ happens to be quasi-unital already then this definition of $f_{*}$ agrees with the natural one.

Proof. Consider first the composition

$$
\begin{equation*}
\mathscr{M}\left(\mathscr{K} \otimes \widetilde{B}_{1}\right) \xrightarrow{f} \mathscr{M}\left(\mathscr{K} \otimes \widetilde{B}_{2}\right) \xrightarrow{\rho} \mathscr{M}\left(\mathscr{K} \otimes B_{2}\right) . \tag{5.2.6}
\end{equation*}
$$

If $p \in \mathscr{M}\left(B_{2}\right)$ is the projection such that $f\left[B_{1}\right]$ generates $p B_{2} p$ as a hereditary subalgebra, then $1 \otimes p \in \mathscr{M}\left(\mathscr{K} \otimes B_{2}\right)$ commutes with the image of the map (5.2.6). Thus we may write $\rho f=g+h$, where $g$ maps into $\mathscr{M}\left(\mathscr{K} \otimes p B_{2} p\right)$ and $h$ maps into $\mathscr{M}\left(\mathscr{K} \otimes(1-p) B_{2}(1-p)\right)$. Consider now the induced maps

$$
g_{*}, h_{*}: \quad K_{*}\left(\mathscr{M}\left(\mathscr{K} \otimes \widetilde{B}_{1}\right) / \mathscr{K} \otimes B_{1}\right) \rightarrow K_{*}\left(\mathscr{M}\left(\mathscr{K} \otimes \widetilde{B}_{2}\right) / \mathscr{K} \otimes B_{2}\right) .
$$

By Lemma 2.1.18, $\rho_{*} f_{*}=g_{*}+h_{*}$. But $h_{*}=0$ : the reason is that $\mathscr{K} \otimes B_{1}$ is contained in the kernel of $h$; therefore the induced map on $\mathscr{M}\left(\mathscr{K} \otimes \widetilde{B}_{1}\right) / \mathscr{K} \otimes B_{1}$ factors through $\mathscr{M}\left(\mathscr{K} \otimes \widetilde{B}_{2}\right)$,

and so we get $h_{*}=0$ from the fact that $K_{*}\left(\mathscr{M}\left(\mathscr{K} \otimes \widetilde{B}_{2}\right)\right)=0$. So it remains to show that $g_{*}=f_{*} \rho_{*}$. In fact $g=f \rho$ : both $g$ and $f \rho$ map into
$\mathscr{M}\left(\mathscr{K} \otimes p B_{2} p\right)$, and since $f\left[B_{1}\right]$ generates $p B_{2} p$ as a hereditary subalgebra, it suffices to show that

$$
\begin{equation*}
g(x) f(b)=f(\rho(x)) f(b) \tag{5.2.7}
\end{equation*}
$$

for every $x \in \mathscr{M}\left(\mathscr{K} \otimes \tilde{B}_{1}\right)$ and every $b \in \mathscr{K} \otimes B_{1}$. Since $\rho$ is the identity on $\mathscr{K} \otimes B_{1}$, the right-hand side of (5.2.7) is equal to $f(x b)$. The left-hand side is equal to $g(x) g(b)$, since $g$ is equal to $f$ on $\mathscr{K} \otimes B_{1}$; this is, in turn, equal to $g(x b)$, because $x b \in \mathscr{K} \otimes B_{1}$.

Remark. By the same technique we may, for example, make Ext ${ }^{-1}(C, A)$ functorial in the second variable, as long as $C$ is separable and $A$ is $\sigma$-unital. We leave the details to the reader, but note that the reason this works is that conjugation with an isometry gives the identity map on extension groups.

Theorem 5.2.5. With respect to ideals $J \subset B$ for which $J, B$, and $\operatorname{Ann}(J)$ are $\sigma$-unital, the functor $B \mapsto K_{*}(\mathscr{2}(B))$ is split exact.

Proof. Suppose that

$$
0 \rightarrow J \rightarrow B \leftrightarrows B / J \rightarrow 0
$$

is a split exact sequence with $J, \operatorname{Ann}(J)$, and $B$ all $\sigma$-unital. By Lemma 5.2.4, the map $p_{*}: K_{*}(\mathscr{2}(B)) \rightarrow K_{*}(\mathscr{Q}(B / J))$ has a right inverse. Therefore, it follows from the long exact sequence of Theorem 5.2.2, together with Theorem 5.2.3, that the sequence

$$
0 \rightarrow K_{*}(2(J)) \rightarrow K_{*}(2(B)) \xrightarrow{p_{*}} K_{*}(2(B / J)) \rightarrow 0
$$

is split exact, where the map from $K_{*}(\mathscr{Q}(J))$ to $K_{*}(\mathscr{2}(B))$ is equal to

$$
\begin{equation*}
K_{*}(\mathscr{2}(J)) \xrightarrow{\stackrel{p}{*}_{-1}^{\longrightarrow}} K_{*}(\mathscr{M}(\mathscr{K} \otimes B) / \mathscr{K} \otimes J) \rightarrow K_{*}(\mathscr{Q}(B)) . \tag{5.2.8}
\end{equation*}
$$

(We use the isomorphism (5.2.5) to replace $\mathscr{X}$ with $\mathscr{M}(\mathscr{K} \otimes B) / \mathscr{K} \otimes J$ in Theorems 5.2.3 and 5.2.2.) So the proof is a matter of showing that this is the same as the map $j_{*}: K_{*}(2(J)) \rightarrow K_{*}(\mathscr{2}(B))$, defined as in (5.2.4). Consider the diagram

where the various maps named $\rho$ are all obtained from restriction, while the maps $\pi$ are projections into quotients. The left-hand square commutes because in the analogous square

both ways around the diagram, from $\mathscr{M}(\mathscr{K} \otimes \widetilde{J})$ to $\mathscr{M}(\mathscr{K} \otimes J)$, are extensions of the inclusion of the ideal $\mathscr{K} \otimes J$ of $\mathscr{M}(\mathscr{K} \otimes \widetilde{J})$ into $\mathscr{M}(\mathscr{K} \otimes J)$ : hence by Theorem 1.1.2 the two maps are equal. The right-hand square certainly commutes too, and so since the top map of the diagram gives the map $j_{*}$ as in (5.2.4), while the bottom gives (5.2.8), these two maps are equal.

### 5.3. Isomorphism with Topological $K$-Theory

We begin by proving homotopy invariance for the algebraic $K$-theory groups of generalized Calkin algebras.

Lemma 5.3.1. The functor $B^{\prime} \mapsto K_{*}\left(2\left(B^{\prime}\right)\right)$ is stable, and also half exact with respect to split exact sequences of the form

$$
0 \rightarrow B \otimes I \rightarrow B \otimes A \leftrightarrows B \otimes A / I \rightarrow 0
$$

where $B$ is $\sigma$-unital, $A$ is separable, and the sequence

$$
0 \rightarrow \operatorname{Ann}(I) \rightarrow A \rightarrow A / \operatorname{Ann}(I) \rightarrow 0
$$

admits a completely positive section.
Proof. After identifying $\mathscr{2}\left(\mathscr{K} \otimes B^{\prime}\right)$ with $M_{2}\left(\mathscr{2}\left(B^{\prime}\right)\right)$ appropriately, the canonical map $e: \mathscr{Q}\left(B^{\prime}\right) \rightarrow \mathscr{2}\left(\mathscr{K} \otimes B^{\prime}\right)$ is equal to the map embedding $\mathscr{Q}\left(B^{\prime}\right)$ in the top-left-hand corner of $M_{2}\left(2\left(B^{\prime}\right)\right)$. By Theorem 2.6.11 this induces an isomorphism on $K$-theory. As for the split exactness part of the theorem, it suffices to show that the hypotheses of Theorem 5.2 .5 are satisfied. Thus we need to show that $\operatorname{Ann}(B \otimes I)$ is $\sigma$-unital. However, $\operatorname{Ann}(B \otimes I)=B \otimes \operatorname{Ann}(I)$ by the following argument. By Theorem 1.3.5, the sequence

$$
0 \rightarrow B \otimes \operatorname{Ann}(I) \rightarrow B \otimes A \rightarrow B \otimes A / \operatorname{Ann}(I) \rightarrow 0
$$

is exact. Therefore $B \otimes \operatorname{Ann}(I)$ is the kernel of the restriction map

$$
B \otimes A \rightarrow B \otimes A / \operatorname{Ann}(I) \subsetneq B \otimes \mathscr{M}(I) \subsetneq \mathscr{M}(B \otimes I) .
$$

In other words, it is the annihilator ideal of $B \otimes I$.
TheOrem 5.3.2. The functor $B \mapsto K_{*}(\mathscr{2}(B))$ (where $B$ is $\sigma$-unital) is homotopy invariant.

Proof. We apply Theorem 3.2.2 to the functor $A \mapsto K_{*}(2(B \otimes A))$, keeping $B$ fixed. As we pointed out in Remark 3.2.4, it suffices to know that the functor is split exact with respect to split exact sequences of the form described in Lemma 5.3.1. Hence homotopy invariance follows from the lemma and Theorem 3.2.2.

Theorem 5.3.3. For every $\sigma$-unital $C^{*}$-algebra $B$ the homomorphism

$$
\alpha: K_{*}(2(B)) \rightarrow K_{*}^{t}(\mathscr{2}(B))
$$

is an isomorphism.
Proof. Let $J^{\prime}$ be a $\sigma$-unital ideal in a $\sigma$-unital $C^{*}$-algebra $B^{\prime}$. Let $\mathscr{X}^{\prime}$ denote the kernel of $\mathscr{2}\left(B^{\prime}\right) \rightarrow \mathscr{2}\left(B^{\prime} / J^{\prime}\right)$. By Theorem 2.6.13, the diagram

which relates the long exact sequences in algebraic and topological $K$-theory, commutes. Apply this to the short exact sequence

$$
0 \rightarrow B \otimes C_{0}(\mathbf{R}) \rightarrow B \otimes C_{0}(0,1] \rightarrow B \rightarrow 0
$$

Since $K_{*}\left(\mathscr{2}\left(B \otimes C_{0}(0,1]\right)\right)=0=K_{*}^{\prime}\left(\mathscr{2}\left(B \otimes C_{0}(0,1]\right)\right)$, we get the commuting square


But since in the case we are considering, the annihilator ideal $\operatorname{Ann}\left(J^{\prime}\right)$ is $\sigma$-unital, (it is trivial, in fact), we also have the commuting square

where the top map is an isomorphism by Theorem 5.2.3, and the bottom is an isomorphism by applying the arguments of Theorem 5.2.3 to topological $K$-theory. Therefore, under the inductive hypothesis that the map $\alpha_{n-1}$ is an isomorphism for every $\sigma$-unital $C^{*}$-algebra $B$, it follows that $\alpha_{n}$ is an isomorphism for every $\sigma$-unital $C^{*}$-algebra. So it remains only to consider the case of $n=1$ to start off the induction. We know that $\alpha: K_{1}(\mathscr{Q}(B)) \rightarrow K_{1}^{\prime}(\mathscr{2}(B))$ is onto (this is true for any Banach algebra). We must show that any element of GL $\mathscr{Q}(B)$ which is connected to the identity is actually in the commutator subgroup. For this we can either argue from homotopy invariance or more directly, along the lines of the infinite dimensions argument of Theorem 2.4.7. Since the argument of Theorem 2.4.7 works, verbatim, we will not bother to repeat it here.

