AN APPROACH TO $\mathbb{Z}/k$-INDEX THEORY

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Introduction

Let Q be a smooth, compact, spin^c-manifold whose boundary consists of k isomorphic components (we fix suitable isomorphisms, and suppose that these preserve the metric and spin^c-structures on the boundary). In recent work [6], D. Freed has considered an index theory appropriate to such manifolds. In collaboration with R. Melrose, he has proved an index theorem along the lines of the Atiyah-Singer Index Theorem [4]. Freed analyzes the index of the boundary problem on Q obtained by imposing Atiyah-Patodi-Singer boundary conditions on the Dirac operator. Because of the boundary conditions, this index is sensitive to deformations in the geometry of Q, and so is not a topological invariant. However, Freed and Melrose show that the congruence class, mod k, of the index is given by a natural K-theoretic formula.

The essential problem in adapting the argument of Atiyah and Singer to the present situation lies in demonstrating various stability properties for the analytic index: whereas Atiyah and Singer appeal to the theory of pseudodifferential operators on a smooth, closed manifold, the corresponding theory for a manifold with boundary is much more complicated and subtle. The purpose of this article is to describe a new approach to $\mathbb{Z}/k$-index theory, in which this issue is more or less circumvented by realizing the analytic index as an element in a certain C^*-algebra K-theory group (we will make use of C^*-algebra K-theory in only a rather mild way, and in particular we will not need K-homology or KK-theory). It follows from the elementary rigidity properties of K-theory that our index has the desired stability properties (and also that it is equal to Freed's index).

Our proof will follow the general strategy of [4], although in several places the details will be different. Rather than introduce pseudodifferential operators, we shall work entirely with first order differential operators, using nothing more sophisticated than the basic elliptic estimates. (A discussion of the Atiyah-Singer Theorem from this point of view will be given in [10].)

We shall assume some familiarity with $\mathbb{Z}/2$-gradings, Clifford algebras, spin^c-structures, and so on. In Secs. 1 and 2 we shall review the construction of analytic indices in C^*-algebra K-theory and certain rudimentary properties of Dirac operators that we shall need.

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1. Analytic Indices and K-Theory

As mentioned in the introduction, we shall make only limited use of K-theory. Here we shall use nothing more than the most basic properties of K_0-groups (in the computations of the next section we shall use the long exact sequence in C^*-algebra K-theory, and as in [4], the proof of the Index Theorem will rest ultimately on the Bott Periodicity Theorem in Atiyah-Hirzebruch K-theory).

Let $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ be a $\mathbb{Z}/2$-graded Hilbert space. Denote by $\varepsilon : \mathscr{H} \to \mathscr{H}$ the grading operator, so that the operators

$$Q_+ = 1/2(1+\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$Q_- = 1/2(1-\varepsilon) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

are the orthogonal projections onto the subspaces $\mathscr{H}_+$ and $\mathscr{H}_-$ of $\mathscr{H}$. Let D be a self-adjoint operator on $\mathscr{H}$ of odd grading degree. To be precise, we are supposing that ε maps domain(D) into itself, and $\varepsilon D = -D\varepsilon$ on domain (D). Thus D is of the form

$$D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix}, \tag{1.1}$$

for some closed operator D_+ from $\mathscr{H}_+$ to $\mathscr{H}_-$.

Assume that a C^*-algebra $\mathscr{C}$ has been specified so that

$$(D \pm i)^{-1} \in \mathscr{C}. \tag{1.2}$$

Suppose that $\varepsilon\mathscr{C} \subset \mathscr{C}$ and, for the sake of simplicity, that neither of the projections Q_+ or Q_- is an element of $\mathscr{C}$. Denote by $\tilde{\mathscr{C}} \subset \mathscr{B}(\mathscr{H})$ the C^*-algebra generated by $\mathscr{C}$ and ε. From the inclusion $\varepsilon\mathscr{C} \subset \mathscr{C}$ it follows that $\mathscr{C}$ is an ideal in $\tilde{\mathscr{C}}$; the quotient $\tilde{\mathscr{C}}/\mathscr{C}$ is of course generated by the image of ε and is isomorphic to $C^*(\varepsilon)$ in this natural way. Thus the short exact sequence

$$0 \to \mathscr{C} \to \tilde{\mathscr{C}} \to C^*(\varepsilon) \to 0$$

splits, and gives rise to an exact sequence of K-theory groups

$$0 \to K_0(\mathscr{C}) \to K_0(\tilde{\mathscr{C}}) \to K_0\big(C^*(\varepsilon)\big) \to 0. \tag{1.3}$$

Now, let $f : \mathbb{R} \to [-1, 1]$ be any continuous function satisfying the conditions:

(a) f is odd (that is, $f(x) = -f(-x)$); and

(b) $\lim_{x \to +\infty} f(x) = 1;$ (1.4) and let $g = (1 - f^2)^{1/2}$. The effect of choosing f to be odd (and hence g even) is that the bounded operators $f(D)$ and $g(D)$ are of odd and even grading degree, respectively. Because of this, the operator

$$U = \varepsilon f(D) + g(D)$$

on $\mathscr{H}$ is unitary. Let us define a projection operator P by the formula

$$\begin{aligned} P &= UQ_+U^* \\ &= g^2(D)\varepsilon - f(D)g(D) + Q_- . \end{aligned} \tag{1.5}$$

From condition (1.2) and the Stone-Weierstrass Theorem it follows that $h(D) \in \mathscr{C}$ for every $h \in C_0(\mathbb{R})$ (the continuous functions from $\mathbb{R}$ to $\mathbb{C}$ which vanish at $\pm\infty$). Thus from (1.5) we see that the projections P and Q_- lie in $\tilde{\mathscr{C}}$, and that $P - Q_- \in \mathscr{C}$. The class $[P] - [Q_-] \in K_0(\tilde{\mathscr{C}})$ lies in the kernel of the map $K_0(\tilde{\mathscr{C}}) \to K_0(C^*(\varepsilon))$, and so $[P] - [Q_-] \in K_0(\mathscr{C})$ by (1.3).

1.1. Lemma. *The class* $[P] - [Q_-]$ *does not depend on the choice of the function* f *satisfying* (1.4).

Proof. We note that if P_1 and P_2 are projections in a C^*-algebra such that $\|P_1 - P_2\| < 1$, then P_1 and P_2 are unitarily equivalent (see [5, Proposition 4.6.6]), and so P_1 and P_2 determine the same K-theory class. We shall refer to this as the *homotopy invariance* of C^*-algebra K-theory, since it implies that all the elements in a norm continuous path of projections have the same K-theory class.

Given f_0 and f_1 satisfying (1.4), let $f_t = (1 - t)f_0 + tf_1$; each of the functions f_t satisfies (1.4). The path $t \to f_t$ is continuous in the supremum norm, and thus the path $t \to f_t(D)$ is norm continuous. Therefore the corresponding path $t \to P_t$ is norm continuous, and so $[P_0] = [P_1]$. QED

We define: $\mathrm{Index}_{\mathscr{C}}(D) = [P] - [Q_+] \in K_0(\mathscr{C})$. This generalizes the usual notion of Fredholm index, as follows.

1.2. Lemma. *Suppose* $\mathscr{C}$ *is* $\mathscr{K}(\mathscr{H})$, *the* C^*-*algebra of compact operators on* $\mathscr{H}$. *Then the operator* D_+ *in* (1.1) *is Fredholm, and if we identify* $K_0(\mathscr{K}(\mathscr{H}))$ *with* $\mathbb{Z}$ *in the natural way* (*via the trace on* $\mathscr{K}(\mathscr{H})$) *then* $\mathrm{Index}_{\mathscr{C}}(D) = \dim\ker D_+ - \dim\ker D_+^*$.

Proof. By assumption, D has compact resolvent, and therefore D, and hence D_+, is Fredholm. We may choose a function f as above which is $+1$ on the positive spectrum of D and -1 on the negative spectrum. Then $f(D)$ is of the form

$$f(D) = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix},$$

where V is a partial isometry whose kernel and cokernel are the same as those of D_+, and $g(D)$ is the projection onto the orthogonal sum of the kernels of D_+ and D_+^*, say

$$g(D) = \begin{pmatrix} p_0 & 0 \\ 0 & p_1 \end{pmatrix}.$$

According to the formula (1.5),

$$P = \begin{pmatrix} p_0 & 0 \\ 0 & 1 - p_1 \end{pmatrix},$$

and so

$$\begin{aligned}
[P] - [Q_1] &= \left[\begin{pmatrix} p_0 & 0 \\ 0 & 1 - p_1 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} p_0 & 0 \\ 0 & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 - p_1 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 - p_1 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & p_1 \end{pmatrix}\right] \\
&= [p_0] - [p_1].
\end{aligned}$$

QED

In Sec. 6 we shall need a mild extension of this index theory. Let us suppose that instead of (1.2) the operator D satisfies the weaker condition that for some $\delta > 0$, $h(D) \in \mathscr{C}$ for every $h \in C_0(\mathbb{R})$ with $\mathrm{supp}(h) \subset [-\delta, \delta]$. If we choose f as in (1.4) such that $f^2 = 1$ outside of the interval $[-\delta, \delta]$, then the projection P in (1.5) will be equal to Q_-, modulo $\mathscr{C}$, and we can still form the index class $[P] - [Q_-]$ in $K_0(\mathscr{C})$. We note that Lemmas 1.1 and 1.2 are still valid.