We remark that another way of getting the case $n=1$ is to introduce the group $K_{0}$ (see [4]). The reduction of dimensions argument may be taken one step further, to $K_{0}$. But the topological and algebraic $K_{0}$-groups are the same, so there is no further work to be done.

### 5.4. The General Case

Throughout this section, $A$ will denote a fixed unital $C^{*}$-algebra. Our goal is to indicate how the results of the previous two sections should be modified so as to yield the following generalization of Theorem 5.3.1.

Theorem 5.4.1. For every $\sigma$-unital $C^{*}$-algebra $B$ the homomorphism

$$
a: \quad K_{*}(A \otimes \mathscr{Q}(B)) \rightarrow K_{*}^{t}(A \otimes \mathscr{Q}(B))
$$

is an isomorphism.
The only non-trivial point has already been dealt with in Section I: by Theorems 1.3 .11 and 1.3 .12 , if $J$ is a $\sigma$-unital ideal in a $\sigma$-unital $C^{*}$-algebra $B$, then we have exact sequences

$$
\begin{equation*}
0 \rightarrow A \otimes \mathscr{X} \rightarrow A \otimes \mathscr{2}(B) \rightarrow A \otimes \mathscr{Q}(B / J) \rightarrow 0 \tag{5.4.1}
\end{equation*}
$$

and
$0 \rightarrow A \otimes \mathscr{X} \rightarrow A \otimes(\mathscr{M}(\mathscr{K} \otimes B) / \mathscr{K} \otimes J) \rightarrow A \otimes \mathscr{M}(\mathscr{K} \otimes B / J) \rightarrow 0$.
Here is the analog of Theorem 5.2.1.
Theorem 5.4.2. If $\mathscr{C}$ is a separable $C^{*}$-suvalgebra of $A \otimes \mathscr{X}$ then there exists an element $e \in A \otimes \mathscr{X}$ such that $1 \geqslant e \geqslant 0$ and $e c=c=c e$ for every $c \in \mathscr{C}$.

Proof. The $C^{*}$-algebra $\mathscr{C}$ is contained in some $A \otimes \mathscr{A}$, where $\mathscr{A}$ is a separable $C^{*}$-subalgebra of $\mathscr{X}$. By applying Lemma 5.2 .1 we obtain a suitable element $e$ of the form $e=1 \otimes u$.

Thus we can apply the excision results of Section 5.1 to the short exact sequences (5.4.1) and (5.4.2), and so obtain the analog of Theorem 5.2.2 for $A \otimes \mathscr{X}$.

Modulo forming the tensor product with $A$, or with the identity map on $A$, as the case may be, all the remaining results of Sections 5.1 and 5.3 follow easily. (There is one additional point, and that is the fact that $K_{*}(A \otimes \mathscr{H}(\mathscr{K} \otimes B))=0$. This is proved exactly as in Theorem 2.6.5, modulo the same modifications: form the tensor product of everything with $A$ or $\mathrm{id}_{A}$.) Then the proof of Theorem 5.4.1, along the lines of the proof of Theorem 5.3.3, follows without further complications.

## Vi. Further Results

We begin by studying the non-stable general linear group $G L_{1} 2(B)$. We prove that if $B$ is unital then the inclusion of $G L_{1} 2(B)$ into $G L_{\infty} \times 2(B)$ indices a homotopy equivalence $B G L_{1} \mathscr{2}(B)^{+} \cong B G L_{\infty} \mathscr{2}(B)^{+}$, and therefore also, for example, an isomorphism in homology. The proof is rather intricate, which should not be too surprising when one considers that the corresponding topological $K$-theory results are not at all trivial either.

Following this, we will consider the $K$-theory of unitary, as opposed to general linear groups. We find that the algebraic and topological groups for the algebras $\mathscr{2}(B)$ are equal; since everything goes through as expected, we will be brief

In Section 6.3 we study the Karoubi-Villamayor $K$-theory. These groups are another solution to the problem of defining the homotopy of the group $G L R$, where $R$ is a discrete ring. Karoubi and Villamayor make use of the equation

$$
\pi_{n}(X)=\pi_{n-1}(\Omega X)
$$

from homotopy theory, where $\Omega$ denotes the loop space, by defining an algebraic version of $\Omega$. In order to make things tick, a suitable notion of fibration is needed, and it must be shown that a short exact sequence of rings gives rise to a fibration

$$
G L J \rightarrow G L B \rightarrow G L B / J .
$$

This is unfortunately not the case for a general short exact sequence of rings, but it is true for stable $C^{*}$-algebras, and using this we are able to show that the Karoubi-Villamayor groups of a stable $C^{*}$-algebra are isomorphic to the topological $K$-theory groups, using the by now familiar technique of proving homotopy invariance and then reducing dimensions.

Karoubi has quite recently shown that the algebraic $\bmod p K$-theory groups of a stable $C^{*}$-algebra are equal to the topological $\bmod p K$-theory groups. We briefly indicate in Section 6.4 how to duplicate this result using our techniques.

We end with a conjecture concerning the algebraic $K$-theory of stable $C^{*}$-algebras.

### 6.1. Non-stable General Linear Groups

We begin by stating a result which effectively answers in the topological context the question about non-stable algebraic $K$-theory that we are addressing. It asserts that the non-stable topological $K_{1}$-group of $\mathscr{Z}(B)$, for a unital $C^{*}$-algebra $B$, is equal to the usual $K$-theory.

Theorem 6.1.1 [41, Theorem 3.7]. The group $G L_{1} 2(B) / G L_{1}^{0} \mathscr{2}(B)$ is naturally isomorphic to the group $K_{0}(B)$, for unital $C^{*}$-algebras $B$.

It is easy to give an explicit description of the above isomorphism: we map $G L_{1} \mathscr{2}(B) / G L_{1}^{0} \mathscr{Q}(B)$ into $G L_{\infty} \mathscr{2}(B) / G L_{\infty}^{0} \mathscr{2}(B)$ in the usual way; the latter group is $K_{1}^{\prime}(\mathscr{2}(B))$, and we pass to $K_{0}(B)$ via the boundary map:

$$
\text { index: } \quad K_{0}^{\prime}(\mathscr{Q}(B)) \rightarrow K_{0}(\mathscr{K} \otimes B)
$$

(see [41] for details).
Theorem 6.1.1 will be of considerable importance to us. We will also need computations of non-stable algebraic $K_{1}$ for various $C^{*}$-algebras $A$; in other words, we will need to compute the maximal perfect subgroup of $G L_{1} A$ for these $A$. We will suppose that $A$ is stable in the following weakened sense: there exist pairwise orthogonal projections $p_{1}, p_{2}, \ldots$ in the multiplier algebra of $A$, such that each $p_{i}$ and each $1-p_{i}$ is equivalent to the identity. Examples are the $C^{*}$-algebras $\mathscr{2}(B)$, and the ideals $\mathscr{X}$ defined in (5.2.1).

Theorem 6.1.2. (Compare [24, Proposition 7.1].) If $A$ is stable in the above sense then the group $G L_{1}^{0} A$ is perfect.

Proof. We give the argument of [24]. It suffices to show that if $x \in G L_{1}^{0} A$ and if $x$ is sufficiently close to the identity-we shall assume that $\|1-x\|<\frac{1}{2}$-then $x$ may be written as a product of commutators in $G L_{1}^{0} A$. With respect to the decomposition of the identity element $1=p_{1}+\left(1-p_{1}\right)$, write $x$ as the matrix $\left(\begin{array}{ll}x_{11} & x_{12} \\ x_{21} & x_{22}\end{array}\right)$. Direct computation reveals that $x=l u d$, where

$$
\begin{gathered}
l=\left(\begin{array}{cc}
1 & 0 \\
x_{21} x_{11}^{-1} & 1
\end{array}\right), \quad u=\left(\begin{array}{cc}
1 & x_{12}\left(x_{22}-x_{21} x_{11}^{-1} x_{12}\right)^{-1} \\
0 & 1
\end{array}\right) \\
d=\left(\begin{array}{cc}
x_{11} & 0 \\
0 & x_{22}-x_{21} x_{11}^{-1} x_{12}
\end{array}\right) .
\end{gathered}
$$

From our assumption $\|1-x\|<\frac{1}{2}$, it follows that $\left\|1-x_{11}\right\|<\frac{1}{2}$, and also

$$
\begin{aligned}
\left\|1-x_{22}+x_{21} x_{11}^{-1} x_{12}\right\| & \leqslant\left\|1-x_{22}\right\|+\left\|x_{21}\right\| \frac{1}{1-\left\|1-x_{11}\right\|}\left\|x_{12}\right\| \\
& <\frac{1}{2}+\frac{1}{2} \times \frac{1}{1 / 2} \times \frac{1}{2} \\
& =1
\end{aligned}
$$

so that $l, u$, and $d$ are all well defined and connected to the identity. Now, both $l$ and $u$ are commutators, as are any upper or lower triangular, unimodular matrices. For example,

$$
\left(\begin{array}{ll}
1 & 0 \\
y & 1
\end{array}\right)=\left[\left(\begin{array}{cc}
\left(1+|y|^{1 / 4}\right)^{-1} & 0 \\
0 & 1
\end{array}\right),\left(\begin{array}{cc}
1 & 0 \\
y|y|^{-1 / 4} & 1
\end{array}\right)\right] .
$$

So it remains to show that $d$ is a product of commutators. However, we can write

$$
d=\left(\begin{array}{cc}
x_{11} & 0 \\
0 & 1
\end{array}\right) \times\left(\begin{array}{cc}
1 & 0 \\
0 & x_{22}-x_{21} x_{11}^{-1} x_{12}
\end{array}\right)
$$

and then the infinite dimensions trick used in the proof of Theorem 2.4.7 shows that each of the terms on the right is a product of two matrices of the form

$$
\left(\begin{array}{ccc}
w & 0 & 0 \\
0 & w^{-1} & 0 \\
0 & 0 & 1
\end{array}\right)
$$

These are products of commutators by the Whitehead lemma (Theorem 2.4.2). In fact, these are products of commutators of elementary matrices, which are of course connceted to the identity.