2. Dirac Operators

Let M be an oriented, even dimensional Riemannian manifold without boundary. Let S be a smooth, Hermitian vector bundle over M, equipped with a Clifford action of the tangent bundle TM and a compatible connection, in the sense of [12, Definition 2.3]. We shall refer to S as a *Dirac bundle* (the name *Clifford bundle* is used in [12], but this is in conflict with the terminology of [2]).

Let E be any Hermitian bundle over M, equipped with a connection compatible with the Hermitian structure. There is a natural Clifford action on the tensor product bundle $S \otimes E$, and this, together with the connection $\nabla_X(\xi \otimes \zeta) = (\nabla_X \xi) \otimes \zeta + \xi \otimes (\nabla_X \zeta)$ on $S \otimes E$ (see [8]), makes $S \otimes E$ into a Dirac bundle.

Every Dirac bundle has a natural $\mathbb{Z}/2$-grading, determined by the formula

$$\varepsilon(\xi) = i^{n/2} e_1 \cdot e_2 \cdot \ldots e_n \cdot \xi,$$

for the grading operator, where $\{e_1, \ldots, e_n\}$ is a local, oriented, orthonormal frame for TM and "$\cdot$" denotes Clifford multiplication. If, however, our Dirac bundle is a tensor product, as above, and if E is equipped with a non-trivial $\mathbb{Z}/2$-grading, then we shall equip $S \otimes E$ with the product grading, $\deg(\xi \otimes \zeta) = \deg(\xi) + \deg(\zeta)$, and use the standard notation $S \hat{\otimes} E$ to signify this. If G is an endomorphism of E of odd grading degree then we shall consider G as acting on $S \hat{\otimes} E$ in the graded fashion: $G(\xi \otimes \zeta) = (-1)^{\deg(\xi)}\xi \otimes G\zeta$.

Given any Dirac bundle S, we may form the associated Dirac operator $D_S = \Sigma_i e_i \cdot \nabla_i$ as in [12, Definition 2.4]. A calculation in local coordinates [12, Proposition 2.9] shows that D_S is a formally self-adjoint differential operator. It is of $\mathbb{Z}/2$-grading degree one with respect to either the usual grading or the tensor product grading (if the Dirac bundle is a graded tensor product). In the later sections of the paper we shall need to consider more general operators, of the form

$$D = \psi D_S \psi + G, \tag{2.1}$$

where G is a self-adjoint endomorphism of S and ψ is a smooth, bounded, real valued function on M (it is a sort of cut-off function, confining the action of D to manageable portions of M). These, too, are formally self-adjoint and of grading degree one.

Let us consider D as a Hilbert space operator, with domain initially the smooth, compactly supported sections of S. A standard argument involving Friedrichs' Mollifiers (compare [12, Proposition 3.20]) shows that if $\xi \in \text{domain}(D^*)$ (that is, $D\xi$, computed in the sense of distributions, is an L^2-section), and if ξ is compactly supported, then $\xi \in \text{domain}(\bar{D})$.

2.1. Lemma. (Compare [9, Theorem 1.17].) *Suppose that the Riemannian metric on M may be altered outside of a neighborhood of* $\text{supp}(\psi)$ *so as to make M complete. Then D is essentially self-adjoint.*

Proof. We may assume that M is complete. There exists a sequence of smooth functions $\phi_k : M \to [0,1]$ which converges uniformly to the constant 1 on compact subsets, such that $\sup_{x \in M} \|\text{grad}(\phi_k)\| \to 0$ as $k \to \infty$. Pointwise multiplication by ϕ_k leaves domain(D^*) and domain $(\bar{D})$ invariant. The commutator $[\phi_k, D]$ is the operator given by Clifford multiplication with $\psi^2\text{grad}(\phi_k)$, and so it is bounded: in fact $\|[\phi_k, D]\| \to 0$ as $k \to \infty$. Suppose that $\xi \in \text{domain}(D^*)$. Then $\phi_k\xi \in \text{domain}(D^*)$ for all k, and so by the above remarks, $\phi_k\xi \in \text{domain}(D^-)$. By taking inner products with any test function we see that

$$D\phi_k\xi = \phi_k D\xi + [D, \phi_k]\xi.$$

Taking limits as $k \to \infty$, we obtain $\phi_k\xi \to \xi$ and $D\phi_k\xi \to D\xi$, so that $\xi \in \text{domain}(\bar{D})$.

QED

All our operators (with the exception of the boundary value problems appearing in Theorem 3.5) will be essentially self-adjoint on the smooth, compactly supported sections. We shall work with the closures of these operators, although to keep our notation uncluttered we shall write D rather than $\bar{D}$. It is easy to check that the self-adjoint Hilbert space operator D is of $\mathbb{Z}/2$-grading degree one, in the sense described in the previous section.

It is useful to investigate the question of domains a little more closely. Let K be a compact subset of M and denote by $H_c^1(K)$ the completion of the space of smooth sections supported in K, with respect to the norm $\|\xi\|_1{}^2 = \|\xi\|^2 + \|\nabla\xi\|^2$ (the quantities on the right are L^2-norms; for convenience we shall omit mention of the bundle S in our notation). The natural map of $H_c^1(K)$ into $L^2(M)$ is an inclusion, and the Rellich Lemma (compare [12, Proposition 3.8]) asserts that this inclusion is compact. If the cut-off function ψ is bounded below on K then we have the "basic elliptic estimate" $\|\xi\|_1 \leq C(\|D\xi\| + \|\xi\|)$ (compare [12, 3.14]). It follows that if ϕ is a smooth function supported in K then

$$\phi \cdot \mathrm{domain}(D) \subset H_c^1(X). \tag{2.2}$$

If X and Y are bounded Hilbert space operators then we shall write $X \sim Y$ if $X - Y$ is compact. We shall need following calculations.

2.2. Lemma. *Let ϕ be a smooth, compactly supported function, or endomorphism of S, such that ϕ vanishes wherever ψ is zero. Then $\phi h(D) \sim 0$ for every $h \in C_0(\mathbb{R})$.*

Proof. There is a sequence $\{\phi_k\}$ of smooth, compactly supported functions converging uniformly to ϕ, such that ψ is bounded below on each $\mathrm{supp}(\phi_k)$, and since the space of compact operators is closed it suffices to prove the Lemma for each ϕ_k. Furthermore, we may assume that h is compactly supported. Then $\mathrm{range}(h(D)) \subset \mathrm{domain}(D)$, and so, by (2.2), $\mathrm{range}(\phi h(D)) \subset H_c^1(\mathrm{supp}(\phi))$. It follows easily from the Closed Graph Theorem that $\phi h(D)$ is continuous as a map from $L^2(M)$ to $H_c^1(\mathrm{supp}(\phi))$, and since the inclusion $H_c^1(\mathrm{supp}(\phi)) \hookrightarrow L^2(M)$ is compact, the composition is compact.

QED

2.3. Lemma. *Let $U_1 \subset M$ and $U_2 \subset M$ be open subsets and let $\gamma: U_1 \to U_2$ be an isometry, lifting to an isomorphism of Dirac bundles, such that $D(\xi \circ \gamma)(x) = (D\xi)(\gamma(x))$ for $x \in U_1$. Denote by $\Gamma: L^2(U_2) \to L^2(U_1)$ the Hillbert space partial isometry induced from γ, and let ϕ_1 be a smooth, bounded function supported within U_1. If* $\mathrm{supp}([D, \phi_1])$ *is compact then $h(D)\phi_1\Gamma \sim \phi_1\Gamma h(D)$ for every $h \in C_0(\mathbb{R})$.*

Proof. By the Stone-Weierstrass Theorem it suffices to consider $h(D) = (D \pm i)^{-1}$. The operator $\Gamma\phi_1$ maps $\mathrm{domain}(D)$ into itself, and so we may write

$$\begin{aligned}\phi_1\Gamma(D \pm i)^{-1} - (D \pm i)^{-1}\phi_1\Gamma &= (D \pm i)^{-1}\big((D \pm i)\phi_1\Gamma - \phi_1\Gamma(D \pm i)\big)(D \pm i)^{-1} \\ &= (D \pm i)^{-1}\big(\Gamma(D \pm i)\phi_2 - \Gamma\phi_2(D \pm i)\big)(D \pm i)^{-1} \\ &= (D \pm i)^{-1}\Gamma[D, \phi_2](D \pm i)^{-1},\end{aligned}$$

where $\phi_2 = \phi_1 \circ \gamma = \Gamma^*\phi_1\Gamma$. The compactness of this operator follows from Lemma 2.2 (note that the endomorphism $[D, \phi_2]$ vanishes wherever ψ is zero). QED

2.4. Lemma. *Let D_1 and D_2 be operators on M, and suppose that ϕ is a smooth, bounded function on M such that $D_1 = D_2$ near* $\mathrm{supp}(\phi)$. *For each fixed $h \in C_0(\mathbb{R})$ and* *every $\varepsilon > 0$ there exists $\delta > 0$, depending only on h, and not on ϕ or D, such that if $\|[D_1, \phi]\| < \delta$ then $\|\phi h(D_1) - \phi h(D_2)\| < \varepsilon$. If $[D_1, \phi]$ is compactly supported then $\phi h(D_1) \sim \phi h(D_2)$ for every $h \in C_0(\mathbb{R})$.*