TheOrem 6.1.3. (i) If $B$ is a unital $C^{*}$-algebra then

$$
G L_{1}^{0} \mathscr{Q}(B)=\left[G L_{1} \mathscr{Q}(B), G L_{1} \mathscr{2}(B)\right] .
$$

(ii) If $B$ is a unital $C^{*}$-algebra and if $\mathscr{X}$ denotes the kernel of the map from $\mathscr{Q}(B \otimes C[0,1])$ to $\mathscr{2}(B)$ induced from evaluation at $1 \in[0,1]$, then

$$
G L_{1} \mathscr{X}=G L_{1}^{0} \mathscr{X}=\left[G L_{1} \mathscr{X}, G L_{1} \mathscr{X}\right] .
$$

Proof. For (i), it follows from Theorem 6.1.2 that $G L_{1}^{0} \mathscr{Z}(B)$ is contained in the commutator subgroup. On the other hand, according to Theorem 6.1.1, the quotient group $G L_{1} 2(B) / G L_{1}^{0} \mathscr{2}(B)$ is isomorphic to $K_{0}(B)$; so it is, in particular, abelian. Therefore, the connected component of the identity contains the commutator subgroup. As for (ii), it follows from Theorem 6.1.1 that the homomorphism

$$
G L_{1} \mathscr{Q}(B \otimes C[0,1]) / G L_{1}^{0} \mathscr{2}(B \otimes C[0,1]) \rightarrow G L_{1} \mathscr{2}(B) / G L_{1}^{0} \mathscr{Q}(B),
$$

induced from evaluation at 1 , is an isomorphism, since the corresponding map in topological $K$-theory is. Therefore, if $x \in \mathrm{GL}_{1} \mathscr{X}$ then $x$ is connected to the identity by some path $\gamma$ in $G L_{1} \mathscr{2}(B \otimes C[0,1])$. If $\bar{\gamma}$ denotes the path in $G L_{1} \mathscr{2}(B \otimes C[0,1])$ which is the image of $\gamma$ under the endomorphism of $G L_{1} \mathscr{Q}(B \otimes C[0,1])$ induced from the map

$$
B \otimes C[0,1] \rightarrow B \rightarrow B \otimes C[0,1]
$$

then $\gamma \bar{\gamma}^{-1}$ is a path in $G L_{1} \mathscr{X}$ connecting $x$ to the identity. Thus $G L_{1} \mathscr{X}=G L_{1}^{0} \mathscr{X}$. The rest of the assertion of part (ii) follows from Theorem 6.1.2.

From now on we will consider $G L_{1} \mathscr{X}$ as a discrete group. We wish to show that the space $B G L_{1} \mathscr{X}^{+}$is contractible. By Theorem 6.1.3, together with the definition of the plus construction, $B G L_{1} \cdot \mathscr{P}^{+}$is simply connected. Therefore, by the Whitehead theorem (Theorem2.5.3) together with the fact that $H_{*}\left(B G L_{1} \mathscr{X}\right) \cong H_{*}\left(B G L_{1} \mathscr{X}^{+}\right)$, the following result suffices.

Theorem 6.1.4. The space $B G L_{1} \mathscr{X}$ is acyclic.
The proof is rather long and complicated, so let us put it to one side for the moment and see how we can prove from it the main theorem of the section:

Theorem 6.1.5. If $B$ is a unital $C^{*}$-algebra then the natural inclusion $j$ of $G L_{1} \mathscr{2}(B)$ into $G L_{\infty} \mathscr{2}(B)$ induces a homotopy equivalence

$$
j_{+}: B G L_{1} \mathscr{2}(B)^{+} \xlongequal{\cong} B G L_{\infty} \mathscr{2}(B)^{+} .
$$

We will need the following strengthening of the excision results of Section 5.1. The theorem is due to Berrick [7, 8].

Theorem 6.1.6. Let

$$
1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1
$$

be an extension of groups. If PH acts trivially on $H_{*}(N)$, and if $N$ is perfect then the sequence

$$
B N^{+} \rightarrow B G^{+} \rightarrow B H^{+}
$$

is a fibration, up to homotopy.
(We remind the reader that $P H$ acts on $H_{*}(N)$ as follows: lift $y \in P H$ to an element $x \in G$; then the action of $y$ is the automorphism of $H_{*}(N)$ induced by the automorphism $z \mapsto x z x^{-1}$ of $N$.)

Lemma 6.1.7. The map $B G L_{1} \mathscr{2}(B \otimes C[0,1])^{+} \rightarrow B G L \mathscr{2}(B)^{+}$is a homotopy equivalence. Thus the two maps from $B G L_{1} \mathscr{2}(B \otimes C[0,1])^{+}$to $B G L 2(B)^{+}$induced from evaluation at $0 \in[0,1\rfloor$ and $1 \in[0,1]$ are homotopic.

Proof. We wish to apply Theorem 6.1.6 to the extension

$$
\begin{equation*}
1 \rightarrow G L_{1} \mathscr{X} \rightarrow G L_{1} \mathscr{Z}(B \otimes C[0,1]) \rightarrow G L_{1} \mathscr{V}(B) \rightarrow 1 \tag{6.1.1}
\end{equation*}
$$

Since the maximal perfect subgroup of $G L_{1} \mathscr{2}(B)$ is equal to the connected component of the identity, and since the connected component of the identity is generated by elements of the form $e^{y}$, where $y \in \mathscr{2}(B)$, in order to show that $P G L_{1} \mathscr{2}(B)$ acts trivially on $H_{*}\left(G L_{1} \mathscr{X}\right)$, it suffices to show that each $e^{y}$ acts trivially. Thus it suffices to show that if $x \in \mathscr{Q}(B \otimes C[0,1])$, then the automorphism $\operatorname{Ad}\left(e^{x}\right)$ of $G L_{1} \mathscr{X}$ is trivial at the level of homology. Using the continuity of homology (Lemma 2.5.2), it suffices to show that the map

$$
\operatorname{Ad}\left(e^{x}\right)_{*}: \quad H_{*}\left(G L_{1} \mathscr{X}\right) \rightarrow H_{*}\left(G L_{1} \mathscr{X}\right)
$$

is trivial on the image of every $H_{*}\left(G L_{1} \mathscr{A}\right)$ in $H_{*}\left(G L_{1} \mathscr{X}\right)$, where $\mathscr{A}$ is a separable $C^{*}$-subalgebra of $\mathscr{X}$. Define $C^{*}$-subalgebras of $\mathscr{M}(\mathscr{K} \otimes B \otimes C[0,1])$ as follows:
(1) $E_{1}=\mathscr{K} \otimes B \otimes C[0,1]$.
(2) $E=\mathscr{K} \otimes B \otimes C[0,1)$.
(3) $E_{2}$ is a separable subalgebra of $\mathscr{M}(\mathscr{K} \otimes B \otimes C[0,1] ;$ $\mathscr{K} \otimes B \otimes C[0,1))$ which maps onto $\mathscr{A}$ in the quotient $\mathscr{X}$.
(4) $\mathscr{F}$ is a separable subalgebra of $\mathscr{M}(\mathscr{K} \otimes B \otimes C[0,1])$ which maps onto the $C^{*}$-algebra generated by $x$ in $2(B \otimes C[0,1])$.

Then by Theorem 1.1.12, there exists an element $M \in \mathscr{M}(\mathscr{K} \otimes B \otimes$ $C[0,1]$ ) such that:
(i) $1 \geqslant M \geqslant 0$.
(ii) $M \cdot E_{1} \subset E$.
(iii) $(1-M) \cdot E_{2} \subset E$.
(iv) $[M, \mathscr{F}] \subset E$.

If $u$ denotes the image of $M$ in $2(B \otimes C[0,1])$, then condition (ii) implies that $u \in \mathscr{X}$; condition (iii) implies that $u$ acts as a unit for $\mathscr{A}$, and condition (iv) implies that $u$ commutes with $x$. It follows that on the image of $\mathscr{A}$ in $\mathscr{X}$, the automorphism $\operatorname{Ad}\left(e^{x}\right)$ is equal to $\operatorname{Ad}\left(e^{u^{1 / 2 x}}{ }^{1 / 2}\right)$. But $e^{u^{1 / 2 x u^{1 / 2}}} \in G L_{1} X$, and so this last automorphism is inner, which implies by Lemma 2.5.1 that it acts trivially on $H_{*}\left(G L_{1} \mathscr{X}\right)$, and therefore $\operatorname{Ad}\left(e^{x}\right)_{*}$ is trivial on the image of $H_{*}\left(G L_{1} \mathscr{A}\right)$ in $H_{*}^{*}\left(G L_{1} \mathscr{X}\right)$. This shows that $P G L_{1} \mathscr{Q}(B)$ acts trivially on $H_{*}\left(G L_{1} \mathscr{X}\right)$; by Theorem 6.1.3, $P G L_{1} \mathscr{X}=G L_{1} \mathscr{X}$. So we get that the sequence

$$
B G L_{1} \mathscr{X}{ }^{+} \rightarrow B G L_{1} \mathscr{\mathscr { }}(B \otimes C[0,1])^{+} \rightarrow B G L_{1} \mathscr{Z}(B)^{+}
$$

is a fibration, up to homotopy. Since by Theorem 6.1.4, the space $B G L_{1} \mathscr{X}^{+}$ is contractible, it follows that the map

$$
B G L_{1} \mathscr{Q}(B \otimes C[0,1])^{+} \rightarrow B G L_{1} \mathscr{2}(B)^{+}
$$

is a homotopy equivalence.
Proof of Theorem 6.1.5. Define an embedding $g$ of $M_{\infty} \mathscr{2}(B)$ into $2(B)$ as follows. Choose pairwise orthogonal projections $P_{i} \otimes 1 \in \mathscr{M}(\mathscr{H} \otimes B)$, each equivalent to the identity, and then embed $M_{n} \mathscr{Q}(B)$ into $\mathscr{Z}(B)$, using these equivalences, by sending the matrix unit $e_{i i} \in M_{n}$ to the projections $P_{i} \otimes 1$ (or, strictly speaking, the image of this projection in the quotient
$\mathscr{Q}(B)$ of $\mathscr{M}(\mathscr{K} \otimes B)$ ). This passes to a homomorphism of groups from $G L_{\infty} \mathscr{2}(B)$ to $G L_{1} \mathscr{2}(B)$. The composition

$$
G L_{\infty} \mathscr{2}(B) \xrightarrow{g} G L_{1} \mathscr{Q}(B) \xrightarrow{j} G L_{\infty} \mathscr{Q}(B)
$$

is easily seen to pass to the identity map on $K_{*}(2(B))$ by the technique used in the proof of Lemma 2.6.12. As for the composition

$$
\begin{equation*}
G L_{1} \mathscr{2}(B) \xrightarrow{j} G L_{\infty} \mathscr{Q}(B) \xrightarrow{g} G L_{1} \mathscr{Q}(B), \tag{6.1.2}
\end{equation*}
$$

it is given by conjugation with an isometry $V \otimes 1 \in \mathscr{M}(\mathscr{K} \otimes B)$ whose final space is equal to the projection $P_{1} \otimes 1$. Now, by connecting $V$ to the identity in $\mathscr{B}(\mathscr{H})$ by a strongly continuous path $\left\{V_{t}\right\}_{\in \in[0,1]}$ of isometries, we obtain a $*$-homomorphism from $\mathscr{K} \otimes B$ to $\mathscr{K} \otimes B \otimes[0,1]$ which is the identity at $0 \in[0,1]$ and which is equal to conjugation with the isometry $V \otimes 1$ at $1 \in[0,1]$. It is easily verified that this map is quasi-unital: for instance, the projection of Definition 1.1.6 is equal to the tensor product of $1 \in B$ with the projection $\left\{V_{t} V_{t}^{*}\right\}_{t \in[0,1]} \in \mathscr{M}(\mathscr{K} \otimes C[0,1])$. So we obtain from it a *-homomorphism from $2(B)$ to $2(B \otimes C[0,1])$ which is the identity over $0 \in[0,1]$, and which is the map induced by conjugation with $V \otimes 1$ over $1 \in[0,1]$. Therefore, it follows from Theorem 6.1.7 that the map

$$
B G L_{1} \mathscr{2}(B)^{+} \xrightarrow{i_{+}} B G L_{\infty} \mathscr{Q}(B)^{+} \xrightarrow{g_{+}} B G L_{1} \mathscr{Q}(B)^{+}
$$

is homotopic to the identity. Thus the map

$$
g_{+}: B G L_{x} \mathscr{V}(B)^{+} \rightarrow B G L_{1} \mathscr{Q}(B)^{+}
$$

is a homotopy inverse to $j_{+}: B G L_{1} \mathscr{Q}(B)^{+} \rightarrow B G L_{\infty} \mathscr{Q}(B)^{+}$.
Corollary 6.1.8. The natural map from $B G L_{1} \mathscr{Q}(B)^{+}$to $B^{\prime} G L_{1} \mathscr{Q}(B)$ is a homotopy equivalence.

Here $B^{\prime} G L_{1} \mathscr{Q}(B)$ denotes the classifying space of the group considered as a topological group.

Proof. Consider the homotopy commutative diagram


By the theorem we have just finished proving, the top map is a homotopy equivalence. It follows from [41] that the bottom map is a homotopy equivalence. Finally, Theorem 5.3 .3 asserts that the right-hand map is a homotopy equivalence. Three maps in the diagram being homotopy equivalences, so is the fourth.

It remains then to prove Theorem 6.1.4. The basic idea behind the proof is to construct a model for $B G L_{1} \mathscr{X}$ which is easily comparable to $B G L_{\infty} \mathscr{X}$, this latter space being acyclic by the results of Section V. We will follow [23], where the same ideas are used to show that $B G L_{1} \mathscr{B}(\mathscr{H})$ is acyclic; the technique is credited there to a paper of Segal [48]. For simplicity's sake, we will let $B^{\prime}$ denote $B \otimes C[0,1]$, and let $J^{\prime}$ denote the ideal $B \otimes C[0,1)$. Fix for the rest of this section a maximal set $\left\{e_{1}, e_{2}, \ldots\right\}$ of pairwise orthogonal rank one projections in $\mathscr{K}$.

Definiton 6.1.9. Let us say that a projection $P \in \mathscr{M}\left(\mathscr{K} \otimes B^{\prime}\right)$ is standard if it is of the form

$$
P=\sum_{n=1}^{\infty} e_{i_{n}} \otimes 1
$$

where $\left\{e_{i_{1}}, e_{i_{2}}, \ldots\right\}$ is a countably infinite subset of $\left\{e_{1}, e_{2}, \ldots\right\}$.
It will be convenient to denote also by $P$ the image of the standard projection $P$ in the quotient algebra $\mathscr{2}\left(B^{\prime}\right)$.

Lemma 6.1.10. Let $P_{1}, \ldots, P_{n}$ be standard projections and let $\mathscr{A}$ be $a$ separable $C^{*}$-subalgebra of $\mathscr{X}$. There exist standard projections $P_{i}^{\prime} \leqslant P_{i}$ which are mutually orthogonal, and for which $P_{i}^{\prime} \mathscr{A} P_{j}^{\prime}=0$, if $i \neq j$.

Proof. Write $P_{i}=\sum_{j=1}^{\infty} p_{i j}$, where each $p_{i j}$ is some $e_{k} \otimes 1$. As a first step, we may assume that the $P_{i}$ are orthogonal, that is, that the $p_{i j}$ are distinct from one another. In order to define $P_{i}^{\prime}$ observe first that for every $p_{i j}$, and any element $X \in \mathscr{M}\left(\mathscr{K} \otimes B^{\prime} ; \mathscr{K} \otimes J^{\prime}\right)$, the element $p_{i j} X$ is contained in $\mathscr{K} \otimes J^{\prime}$, by definition of $\mathscr{M}\left(\mathscr{K} \otimes B^{\prime} ; \mathscr{K} \otimes J^{\prime}\right)$. Next, since in particular, $p_{i j} X \in \mathscr{K} \otimes B^{\prime}$, it follows that for any index $k$, the sequence $\left\{p_{i j} X p_{k h}\right\}_{h=0}^{\infty}$ converges in norm to zero as $h \rightarrow \infty$. It is convenient to use the reverse lexicographic ordering on the set of all pairs $(i, j)$ of natural numbers, which is defined as

$$
(a, b)<(c, d) \begin{cases}\text { if } & b>d \text { or } \\ \text { if } & b=d \text { and } a<c\end{cases}
$$

In our case, the first number in the pair will be limited to $1,2, \ldots, n$. It follows that any ( $i, j$ ) has only a finite number of predecessors in this
ordering. Now, let $\left\{X_{1}, X_{2}, \ldots\right\}$ be a countable subset of $\mathscr{M}\left(\mathscr{K} \otimes B^{\prime} ; \mathscr{K} \otimes J^{\prime}\right)$ whose image in $\mathscr{X}$ is a dense subset of $\mathscr{A}$. Define $P_{i}^{\prime}=\sum_{j} p_{i j}^{\prime}$, where each $p_{i j}^{\prime}$ is chosen recursively from among the projections $\left\{p_{i 1} p_{i 2}, \ldots\right\}$ as follows:
(i) $p_{11}^{\prime}=p_{11}$.
(ii) Having chosen projections $p_{i j}^{\prime}$ for all $(i, j)<(k, l)$, choose the projection $p_{k l}^{\prime}$ such that

$$
\left\|p_{i j}^{\prime} X_{h} p_{k}^{\prime}\right\|<2^{-(l+j)}
$$

for all $(i, j)<(k, l)$ and all $h=1, \ldots, l+j$.
(Note that we are only asking each $p_{k l}^{\prime}$ to do a finite number of things, and so its existence is not in doubt.) To show that $P_{i}^{\prime} \mathscr{A} P_{k}^{\prime}=0$, it suffices to show that $P_{i}^{\prime} X_{h} P_{k}^{\prime} \in \mathscr{K} \otimes J^{\prime}$ for every $h$. We have

$$
\begin{aligned}
P_{i}^{\prime} X_{h} P_{k}^{\prime} & =\sum_{j} p_{i j}^{\prime} X_{h} \sum_{l} p_{k l}^{\prime} \\
& =\sum_{j}\left(\sum_{l} p_{i j}^{\prime} X_{h} p_{k l}^{\prime}\right)
\end{aligned}
$$

by continuity of multiplication on bounded subsets in the strict topology. All of the terms in this double series are elements of $\mathscr{K} \otimes J^{\prime}$, and apart from the finitely many terms for which $h>j+l$, the $(i, j)$ th term is bounded in norm by $2^{-(j+1)}$. Consequently, the series converges in norm (absolutely, in fact), and so the limit, $P_{i}^{\prime} X_{h} P_{k}^{\prime}$, is an element of $\mathscr{K} \otimes J^{\prime}$.

Next, we need a generalization of Theorem 1.1.11.
Lemma 6.1.11. Let $\mathscr{A}$ be a separable $C^{*}$-subalgebra of $\mathscr{X}$, and let $E_{1}, \ldots, E_{n}$ be separable C$C^{*}$-subalgebras of $\mathscr{X}$ such that $\mathscr{A} \cdot E_{i} \subset E_{i}$ for all $i$, and $E_{i} \cdot E_{j}=0$, if $i \neq j$. There exist elements $M_{1}, \ldots, M_{n}$ of $X$ such that
(i) Each $M_{i}$ commutes with $\mathscr{A}$.
(ii) If $i \neq j$, then $M_{i} \cdot E_{j}=0$.
(iii) For every $i,\left(1-M_{i}\right) \cdot E_{i}=0$.

Proof. We proceed by induction. The case $n=1$ follows from the fact that we may find $M_{1} \in \mathscr{X}$ such that $M x=x=x M$ for every element $x$ of $\mathscr{A}$ and $E_{1}$ (see Theorem 5.2.1). Suppose then that we are given elements $M_{1}^{\prime}, \ldots, M_{n-1}^{\prime}$ in $\mathscr{X}$ with the required properties for the $C^{*}$-algebras $E_{1}, \ldots, E_{n-1}$. Apply Theorem 1.1.11 to the $C^{*}$-subalgebras $\tilde{E}_{1}, \tilde{E}_{2}, \tilde{E}$, and $\mathscr{F}$ of $\mathscr{M}\left(\mathscr{K} \otimes B^{\prime}\right)$, where:
(i) $\mathscr{F}$ is a separable $C^{*}$-subalgebra of $\mathscr{M}\left(\mathscr{K} \otimes B^{\prime} ; \mathscr{K} \otimes J^{\prime}\right)$, the image of which in $\mathscr{X}$ is the $C^{*}$-algebra generated by $M_{1}^{\prime}, \ldots, M_{n-1}^{\prime}$, and $\mathscr{A}$.
(ii) $\widetilde{E}_{1}$ is the $C^{*}$-algebra generated by $\mathscr{K} \otimes B^{\prime}$, along with a separable $C^{*}$-subalgebra of $\mathscr{M}\left(\mathscr{K} \otimes B^{\prime} ; \mathscr{K} \otimes J^{\prime}\right)$ which maps onto the $C^{*}$-subalgebra of $\mathscr{X}$ generated by $E_{1}, \ldots, E_{n-1}$.
(iii) $\tilde{E}_{2}$ is a separable $C^{*}$-subalgebra of $\mathscr{M}\left(\mathscr{K} \otimes B^{\prime} ; \mathscr{K} \otimes J^{\prime}\right)$ which maps onto $E_{n}$.
(iv) $\tilde{E}=\mathscr{K} \otimes J^{\prime}$.

Then $\tilde{E}_{1} \cdot \tilde{E}_{2} \subset \tilde{E}$ and $\mathscr{F} \cdot \tilde{E}_{1} \subset \tilde{E}_{1}$, and so there exists an element $\tilde{M} \in \mathscr{M}\left(\mathscr{H} \otimes B^{\prime}\right)$ such that:
(1) $\tilde{M}$ annihilates $\tilde{E}_{1}$, modulo $\mathscr{K} \otimes J^{\prime}$.
(2) $\tilde{M}$ commutes with $\mathscr{F}$, modulo $\mathscr{K} \otimes J^{\prime}$.
(3) $1-\tilde{M}$ annihilates $\widetilde{E}_{2}$, modulo $\mathscr{K} \otimes J^{\prime}$.

Condition (1) implies that $\tilde{M} \in \mathscr{M}\left(\mathscr{K} \otimes B^{\prime} ; \mathscr{K} \otimes J^{\prime}\right)$. If we let $M_{n}$ denote the image of $\tilde{M}$ in $X$, then by conditions (1), (2), and (3): $M_{n}$ commutes with $\mathscr{A}$ and the $M_{i}^{\prime} ;\left(1-M_{n}\right) \cdot E_{n}=0$; and $M_{n} \cdot E_{i}=0$ for $i=1, \ldots, n-1$. Define $M_{i}$, for $i=1, \ldots, n-1$, to be $\left(1-M_{n}\right) M_{i}^{\prime}$.

Theorem 6.1.12. Let $P_{1}, \ldots, P_{n}$ be standard projections and let $Y_{1}, \ldots, Y_{n}$ be elements of $G L_{1} \mathscr{X}$. There exist proper projections $P_{i}^{\prime} \leqslant P_{i}$, for $i=1, \ldots, n$, and an element $Y_{0} \in G L_{1} \mathscr{X}$ such that $Y_{0} P_{i}^{\prime}=Y_{i} P_{i}^{\prime}$ for all $i=1, \ldots, n$.