Proof. It suffices to show that $h(D_1)\phi \sim \phi h(D_2)$ and find $\delta > 0$ so that if $\|[D_1, \phi]\| < \delta$ then $\|h(D_1)\phi - \phi h(D_2)\| < \varepsilon/2$ (to complete the proof of the lemma, apply these conclusions to the case where $D_1 = D_2$). We may assume that the endomorphism $[D_1, \phi]$ is bounded. Then ϕ maps the domains of D_1 and D_2 into themselves, and

$$\phi(D_2 \pm i)^{-1} - (D_1 \pm i)^{-1}\phi = (D_1 \pm i)^{-1}[D_1, \phi](D_2 \pm i)^{-1}.$$

Therefore $\|\phi(D_2 \pm i)^{-1} - (D_1 \pm i)^{-1}\phi\| \leq \|[D_1, \phi]\|$, and furthermore $\phi(D_2 \pm i)^{-1} - (D_1 \pm i)^{-1}\phi \sim 0$ by Lemma 2.2. This proves the lemma for the functions $h(x) = (x \pm i)^{-1}$. The general case follows from this and the Stone-Weierstrass Theorem (compare [1, Lemma to Theorem 2]). QED

2.5. Lemma. *Let W be an open subset of M such that if ξ is a smooth section compactly supported inside W then $\|D\xi\| \geq \|\xi\|$. Let ϕ be a smooth, bounded function supported inside W and let $h \in C_0(\mathbb{R})$ with $\operatorname{supp}(h) \subset [-1/2, 1/2]$. For every $\varepsilon > 0$ there exists $\delta > 0$ (depending on h but not on ϕ or D), such that if $\|[D, \phi]\| < \delta$ then $\|\phi h(D)\| < \varepsilon$. Furthermore, if $\phi\psi$ is compactly supported (with ψ as in (2.2)) then $\phi h(D) \sim 0$.*

Proof. We may assume that $[D, \phi]$ is bounded, so that ϕ maps the domain of D into itself; note that since D is essentially self-adjoint, $\|D\phi\xi\| \geq \|\phi\xi\|$ for every $\xi \in \operatorname{domain}(D)$. It suffices to prove the lemma for $1 \geq h \geq 0$ and $\|\phi\| \leq 1$. Choose N so that $2^{-N} < \varepsilon/5$, and let $g = h^{1/N}$. For every $\zeta \in L^2$ and every $n = 1, \ldots, N$ we have that $g^n(D)\zeta \in \operatorname{domain}(D)$ and

$$\begin{aligned} D\phi g^n(D)\zeta &= [D, \phi]g^n(D)\zeta + \phi D g^n(D)\zeta \\ &= [D, \phi]g^n(D)\zeta + [\phi, Dg(D)]g^{n-1}(D)\zeta + Dg(D)\phi g^{n-1}(D)\xi. \end{aligned} \tag{2.3}$$

Note that since $\operatorname{supp}(g) \subset [-1/2, 1/2]$, $\|Dg(D)\| \leq 1/2\|g(D)\| \leq 1/2$. Given $\varepsilon > 0$, choose $\varepsilon/5 > \delta > 0$ so small that if $\|[\phi, D]\| < \delta$ then $\|[\phi, Dg(D)]\| < \varepsilon/5$. Bearing in mind the inequality $\|\xi\| \leq \|D\xi\|$, it follows from (2.3) that if $\|[\phi, D]\| < \delta$ then

$$\|\phi g^n(D)\zeta\| \leq \|D\phi g^n(D)\zeta\| \leq \varepsilon/5\|\zeta\| + \varepsilon/5\|\zeta\| + 1/2\|\phi g^{n-1}(D)\zeta\|.$$

Iterating this inequality we obtain

$$\begin{aligned} \|\phi g^N(D)\zeta\| &\leq 2\varepsilon/5\|\zeta\| + \varepsilon/5\|\zeta\| + \cdots + 2^{-(N-2)}\varepsilon/5\|\zeta\| + 2^{-N}\|\phi\zeta\| \\ &< \varepsilon\|\zeta\|. \end{aligned}$$

Hence $\|\phi h(D)\| < \varepsilon$ as required. For the second part of the lemma, recall that an operator K is compact iff $\lim_{j\to\infty} \|K\zeta_j\| = 0$ for every orthonormal sequence $\{\zeta_j\}$. By Lemmas 2.2 and 2.3, the operators $[D, \phi]g^n(D)$ and $[\phi, Dg(D)]g^{n-1}(D)$ appearing in Eq. (2.3) are compact. Therefore $\limsup_{j\to\infty} \|\phi g^n(D)\zeta_j\| \leq 1/2 \limsup_{j\to\infty} \|\phi g^{n-1}(D)\zeta_j\|$, from which it follows that $\limsup_{j\to\infty} \|\phi h(D)\zeta_j\| \leq 2^{-N}$. QED

3. The Analytic Index for $\mathbb{Z}/k$-Manifolds

We shall develop an index theory associated with the following geometric objects (compare [6, Definition 1.7]).