We are using $P$ to denote not only a proper projection but also its image in $2\left(B^{\prime}\right)$.

Proof. By Theorem 6.1.3, the group $G L_{1} \mathscr{X}$ is connected, and so each $Y_{i}$ may be written as a product of exponentials

$$
Y_{i}=e^{A_{1} \cdots} \cdots e^{A_{m} m} \quad(i=1, \ldots, n) .
$$

(By adding some $A_{i j}$ equal to zero, we may assume that the number $m$ of exponentials does not depend on the index i.) Now, let $\mathscr{A}$ be the $C^{*}$-algebra generated by the $A_{i j}$, and choose orthogonal projections $P_{i}^{\prime} \leqslant P_{i}$ as in Lemma 6.1.10 such that $P_{i}^{\prime} \mathscr{A} P_{k}^{\prime}=0$ if $i \neq k$. If $E_{i}$ denotes the $C^{*}$-subalgebra of $\mathscr{X}$ gencrated by $\mathscr{A} P_{i}^{\prime}$ then $E_{i} \cdot E_{k}=0$ if $i \neq k$ and $\mathscr{A} \cdot E_{i} \subset E_{i}$. The hypotheses of Lemma 6.1.11 are satisfied, so let $M_{1}, \ldots, M_{n}$ be elements of $\mathscr{X}$ as in the conclusion of the lemma. Define

$$
Y_{0}=\prod_{i=1}^{n}\left(\prod_{j=1}^{m} e^{A_{i j} M_{i}}\right)
$$

(The ordering of the products is from left to right: thus $\prod_{j} x_{j}=x_{1} x_{2} \cdots x_{m}$.) If $i \neq k$ then by expanding $e^{A_{i j} M_{i}}$ as a Taylor series, we see that if $X \in \mathscr{A}$, or $X=1$, then

$$
\begin{equation*}
e^{A_{i j} M_{i}} X P_{k}^{\prime}=X P_{k}^{\prime} \tag{6.1.3}
\end{equation*}
$$

On the other hand,

$$
\begin{equation*}
e^{A_{i j} M_{i}} X P_{i}^{\prime}=e^{A_{i j}} X P_{i}^{\prime} \tag{6.1.4}
\end{equation*}
$$

which again we can verify by expanding the exponential as an infinite series. We now compute $Y_{0} P_{k}^{\prime}$,

$$
\begin{aligned}
Y_{0} P_{k}^{\prime} & =\prod_{i=1}^{n}\left(\prod_{j=1}^{m} e^{A_{i j} M_{i}}\right) P_{k}^{\prime} \\
& =\prod_{i=1}^{k}\left(\prod_{j=1}^{m} e^{A_{i j} M_{i}}\right) P_{k}^{\prime} \\
& =\prod_{i=1}^{k-1}\left(\prod_{j=1}^{m} e^{A_{i, j} M_{i}}\right) \prod_{j=1}^{m} e^{A_{k j}} P_{k}^{\prime} \\
& =\prod_{j=1}^{m} e^{A_{k j}} P_{k}^{\prime} .
\end{aligned}
$$

The second equality above follows from (6.1.3) applied repeatedly, once for each $e^{A_{i j} M_{i}}$ with $i>k$. The third equality follows from (6.1.4) applied one for each $e^{A_{k j} M_{k}}$ (where $j=1, \ldots, m$ ). The last equality follows from another application of (6.1.3). By definition of the $A_{i j}$, we have proved the theorem.

This completes the analysis part of the proof of Theorem 6.1.4. From here on, things are more algebraic in nature, and we are able to follow [23] quite closely. To simplify the notation a little bit, let $G$ denote the group $G L_{1} \mathscr{X}$.

Definition 6.1.13. A flag is a decreasing sequence $\mathscr{P}=\left\{P_{1}, P_{2}, \ldots\right\}$ of standard projections such that each $P_{i-1}-P_{i}$ is a standard projection.

Definition 6.1.14. Let $P$ be a standard projection, and denote by $G_{P}$ the subgroup of $G$ consisting of those elements $x$ for which $x P=P$. If $\mathscr{P}=\left\{P_{1}, P_{2}, \ldots\right\}$ is flag then denote by $G_{\mathscr{P} p}$ the subgroup of $G$ consisting of all elements $x$ for which $x P_{i}=P_{i}=P_{i} x$, for sufficiently large $i$.

Definition 6.1.15. Denote by $H_{P}$ the subgroup of $G_{P}$ consisting of
elements $x$ for which $P x=P=x P$. Denote by $H_{\mathscr{P}}$ the subgroup of $G_{\mathscr{P}}$ consisting of those elements $x$ for which $P_{i} x=P_{i}=x P_{i}$, for sufficiently large $i$.

With respect to the decomposition $1=P+(1-P)$, elements of $G_{P}$ are matrices of the form

$$
\left(\begin{array}{ll}
1 & y \\
0 & x
\end{array}\right)
$$

while elements of $H_{P}$ are of the form

$$
\left(\begin{array}{ll}
1 & 0 \\
0 & x
\end{array}\right)
$$

We note that the groups $G_{\mathscr{y}}$ and $H_{\mathscr{y}}$ may be expressed in terms of the groups $H_{P_{i}}$ and $G_{P_{i}}$ as the direct limits

$$
\begin{align*}
& H_{y p}=\bigcup_{i=1}^{\infty} H_{P_{i}},  \tag{6.1.5}\\
& G_{y p}=\bigcup_{i=1}^{\infty} G_{P_{i}} \tag{6.1.6}
\end{align*}
$$

Lemma 6.1.16. For any flag $\mathscr{P}$, the space $B H_{\mathscr{y}}$ is acyclic.
Proof. It is clear from the description (6.1.5) that the group $H_{\mathscr{P}}$ is isomorphic to $G L_{\infty} \mathscr{X}$. By Theorem 5.2.2, $G L_{\infty} \mathscr{X}$ has the same homology as $G L_{\infty} \mathscr{2}(B \otimes C[0,1))$. But by Theorem 5.3.2, this last group is acyclic.

The next result follows from a computation in group homology using the Hochschild-Lyndon-Serre spectral sequence. We refer the reader to [23] for a proof, pointing out here only the one addition needed to make the proof work in the situation we are considering.

Lemma 6.1.17. The inclusion $H_{P} \subset G_{P}$ induces an isomorphism on homology.

Proof. [23, Lemma 4 and Corollary 5]. The following result is proved in [23]. Let $R$ be a ring and let $G(R)$ be the subgroup of $G L_{2} \tilde{R}$ consisting of matrices of the form

$$
\left(\begin{array}{ll}
1 & y \\
0 & x
\end{array}\right),
$$

where $x$ is equal to 1 , modulo the ideal $R$ of $\tilde{R}$, and $y \in R$. If the action

$$
\left(\begin{array}{ll}
1 & y \\
0 & x
\end{array}\right) \mapsto\left(\begin{array}{ll}
\lambda & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & y \\
0 & x
\end{array}\right)\left(\begin{array}{ll}
\lambda & 0 \\
0 & 1
\end{array}\right)^{-1}=\left(\begin{array}{cc}
1 & \lambda y \\
0 & x
\end{array}\right)
$$

of the multiplicative group $\mathbf{Q}^{*}$ of the rationals $\mathbf{Q}$ on $G(R)$ induces the trivial action of $\mathbf{Q}^{*}$ on homology, then the inclusion of the subgroup of matrices of the form

$$
\left(\begin{array}{ll}
1 & 0 \\
0 & x
\end{array}\right)
$$

into $G(R)$ induces an isomorphism on the level of homology. The paper [23] is concerned with unital rings, but cursory inspection shows that this restriction is only used to show that the above action is trivial-because it is inner. In our case, the action is trivial for the usual reason: for each separable $C^{*}$-subalgebra $\mathscr{A}$ of $\mathscr{X}$, the action may be duplicated on the image of $H_{*}(G(\mathscr{A}))$ in $H_{*}(G(\mathscr{X}))$ by an inner, and hence trivial, action, namely

$$
\left(\begin{array}{ll}
1 & y \\
0 & x
\end{array}\right) \mapsto\left(\begin{array}{cc}
1 & 0 \\
0 & e^{x M}
\end{array}\right)\left(\begin{array}{ll}
1 & y \\
0 & x
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
0 & e^{x M}
\end{array}\right)^{-1}
$$

where $e^{\alpha}=\lambda$ and $M \in \mathscr{T}$ satisfies

$$
(1-M) \mathscr{A}=0=\mathscr{A}(1-M) .
$$

The next result combines these last two lemmas.

Theorem 6.1.18. If $\mathscr{P}$ is any flag then the space $B G_{i p}$ is acyclic.
Proof. By Lemma 6.1.16, it suffices to show that the inclusion of $H_{, p}$ into $G_{\mathscr{y}}$ induces an isomorphism in homology. But in view of (6.1.5) and (6.1.6), together with the continuity of homology (Lemma 2.5.2), this follows from the fact that the inclusion of each $H_{P_{i}}$ into $G_{P_{i}}$ induces an isomorphism in homology, which is the assertion of Lemma 6.1.17.

We are now in a position to construct a suitable model for the classifying space of $G=G L_{1} \mathscr{X}$, and to show that it is acyclic. We can follow [23] almost verbatim. Denote by $E G$ the contractible complex appearing in the Milnor infinite join model of the classifying space $B G$ (see Sect. 2.5). For a flag $\mathscr{P}$ denote by $E_{\mathscr{\mathscr { L }}}$ the subcomplex of $E G$ consisting of all simplices $\left(g_{0}, \ldots, g_{k}\right)$ such that for large enough $i, g_{r} P_{i}=g_{s} P_{i}$ for every $r$ and $s$ (in other words, $g_{r}^{-1} g_{s} \in G_{\mathscr{y}}$ for all $r$ and $s$ ). Since $G$ acts on $E G$ by left multiplication, it is clear that $E_{; p}$ is $G$-invariant; furthermore, the quotient
$G \backslash E_{\mathscr{P}}$ is equal to $B G_{3 p}$ (see [23, Lemma 9]). Indeed, the connected component of the base point in $E_{\mathscr{}}$ is equal to the space $E G_{\mathscr{P}}$, while $G$ acts transitively on the components of $E_{o p}$.

Let $E_{*}$ be the union of the $E_{: p}$ over all flags.
Lemma 6.1.19 [23, Lemma 10]. The space $E_{*}$ is contractible.
Proof. Let $\sigma_{1}, \ldots, \sigma_{p}$ be simplices in $E_{*}$ and choose flags $\mathscr{P}_{1}, \ldots, \mathscr{P}_{p}$ such that $\sigma_{i} \in E_{y, 2}$. Thus there is some $k$ for which if $\sigma_{i}=\left(g_{i 1}, \ldots, g_{i n_{i}}\right)$ then

$$
g_{i 1} P_{i k}=g_{i 2} P_{i k}=\cdots=g_{i n_{i}} P_{i k}
$$

for all $i=1, \ldots, p$. We now apply Theorem 6.1.12 to the standard projections $P_{i k}$ for $i=1, \ldots, p$, and the elements $Y_{i}=g_{i 1}$, to obtain standard projections $P_{i}^{\prime} \leqslant P_{i k}$ and an element $g_{0} \in G$ such that $g_{i 1} P_{i}^{\prime}=g_{0} P_{i}^{\prime}$ for all $i$. It follows that for every $i$, the simplex $\left(g_{0}, g_{i 1}, \ldots, g_{i n_{i}}\right)$ is contained in $E_{*}$. Indeed, it is contained in $E_{\mathscr{P}_{i}^{\prime}}$, where the initial projection for the flag $\mathscr{P}_{i}^{\prime}$ is $P_{i}^{\prime}$, and the rest of the sequence of projections is irrelevant (but note that a sequence exists since $P_{i}^{\prime}$ is a standard projection). Thus for any $\sigma_{1}, \ldots, \sigma_{p}$, the space $E_{*}$ contains the cone over the subspace $\sigma_{1} \cup \sigma_{2} \cup \cdots \cup \sigma_{p}$, and so any finite subcomplex of $E_{*}$ is contractible in $E_{*}$. It follows that $E_{*}$ is contractible (see [50, 7.6.24]).

Corollary 6.1.20 [23, Lemma 11]. The inclusion of $B_{*}=G \backslash E_{*}$ into $B G=G \backslash E G$ is a homotopy equivalence.

Proof. This follows from comparing the homotopy sequences for the fibrations $G \rightarrow E G \rightarrow B G$ and $G \rightarrow E_{*} \rightarrow B_{*}$.

Finally, we can turn to the proof of Theorem 6.1.4. We need to make note of onc or two constructions involving flags. Two flags $\mathscr{P}$ and $\mathscr{R}$ are orthogonal if the projections $P_{1}$ and $R_{1}$ are orthogonal, in which case the sum of $\mathscr{P}$ and $\mathscr{R}$ is defined to be the flag

$$
\mathscr{P} \oplus \mathscr{R}=\left\{P_{1}+R_{1}, P_{2}+R_{2}, \ldots\right\} .
$$

If $\mathscr{P}_{i}=\left\{P_{i 1}, P_{i 2}, \ldots\right\}$, are flags then let us write $\mathscr{P}_{i} \leqslant \mathscr{P}_{j}$ if for every $k$, $P_{i k} \leqslant P_{j k}$. It is easy to see that for any collection $\mathscr{P}_{1}, \ldots, \mathscr{P}_{n}$ of flags there exist pairwise orthogonal flags $\mathscr{P}_{1}^{\prime}, \ldots, \mathscr{P}_{n}^{\prime}$ such that $\mathscr{P}_{i}^{\prime} \leqslant \mathscr{P}_{i}$, for every $i$.

Note that if $\mathscr{P}_{i} \leqslant \mathscr{P}_{j}$ then $E_{\mathscr{O}_{j}} \subset E_{\mathscr{P}_{i}}$; and also, if $\mathscr{P}_{i}$ and $\mathscr{P}_{j}$ are orthogonal flags then $E_{\mathscr{P}_{i}} \cap E_{\mathscr{y}_{i}}=E_{\mathscr{P}_{i} \oplus \not z_{j}}$.

Proof of Theorem 6.1.4. (Compare [23, Theorem 13].) By Corollary 6.1.20, it suffices to show that the space $B_{*}$ is acyclic. Since

$$
B_{*}=\bigcup B G_{\mathscr{}},
$$

where the union is taken over all flags, and since homology is continuous, it suffices to show that for any finite set of flags $\mathscr{P}_{1}, \ldots, \mathscr{P}_{n}$, the space

$$
B G_{\mathscr{p}_{1}} \cup \cdots \cup B G_{\mathscr{P}_{n}}
$$

is contained in an acyclic subspace of $B_{*}$. Choose flags $\mathscr{P}_{i}^{\prime} \leqslant \mathscr{P}_{i}$ such that $\mathscr{P}_{i}^{\prime}$ is orthogonal to $\mathscr{P}_{j}^{\prime}$, if $i \neq j$. Since $B G_{\mathscr{P}_{i}} \supset B G_{\mathscr{P}_{j}}$, it suffices to show that

$$
\begin{equation*}
B G_{\mathscr{f}_{i}} \cup \cdots \cup B G_{\mathscr{P}_{n}^{\prime}} \tag{n}
\end{equation*}
$$

is an acyclic space. This we do by induction on $n$, the case $n=1$ following from Theorem 6.1.18. It is convenient to suppose a little more, and so here are our induction hypotheses: we suppose that the space ( $6.1 .7_{n}$ ) is acyclic for any $n$ orthogonal flags, and furthermore, that if $\mathscr{P}$ is any flag orthogonal to all the $\mathscr{P}_{i}^{\prime}$, then the space

$$
\begin{equation*}
B G_{\mathscr{p} \oplus \mathscr{M}_{1}^{\prime}} \cup \cdots \cup B G_{\mathscr{P} \oplus \mathscr{P}_{n-1}^{\prime}} \tag{n}
\end{equation*}
$$

is acyclic. Note that this vacuously true for $n=1$. Given these two hypotheses, we prove the acyclicity of first ( $6 \cdot 1.8_{n+1}$ ), and then ( $6 \cdot 1.7_{n+1}$ ). The two spaces $B G_{\mathscr{p _ { 1 } ^ { \prime } \oplus \mathscr { p }}} \cup \cdots \cup B G_{\mathscr{P}_{n-1}^{\prime} \oplus \mathscr{p}}$ and $B G_{\mathscr{p _ { n } ^ { \prime } \oplus \mathscr { p }}}$ are acyclic, as is their intersection

$$
B G_{\mathscr{\phi _ { 1 } ^ { \prime } \oplus ( \varphi _ { n } ^ { \prime } \oplus \mathscr { P }}, \cup \cdots \cup B G_{\mathscr{p}_{n-1}^{\prime} \oplus\left(\mathscr{w}_{n}^{\prime} \oplus \mathscr{\prime}\right)} . . . . ~}
$$

It follows from the Mayer-Vietoris sequence that their union is acyclic as well. But the union is

$$
B G_{\mathscr{P}_{1}^{\prime} \oplus \mathscr{P}} \cup \cdots \cup B G_{\mathscr{P}_{n}^{\prime} \oplus \mathscr{H}},
$$

which is the space (6.1.8 $n_{n+1}$ ). The acyclicity of $\left(6 \cdot 1 \cdot 7_{n+1}\right)$, follows from the Mayer-Vietoris sequence for the pair of acyclic spaces

$$
B G_{\mathscr{y}_{1}^{\prime}} \cup \cdots \cup B G_{\mathscr{P}_{n}^{\prime}} \quad \text { and } \quad B G_{\mathscr{Y}_{n+1}}
$$

whose intersection is

$$
B G_{\mathscr{p}_{1}^{\prime} \oplus: \mathscr{n}_{n+1}^{\prime}} \cup \cdots \cup B G_{\mathscr{P}_{n}^{\prime} \oplus \mathscr{P}_{n+1}^{\prime}},
$$

which is the acyclic space $\left(6.1 .8_{n+1}\right)$.

### 6.2. Unitary Groups

Let $A$ be a $C^{*}$-algebra. Denote by $U A$ the subgroup of $G L A$ consisting of all unitary matrices. We want to make a remark or two about proving the following result.

Theorem 6.2.1. If $B$ is a $\sigma$-unital $C^{*}$-algebra then the natural map

$$
\alpha: B^{d} U \mathscr{Q}(B)^{+} \rightarrow B^{t} U \mathscr{Q}(B)^{+}=B^{t} U \mathscr{Q}(B)
$$

is a homotopy equivalence.
Here, $B^{d}$ denotes the classifying space, considering $U A$ as a discrete group, and $B^{t}$ denotes the classifying space, considering $U A$ as a topological group in the usual way. We should make a remark about the latter construction. The family of maps

$$
F_{t}: \quad x \mapsto x|x|^{-1}(t+(1-t)|x|) \quad(x \in G L A)
$$

from $G L A$ to itself, parametrized by $t \in[0,1]$, is a homotopy from the identity to a retraction of $G L A$ onto $U A$. It follows of course that the inclusion $U A \subsetneq G L A$ is a homotopy equivalence, and from this it follows that the natural map $B^{t} U A \rightarrow B^{t} G L A$ is a homotopy equivalence. In particular, $\pi_{1}\left(B^{t} U A\right)=K_{1}^{t}(A)$, which is abelian, and so $B^{t} U A \cong B^{t} U A^{+}$, and the natural map mentioned in the theorem makes sense.

The proof begins by identifying the maximal perfect subgroup of $U \mathscr{Q}(B)$. For this we simply appeal to [24], in which is proved a unitary version of Theorem 2.4.7: if $A$ is a stable $C^{*}$-algebra then the connected component of the identity in $U A$ is perfect (see [24, Proposition 7.6]). On the other hand, the quotient $U A / U^{0} A$ is equal to $K_{1}(A)$, which is abelian, so that $U^{0} A$ contains the commutator subgroup. Therefore $U^{0} A$ is equal to [ $U A, U A$ ], which is, in turn, equal to $P U A$. These results are proved for stable $C^{*}$-algebras; they hold, and the same proofs work equally well, for $C^{*}$-algebras of the form $\mathscr{2}(B)$. Hence the maximal perfect subgroup of $U \mathscr{2}(B)$ is what it should be, and the spaces $B^{d} U \mathscr{Q}(B)^{+}$and $B^{t} U \mathscr{Z}(B)$ have the same fundamental group. From here on, the proof of Theorem 6.2.1 is exactly the same as the proof of Theorem 5.3.3: the reader can see for himself that throughout Section $V$ we may substitute $U \mathscr{Z}(B)$ where ever $G L \mathscr{2}(B)$ appears, with no ill effect.

The result of the previous section holds for the unitary group in place of $G L_{1}$. In fact, the proof becomes in one respect a little simpler: the groups $G_{\mathscr{P}}$ and $H_{\mathscr{y}}$ are equal, and so the spectral sequence argument (Lemma 6.1.16) is not needed. However, it is somewhat more complicated to show that the group $U \mathscr{X}$ is perfect (see [24]).

### 6.3. Karoubi-Villamayor Theory

The basic reference for this section is the paper [32] by Karoubi and Villamayor. We will use the following notation throughout this section: if $R$
is any ring then $R\left[t_{1}, \ldots, t_{n}\right]$ will denote the ring of polynomials in the (commuting) indeterminates $t_{1}, \ldots, t_{n}$, with coefficients in $R$. All rings will be considered as discrete.

Definition 6.3.1 [32, Definition 3.5]. An element $x \in G L R$ is said to be connected to the identity if there exists an element $\tilde{x} \in G L R[t]$ such that $\tilde{x}(0)=1$ and $\tilde{x}(1)=x$. Denote by $G L^{0} R$ the set of all elements connected to the identity.

It is easily verified that $G L^{0} R$ is a normal subgroup of $G L R$. As we pointed out in Section II, cvery elementary matrix $e_{i j}^{r}$ is connected to the identity by the part $e_{i j}^{r t}$. Thus, for example, if $R$ is a unital ring, then since

$$
[G L R, G L R]=E R \subset G L^{0} R
$$

it follows that the quotient $G L R / G L^{0} R$ is abelian. In fact, this last result is true for any $R$ (see [32, Theorem 3.6]).

Definition 6.3.2. Denote by $K V_{1}(R)$ the quotient group $G L R / G L^{0} R$.
This is a natural enough algebraic analog of the topological $K_{1}$-group defined in Section II. The higher $K V$-groups are defined by means of a loop space construction, also motivated by topological considerations.