3.1. Definition. (i) Let Q be an oriented, smooth manifold whose boundary is decomposed into k disjoint manifolds $(\partial Q)_1, \ldots, (\partial Q)_k$ (the $(\partial Q)_i$ are not necessarily connected, and the choice of labelling of $(\partial Q)_1, \ldots, (\partial Q)_k$ is unimportant). We shall say that Q *admits a $\mathbb{Z}/k$-structure* if there exists an oriented manifold P and orientation preserving diffeomorphisms $\gamma_i : V_i \to (0, 1] \times P$, where the V_i are disjoint collaring neighborhoods of the $(\partial Q)_i$ $(i = 1, \ldots, k)$. *A $\mathbb{Z}/k$-structure* on Q consists of a particular choice of such data. By a *Riemannian metric* on Q we shall mean a choice of Riemannian metrics on Q and P such that the γ_i are isometries (we put the product metric on $(0, 1] \times P$).

(ii) A *$\mathbb{Z}/k$-bundle E* over Q is a vector bundle, together with a vector bundle F over P, and liftings of the isomorphisms $\gamma_i : V_i \to (0, 1] \times P$ to isomorphisms $E|_{V_i} \cong \pi^*(F)$ (where $\pi^*(F)$ is the pull-back to $(0, 1] \times P$). Any additional structure on E (for example, a metric), will be assumed to be compatible with these isomorphisms.

(iii) A *spin^c-structure on* an oriented, even dimensional, orthogonal $\mathbb{Z}/k$-bundle V is a Hermitian $\mathbb{Z}/k$-bundle S, equipped with a Clifford action of V (as skew-adjoint endomorphisms) which is fibre-wise irreducible (see [2]). On the open sets V_i we assume that the Clifford action is pulled back from an action over P. A spin^c-structure on an even dimensional, Riemannian $\mathbb{Z}/k$-manifold is a spin^c-structure on the tangent bundle.

This definition has the following counterpart in operator theory.

3.2. Definition. (i) A *$\mathbb{Z}/k$-structure* on Hilbert space $\mathscr{H}$ consists of a separable, infinite dimensional Hilbert space $\mathscr{L}$, together with k isometries $e_i : \mathscr{L} \to \mathscr{H}$ $(i = 1, \ldots, k)$ whose ranges are pairwise orthogonal. We shall denote by e_{ij} the partial isometries $e_i e_j^*$ $(i, j = 1, \ldots, k)$, and by p_i the projections $e_i e_i^* = e_{ii}$ $(i = 1, \ldots, k)$. Let p_0 be the complementary projection $1 - p_1 - \cdots - p_k$.

(ii) *If $\mathscr{H}$* is a $\mathbb{Z}/k$-Hilbert space then denote by $\mathscr{D}_k(\mathscr{H})$ the C*-algebra

$$\mathscr{D}_k(\mathscr{H}) = \{X \in \mathscr{B}(\mathscr{H}) \mid [X, e_{ij}] \sim 0 \text{ and } Xp_0 \sim 0\}.$$

Let Q be an even-dimensional, compact, spin^c-$\mathbb{Z}/k$-manifold, and let E be a hermitian $\mathbb{Z}/k$-bundle over Q. Equip E and the spinor bundle S with connections which are compatible with the metrics, as well as with the $\mathbb{Z}/k$-structures, in the sense that over the open sets V_i the connections are pulled back from connections defined over P. Furthermore, choose the connection for S so that it is compatible with Clifford multiplication and the Riemannian connection on the tangent bundle.

Form a complete manifold M from Q by attaching cyclinders $[1, \infty) \times P$ at the boundaries $(\partial Q)_i$, using the isometries $\gamma_i \cdot (i = 1, \ldots, k)$, and extend E and the Dirac bundle S to M by using the product structure near ∂Q. Denote by D_E the Dirac operator on M with coefficients in E (that is, the Dirac operator for $S \otimes E$).

Denote by $L^2(M)$ the Hilbert space of L^2-sections of $S \otimes E$. This has a natural $\mathbb{Z}/k$-structure (take $\mathscr{L} = L^2([1, \infty) \times P)$ and let $e_1, \ldots, e_k$ be induced by the k inclusions of $[1, \infty) \times P$ into M).