Definition 6.3.3 [32, p. 269]. Denote by $\Omega R$ the ideal in $R[t]$ consisting of all polynomials $f$ such that $f(0)=0=f(1)$. Define $\Omega^{n} R$ inductively by $\Omega^{n} R=\Omega\left(\Omega^{n-1} R\right)$, and $\Omega^{1} R=\Omega R$.

Definition 6.3.4. Denote by $K V_{n}(R)$ the group $K V_{1}\left(\Omega^{n-1} R\right)$.
The higher $K V$-groups are related to $K V_{1}$ by means of a long exact sequence analogous to the long exact sequence of topological $K$-theory. As in the topological case, we need to introduce a suitable notation of fibration.

Definition 6.3.5 [32, Definition 2.2]. A ring homomorphism $\varphi: R \rightarrow S$ is called a GL-fibration if, for every $\beta \in G L S\left[t_{1}, \ldots, t_{n}\right]$ such that $\beta(0, \ldots, 0)=1$, there exists an element $\alpha \in G L R\left[t_{1}, \ldots, t_{n}\right]$ such that $\varphi(\alpha)=\beta$.

This definition should be compared with the hypothesis of Lemma 2.1.7. Suppose that $R$ is a $C^{*}$-algebra. Then $R\left[t_{1}, \ldots, t_{n}\right]$ embeds in $R \otimes C\left(I^{n}\right)$ (where $I$ denotes the unit interval) as the subring of all polynomial
functions from $I^{n}$ to $R$. The hypothesis of Lemma 2.1.7 is that any continuous function $\beta$ from $I^{n}$ to $G L S$, such that $\beta(0, \ldots, 0)=1$, lifts to a continuous function $\alpha: I^{n} \rightarrow G L R$. If $\varphi: R \rightarrow S$ is a $G L$-fibration then any polynomial function $\beta$ such that $\beta(0, \ldots, 0)=1$ lifts to a polynomial function $\alpha: I^{n} \rightarrow G L R$. Going back to general rings, by following the argument of Theorem 2.1.5, we can prove the following result.

Theorem 6.3.6. If $\varphi: R \rightarrow S$ is a GL-fibration then given any elements $\beta \in G L S\left[t_{0}, t_{1}, \ldots, t_{n}\right]$ and $\alpha_{0} \in G L R\left[t_{1}, \ldots, t_{n}\right]$ such that $\varphi\left(\alpha_{0}\right)=\beta\left(0, t_{1}, \ldots, t_{n}\right)$, there exists an element $\alpha \in \operatorname{GLR}\left[t_{0}, t_{1}, \ldots, t_{n}\right]$ such that $\varphi(\alpha)=\beta$ and furthermore, $\alpha\left(0, t_{1}, \ldots, t_{n}\right)=\alpha_{0}\left(t_{1}, \ldots, t_{n}\right)$.

In other words, every $G L$-fibration is an "algebraic Serre fibration." Now, every $G L$-fibration $\varphi: R \rightarrow S$ is onto (see [32]). If $J$ is the kernel of $\varphi$, then from the short exact sequence

$$
\begin{equation*}
0 \rightarrow J \rightarrow R \rightarrow S \rightarrow 0 \tag{6.3.1}
\end{equation*}
$$

we obtain a long exact $K V$-theory sequence

$$
\cdots \rightarrow K V_{n}(R) \rightarrow K V_{n}(S) \xrightarrow{\partial} K V_{n-1}(J) \rightarrow K V_{n-1}(R) \rightarrow \cdots
$$

in a manner exactly analogous to the construction of the long exact homotopy sequence associated with a Serre fibration. Let us describe the boundary map for the case $n=2$. As in the topological case, we can get the higher boundary maps $\partial: K V_{n}(S) \rightarrow K V_{n-1}(J)$ by identifying $K V_{n}(S)$ and $K V_{n-1}(J)$ with $K V_{n-1}(\Omega S)$ and $K V_{n-2}(\Omega J)$, respectively (if $R \rightarrow S$ is a $G L$-fibration then so is $\Omega R \rightarrow \Omega S$-see [32, Proposition 2.10]). Given an element $\beta \in G L \Omega S \subset G L S[t]$, we lift to an element $\alpha \in G L R[t]$ such that $\alpha(0)=1$. Then we define $\partial[\beta]=[\alpha(0)]$. See $[32$, Sect. 4] for further details. Jumping a little bit ahead, suppose that the exact sequence (6.3.1) is an exact sequence of $C^{*}$-algebras. We map the algebraic $\Omega^{n}$ into the topological one as the polynomial functions into the continuous functions, and then we obtain the commutative diagram

comparing long exact sequences in $K V$-theory and topological $K$-theory. This sets the stage for a reduction of dimensions argument to show that $\alpha: K V_{n}(\mathscr{K} \otimes A) \rightarrow K_{n}^{t}(\mathscr{K} \otimes A)$ is an isomorphism for all $n$, which is the goal of this section. We must do the following:
(i) Show that $A \mapsto K V_{*}(\mathscr{K} \otimes A)$ is stable.
(ii) Show that it is split exact.
(iii) Show that $\mathscr{K} \otimes A \rightarrow \mathscr{K} \otimes A / J$ is a $G L$-fibration.
(iv) Show $\alpha: K V_{1}(\mathscr{K} \otimes A) \rightarrow K_{1}^{t}(\mathscr{K} \otimes A)$ is an isomorphism.

We will need to use the fact that the functor $K V_{*}$ is homotopy invariant in the obvious algebraic sense:

Lemma 6.3.7 [32, Proposition 4.3]. The homomorphisms $\varepsilon_{0}, \varepsilon_{1}$ : $R[t] \rightarrow R$, given by evaluation at $t=0$ and $t=1$, induce the same map from $K V_{*}(R[t])$ to $K V_{*}(R)$.

The result is neither surprising nor difficult to prove, given the definition of $K V_{*}$.

Theorem 6.3.8. If $J$ is an ideal in a ring $B$ and $X \in G L B$, then conjugation with $X$ is an automorphism of $G L J$ which induces the trivial automorphism of $K V_{*}(J)$.

Proof. We may assume that $B$ is unital. Suppose first that $X$ is an elementary matrix $e_{i j}^{b}$. Define a map

$$
G L \Omega^{n} A \rightarrow G L \Omega^{n} A[t]
$$

by $f\left(t_{1}, t_{2}, \ldots, t_{n}\right) \mapsto e_{i j}^{t b} f\left(t_{1}, t_{2}, \ldots, t_{n}\right) e_{i j}^{-t b}$. Then composition with evaluation at $t=0$ gives the identity map on $G L \Omega^{n} A$, whilst composition with evaluation at $t=1$ gives conjugation with $X$ on $G L \Omega^{n} A$. Therefore, by Lemma 6.3.7, conjugation with $X$ gives the identity map on $K V_{*}(A)$. It follows that the whole of the commutator subgroup [GLB, GLB] acts trivially on $K V_{*}(A)$. Given an arbitrary $X$ and an element $f \in G L \Omega^{n} A$, since $f$ is contained in some $G L_{k} \Omega^{n} A$, it suffices to show that $X$ acts trivially on the image of each $G L_{k} \Omega^{n} A$, where $k=1,2, \ldots$. But for some $l, X \in G L_{l} B$, and we may assume that $l \geqslant k$. Then the action of $X$ on $G L_{k} \Omega^{n} A$ is equal to the action of $\left(\begin{array}{cc}x & 0 \\ 0 & x^{-1}\end{array}\right)$ on the same, and by what we have already shown, this passes to the trivial action on $K V_{*}(A)$.

TheOrem 6.3.9. The functor $A \mapsto K V_{*}(\mathscr{K} \otimes A)$ is split exact and stable.
Proof. Suppose that

$$
0 \rightarrow J \rightarrow A \underset{p}{\stackrel{s}{\leftrightarrows}} A / J \rightarrow 0
$$

is a split exact sequence of $C^{*}$-algebras and $*$-homomorphisms. We claim that $p: \mathscr{H} \otimes A \rightarrow \mathscr{K} \otimes A / J$ is a $G L$-fibration. We must show that if $\beta$ is an
element of $G L \mathscr{K} \otimes A / J\left[t_{1}, \ldots, t_{n}\right]$, and if $\beta(0, \ldots, 0)=0$, there is some $\alpha \in G L \mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right]$ such that $p(\alpha)=\beta$. But in fact any $\beta$ lifts to some $\alpha$, namely, for example the element $\alpha=s(\beta)$. As for stability, we apply the usual technique: by identifying $\mathscr{K} \otimes \mathscr{K} \otimes A$ with $M_{2}(\mathscr{K} \otimes A)$ appropriately, we see that it suffices to show that the inclusion of a ring $R$ into the top-left-hand corner of $M_{2} R$ induces an isomorphism $K V_{*}(R) \rightarrow \cong K V_{*}\left(M_{2} R\right)$. Define an isomorphism of groups from $G L \Omega^{n} M_{2} R$ to $G L \Omega^{n} R$ by identifying in the obvious manner $G L_{k} \Omega^{n} M_{2} R$ with $G L_{2 k} \Omega^{n} R$. Then the composition

$$
\begin{equation*}
G L \Omega^{n} R \rightarrow G L \Omega^{n} M_{2} R \xrightarrow{\cong} G L \Omega^{n} R \tag{6.3.3}
\end{equation*}
$$

is the endomorphism of $G L \Omega^{n} R$ given by mapping the $n$th row/column to the $2 n$th row/column. On the subgroup $G L_{k} \Omega^{n} R \subset G L \Omega^{n} R$, this coincides with conjugation by a $2 k \times 2 k$ permutation matrix. Consequently by Theorem 6.3.8, (6.3.3) gives the trivial map on $K V_{*}$, and therefore the map $R \rightarrow M_{2} R$ induces an isomorphism from $K V_{*}(R)$ to $K V_{*}\left(M_{2} R\right)$.

Corollary 6.3.10. The functor $A \mapsto K V_{*}(\mathscr{K} \otimes A)$ is homotopy invariant.

Next, we turn to the problem of showing that short exact sequences of stable $C^{*}$-algebras are $G L$-fibrations.

Lemma 6.3.11. For any $C^{*}$-algebra $A$, the maps

$$
\varepsilon_{0}, \varepsilon_{1}: \quad K_{1}\left(\mathscr{K} \otimes A\left[t_{0}, t_{1}, \ldots, t_{n}\right]\right) \rightarrow K_{1}\left(\mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right]\right),
$$

given by evaluation at $t_{0}=0$ and $t_{0}=1$, are equal.
Here, $K_{1}$ denotes the ordinary algebraic $K_{1}$ functor, as defined in Section II.

Proof. It is easily verified that the functor

$$
A \mapsto K_{1}\left(\mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right]\right)
$$

is stable, and split exact (the latter fact follows from Theorem 2.4.14; the former by the same sort of computation that we have just completed). Consequently, it is homotopy invariant. Consider then the commuting diagram

where the map $\varphi: \mathscr{K} \otimes A\left[t_{0}\right] \rightarrow \mathscr{K} \otimes A \otimes C[0,1]$ sends the indeterminate $t_{0}$ to the canonical generator $x \mapsto x$ of $C[0,1]$, and the maps $e_{i}$ are cvaluation at $i=0,1$. Since $e_{0_{*}}=e_{1_{*}}$ it follows that $\varepsilon_{0_{*}}=\varepsilon_{1_{*}}$.

Lemma 6.3.12. For every $C^{*}$-algebra $A$, the map

$$
\theta_{*}: K_{1}\left(\mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right]\right) \rightarrow K_{1}(\mathscr{K} \otimes A)
$$

is an isomorphism, where $\theta$ is given by evaluation at $\left(t_{1}, \ldots, t_{n}\right)=(0, \ldots, 0)$.
Proof. Define a map $\psi: \mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right] \rightarrow \mathscr{K} \otimes A\left[t_{0}, t_{1}, \ldots, t_{n}\right]$ by

$$
t_{i} \mapsto t_{0} t_{i}+\left(1-t_{0}\right) \quad(i=1, \ldots, n) .
$$

Composition with the map from $\mathscr{K} \otimes A\left[t_{0}, t_{1}, \ldots, t_{n}\right]$ to $\mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right]$ given by evaluation at $t_{0}=1$ gives the identity map on $\mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right]$, while composition with evaluation at $t_{0}=0$ gives the map

$$
\begin{equation*}
\mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right] \xrightarrow{\theta} \mathscr{K} \otimes A \leftrightarrows \mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right] . \tag{6.3.4}
\end{equation*}
$$

It follows from Lemma 6.3.11 then, that (6.3.4) passes to the identity map on the group $K_{1}\left(\mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right]\right)$, since it is algebraically homotopic to the identity. The reverse composition of the two maps in (6.3.4),

$$
\mathscr{K} \otimes A \rightarrow \mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right] \xrightarrow{\ominus} \mathscr{K} \otimes A,
$$

is the identity, and so it follows that $\theta_{*}$ is an isomorphism since the homomorphism $K_{1}(\mathscr{K} \otimes A) \rightarrow K_{1}\left(\mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right]\right)$ is an inverse.

TheOrem 6.3.13. (Compare [32, théorème 2.6].) If $R$ is any ring then any surjection $R \rightarrow \mathscr{K} \otimes A$ is a GL-fibration.

Proof. Suppose that $X \in G L \mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right]$ and $X(0, \ldots, 0)=1$. Then, of course, $X$ determines the trivial element of $K_{1}(\mathscr{K} \otimes A)$ via the map $\theta_{*}$ of Lemma 6.3.12. Consequently, $X$ determines the trivial element of $K_{1}\left(\mathscr{K} \otimes A\left[t_{1}, \ldots, t_{n}\right]\right)$, since by the lemma, $\theta_{*}$ is an isomorphism. Thus we may express $X$ as a product of elementary matrices $e_{i j}^{a}$, and since each elementary matrix lifts to $G L R$ (to an element of the form $e_{i j}^{r}$, where $r$ maps onto $a$ ), it follows that $X$ lifts to some element in $G L R$-the product of the liftings for the $e_{i j}^{a}$, for example.

TheOrem 6.3.14. The natural homomorphism

$$
\alpha: \quad K V_{*}(\mathscr{K} \otimes A) \rightarrow K_{*}^{\prime}(\mathscr{K} \otimes A)
$$

is an isomorphism.

Proof. Since $\alpha$ commutes with the boundary maps in the long exact sequences for $K_{*}^{t}$ and $K V_{*}$, by examining the long exact sequences for the short exact sequence

$$
0 \rightarrow \mathscr{K} \otimes A \otimes C[0,1] \rightarrow \mathscr{K} \otimes A \otimes C[0,1) \rightarrow \mathscr{K} \otimes A \rightarrow 0
$$

we see that the result for $\alpha: K V_{n} \rightarrow K_{n}^{t}$ follows from the result for $\alpha: K V_{n-1} \rightarrow K_{n-1}^{t}$. So to prove the theorem it suffices to show that the map $\alpha: K V_{1}(\mathscr{K} \otimes A) \rightarrow K_{1}^{t}(\mathscr{K} \otimes A)$ is an isomorphism. In other words, we must show that $G L^{0} \mathscr{K} \otimes A$, as given in Definition 6.3.1, is equal to the topological connected component of the identity. Clearly it is contained in the topological connected component of the identity. On the other hand, we have observed that the algebraic $G L^{0}$ contains the commutator subgroup, and since in the case of a stable $C^{*}$-algebra the commutator subgroup is equal to the topological connected component of the identity (by Theorem 2.4.6), the result follows.

### 6.4. Mod p K-Theory

We wish to indicate very briefly how our techniques may be used to prove the following result of Karoubi [31].

Thforfm 6.4.1 [31, théorème 2.4]. If $A$ is any $C^{*}$-algebra then

$$
K_{*}(\mathscr{K} \otimes A ; \mathbf{Z} / p) \cong K_{*}^{t}(\mathscr{K} \otimes A ; \mathbf{Z} / p)
$$

The functors $K_{*}(\cdot ; \mathbf{Z} / p)$ and $K_{*}^{t}(\cdot ; \mathbf{Z} / p)$ denote respectively algebraic and topological $K$-theory with coefficients in the group $\mathbf{Z} / p$ of integers modulo $p$. We do not want to go into the details of this; however, at least some words of explanation are in order. For $n \geqslant 2$, the $\bmod p$ homotopy groups $\pi_{n}(X ; \mathbf{Z} / p)$ of a space $X$ are defined to be

$$
\pi_{n}(X ; \mathbf{Z} / p)=\left[P^{n}(\mathbf{Z} / p), X\right]
$$

where $[\cdot \cdot \cdot]$ denotes the set of homotopy classes of maps, and $P^{n}(\mathbf{Z} / p)$ is the space obtained by attaching an $n$-cell to $S^{n-1}$ by a degree $p$ map $S^{n-1} \rightarrow S^{n-1}$. For details the reader is referred to [55], and the references therein. For $n \geqslant 2$ then, we define for a unital $C^{*}$-algebra $A$,

$$
\begin{aligned}
& K_{n}(A ; \mathbf{Z} / p)=\pi_{n}\left(B^{d} G L A^{+} ; \mathbf{Z} / p\right) \\
& K_{n}^{t}(A ; \mathbf{Z} / p)=\pi_{n}\left(B^{\imath} G L A ; \mathbf{Z} / p\right)
\end{aligned}
$$

For $n=1$ a slightly different definition is desirable; we are not going to give it here, but at any rate, there is a universal coefficient sequence

$$
0 \rightarrow K_{n}(A) \otimes \mathbf{Z} / p \rightarrow K_{n}(A ; \mathbf{Z} / p) \rightarrow \operatorname{Tor}_{\mathbf{z}}\left(K_{n-1}(A), \mathbf{Z} / p\right) \rightarrow 0
$$

and a similar sequence in topological $\bmod p K$-theory, by means of which we may compare $K_{1}(A ; \mathbf{Z} / p)$ with $K_{1}^{t}(A ; \mathbf{Z} / p)$. Again, we refer the reader to [55] for details. For our purposes it really suffices to work with indices $n \geqslant 2$ anyway. Now, for a non-unital $C^{*}$-algebra, we define $K_{*}(A ; \mathbf{Z} / p)$ to be the kernel of the map $K_{*}(\tilde{A} ; \mathbf{Z} / p) \rightarrow K_{*}(\mathbf{C} ; \mathbf{Z} / p)$, where $\tilde{A}$ denotes the $C^{*}$-algebra obtained by adjoining a unit to $A$. The proof of Theorem 6.4.1 follows almost immediately from our techniques and the following result.

Theorem 6.4.2 [55, Theorem 1.2]. If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of $\mathbf{C}$-algebras then there is a long exact $\bmod p K$-theory sequence

$$
\cdots K_{n}(B ; \mathbf{Z} / p) \rightarrow K_{n}(C ; \mathbf{Z} / p) \rightarrow K_{n-1}(A ; \mathbf{Z} / p) \rightarrow K_{n-1}(B ; \mathbf{Z} / p) \rightarrow \cdots
$$

Given a fibration there is always a (functorial) long exact $\bmod p$ homotopy sequence. The point of the theorem is that in the case of the map $B^{d} G L B^{+} \rightarrow B^{d} G L C^{+}$, the $\bmod p$ homotopy of the homotopy fibre is identified.

Sketch of the Proof of Theorem 6.4.1. It follows from the universal coefficient sequence and the corresponding fact in ordinary algebraic $K$-theory, that if $A$ is unital then the natural map $A \rightarrow M_{2} A$ induces an isomorphism in algebraic mod $p K$-theory. By the definition for non-unital $A$, the result is true for these algebras too, and so by the usual argument we see that the functor $A \mapsto K_{*}(\mathscr{H} \otimes A ; \mathbf{Z} / p)$ is stable. It follows from Theorem 6.4.2 that it is also split exact. Therefore it is homotopy invariant. The theorem now follows by reduction of dimensions to either $n=1$ or $n=2$, where we know the result to be true by virtue of the universal coefficient theorem and Theorems 2.4.6 or 4.2.7. (We note that in order to be able to reduce dimensions, the technical Theorem 2.6 .12 is needed to ensure that the transformation $\alpha$ between algebraic and topological theories commutes with the boundary maps $\partial$-compare the proof of Theorem 5.3.3.)

As Karoubi [31] observes, the following result can be proved from the above theorem. We denote by $H_{*}(; \mathbf{Z} / p)$ homology with coefficients in $\mathbf{Z} / p$.

Theorem 6.4.3 [31, corollaire 2.8]. If $A$ is any $C^{*}$-algebra then

$$
H_{*}\left(B^{d} G L \mathscr{K} \otimes A ; \mathbf{Z} / p\right) \cong H_{*}\left(B^{\prime} G L \mathscr{K} \otimes A ; \mathbf{Z} / p\right) .
$$

### 6.5. A Conjecture

We end with the following conjecture.
If $A$ is any $C^{*}$-algebra then for every $n=1,2, \ldots$ the natural map

$$
\alpha_{n}: \quad K_{n}(\mathscr{K} \otimes A) \rightarrow K_{n}^{\prime}(\mathscr{K} \otimes A)
$$

is an isomorphism.
It is natural enough to expect that this is true, given the results of this paper. We have shown in Sections II and IV that it is true for $n=1$ and $n=2$, and we can allow ourselves the case $n=0$ as well. The results of Section V assert that the conjecture is true if $K_{*}(\mathscr{K} \otimes A)$ is replaced by a suitable relative group. Altogether, this seems to be reasonably substantial evidence of its veracity.

As the reader will have guessed, the proof comes down to proving good excision results for the algebraic $K$-theory of stable $C^{*}$-algebras. This, however, is much easier said than done. The case $n-1$ is easy enough, as we saw in Section II. The case $n=2$ is a good deal more complicated, as we saw in Section IV. Further progress along the line we have been taking requires a reasonably concrete definition of the group $K_{3}$. While such a definition exists, the computations involved in establishing the results we need would appear to be many orders of magnitude more complicated than those of the $K_{2}$ case (supposing they are accessible at all). It is clear another approach is needed, or at least, the approach used so far must be made much more systematic.

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[^0]:    ${ }^{1}$ Note added in proof. In a revised version of [31] (which has appeared in J. Operator Theory 15 (1986), 109-162), Karoubi proves the main result of the section, Theorem 4.2.7, using techniques which are in several respects comparable to ours.