3.3. Proposition. $(D_E \pm i)^{-1} \in \mathscr{D}_k(L^2(M))$.

Proof. Denote by U_i $(i = 1, \ldots, k)$ the open subsets of M formed from the union of the collar neighborhood V_i of Definition 3.1 and the attached cylinder $[1, \infty) \times P$. Denote by $\gamma_{ij} : U_i \to U_j$ the isometries extending $\gamma_j^{-1} \circ \gamma_i : V_i \to V_j$ and denote by $\Gamma_{ij} : L^2(U_j) \to L^2(U_i)$ the partial isometries induced from the γ_{ij}. Let ϕ_1 be a smooth function supported in U_1 such that $\phi \equiv 1$ on $[1, \infty) \times P$, and let $\phi_i = \phi_1 \circ \gamma_{i1} = \Gamma_{i1}\gamma_1\Gamma_{1i}$ $(i = 1, \ldots, k)$. The operators e_{ij} of Definition 3.2 are given by the formula $e_{ij} = \Gamma_{ij}p_j$, where p_j is (multiplication by) the characteristic function of the j'th cylinder $[1, \infty) \times P$. Now,

$$[e_{ij}, (D_E \pm i)^{-1}] = [\Gamma_{ij}\phi_j, (D_E \pm i)^{-1}] + \Gamma_{ij}(p_j - \phi_j)(D_E \pm i)^{-1} + (D_E \pm i)^{-1}(\phi_i - p_i)\Gamma_{ij},$$

and the first term is compact by Lemma 2.3, whilst the second and third terms are compact by Lemma 2.2, since $p_j - \phi_j$ is compactly supported. Finally, $p_0(D_E \pm i)^{-1} \sim 0$ by another application of Lemma 2.2. QED

3.4. Proposition. *If $\mathscr{H}$ is any $\mathbb{Z}/k$-Hilbert space then* $K_0(\mathscr{D}_k(\mathscr{H})) \cong K_0(\mathscr{K}(\mathscr{H}))/kK_0(\mathscr{K}(\mathscr{H}))$.

Proof. Note that $\mathscr{K}(\mathscr{H})$ is an ideal in $\mathscr{D}_k(\mathscr{H})$. Define a $*$-homomorphism $\alpha : \mathscr{B}(\mathscr{L}) \to \mathscr{D}_k(\mathscr{H})$ by $\alpha(X) = \Sigma e_j X e_j^*$. This maps $\mathscr{K}(\mathscr{L})$ into $\mathscr{K}(\mathscr{H})$, and the induced $*$-homomorphism on quotients is an isomorphism. From the diagram of extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{K}(\mathscr{H}) \arrow[r] & \mathscr{D}_k(\mathscr{H}) \arrow[r] & \mathscr{D}_k(\mathscr{H})/\mathscr{K}(\mathscr{H}) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{K}(\mathscr{L}) \arrow[r] \arrow[u, "\alpha"] & \mathscr{B}(\mathscr{L}) \arrow[r] \arrow[u, "\alpha"] & \mathscr{B}(\mathscr{L})/\mathscr{K}(\mathscr{L}) \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}$$

we obtain a diagram of exact sequences in K-theory, a piece of which is

$$\begin{tikzcd}
K_1(\mathscr{D}_k/\mathscr{K}) \arrow[r] & K_0(\mathscr{K}) \arrow[r] & K_0(\mathscr{D}_k) \arrow[r] & K_0(\mathscr{D}_k/\mathscr{K}) \\
K_1(\mathscr{B}/\mathscr{K}) \arrow[r, "\cong"'] \arrow[u, "\cong"'] & K_0(\mathscr{K}) \arrow[u, "\alpha^*"] & & K_0(\mathscr{B}/\mathscr{K}) = 0 \arrow[u, "\cong"']
\end{tikzcd}$$

The image of $\alpha^* : K_0(\mathscr{K}(\mathscr{L})) \to K_0(\mathscr{K}(\mathscr{H}))$ is easily seen to be $kK_0(\mathscr{K}(\mathscr{H}))$, and so $K_0(\mathscr{D}_k) \cong K_0(\mathscr{K}(\mathscr{H}))/kK_0(\mathscr{K}(\mathscr{H}))$. QED

Identify $K_0(\mathcal{K}(\mathcal{H}))$ with $\mathbb{Z}$ in the usual way and define:

$$\mathrm{Index}_k(D_E) = \mathrm{Index}_{\mathcal{D}_k(L^2(M))}(D_E) \in K_0(\mathcal{D}_k) \cong \mathbb{Z}/k.$$

The following theorem identifies our analytic index with the one considered by Freed in [6], and so gives $\mathrm{Index}_k(D_E)$ a more concrete form. We shall not however use this result in the remainder of the paper.

3.5. Theorem. *The quantity* $\mathrm{Index}_k(D_E)$ *is equal to the congruence class,* mod k, *of the index of the Atiyah-Patodi-Singer boundary value problem for D_E on Q (as described in* [3]).

In order to prove this we shall need the following computations concerning the K-theory of $\mathcal{D}_k$.

3.6. Lemma. (i) *If P is a projection in $\mathcal{D}_k$ such that the operators $[P, e_{ij}]$ and Pp_0 are zero (and not merely compact) then $[P] = 0$ in $K_0(\mathcal{D}_k)$.*

(ii) *If $V: \mathcal{H} \to \mathcal{H}'$ is an isometry of $\mathbb{Z}/k$-Hilbert spaces (meaning that there is an associated isometry $W: \mathcal{L} \to \mathcal{L}'$, and $e_i W = Ve_i$ for $i = 1, \ldots, k$), then the *-homomorphism* $\mathrm{Ad}(V)$ *maps $\mathcal{D}_k(\mathcal{H})$ into $\mathcal{D}_k(\mathcal{H}')$, and the induced map on K-theory corresponds to the identity map on $\mathbb{Z}/k$. Furthermore, if V is in addition an isometry of $\mathbb{Z}/2$-graded spaces, and if D and D' are odd degree, self-adjoint operators such that $Vh(D)V^* = h(D')$ for every $h \in C_0(\mathbb{R})$ with say* $\mathrm{supp}(h) \subset [-1/2, 1/2]$, *then* $\mathrm{Index}_k(D) = \mathrm{Index}_k(D')$ *(assuming that the first, and hence both, of these indices is defined).*

(iii) *Let $\tilde{\mathcal{D}}_k \subset \mathcal{B}(\mathcal{H})$ be obtained from $\mathcal{D}_k$ by adjoining a grading operator, as in Sec. 1, which commutes with the e_{ij}, and let Q_- denote the projection onto the odd-graded subspace of $\mathcal{H}$. Let P_t $(t \in [0, 1])$ be a strongly continuous path of projections in $\tilde{\mathcal{D}}_k$ such that $P_t - Q_- \in \mathcal{D}_k$ for all $t \in [0, 1]$, and such that the paths $[e_{ij}, P_t]$ and $p_0 P_t$ are norm continuous. Then $[P_0] - [Q_-] = [P_1] - [Q_-]$ in $K_0(\mathcal{D}_k)$.*

Proof. Part (i) of the lemma follows from the fact that P is the image of a projection in $\mathcal{B}(\mathcal{L})$, under the *-homomorphism $\alpha: \mathcal{B}(\mathcal{L}) \to \mathcal{D}_k(\mathcal{H})$ defined in the proof of Proposition 3.4, and the fact that $K_0(\mathcal{B}(\mathcal{L})) = 0$.

An easy computation shows that $V\mathcal{D}_k(\mathcal{H})V^* \subset \mathcal{D}_k(\mathcal{H}')$. Since $\mathrm{Ad}(V)$ preserves the rank of projections, and since the isomorphism of $K_0(\mathcal{D}_k)$ with $\mathbb{Z}/k$ is given by the rank, mod k, of finite projections, it follows that $\mathrm{Ad}(V)_*$ corresponds to the identity map on $\mathbb{Z}/k$. The last assertion in part (ii) follows from the formula (1.6) for the projection determining the index.

To prove (iii) we shall show that the K-theory class $[P_0] - [P_1] \in K_0(\mathcal{D}_k)$ is zero. Denote by $\mathcal{D}_k[0, 1)$ the C^*-algebra of those bounded, *-strongly continuous functions X_t from $[0, 1]$ to $\mathcal{D}_k$ which vanish at 1, for which the functions $[X_t, e_{ij}]$ and $p_0 X_t$ are norm-continuous. The formal difference of P_t and the constant projection P_1 determines an element of the group $K_0(\mathcal{D}_k[0, 1))$, which maps to the class $[P_0] - [P_1]$ via the map $K_0(\mathcal{D}_k[0, 1)) \to K_0(\mathcal{D}_k)$ induced from evaluation at 0. Using the *-homomorphism α we see that there is an exact sequence of C^*-algebras

$$0 \to \mathcal{K}_{\mathrm{norm}}[0, 1) \to \mathcal{D}_k[0, 1) \to \mathcal{B}[0, 1)/\mathcal{K}_{\mathrm{norm}}[0, 1) \to 0,$$

where $\mathscr{B}[0,1)$ denotes the bounded, *-strongly continuous maps from $[0,1]$ to $\mathscr{B}(\mathscr{L})$ which vanish at 1, and $\mathscr{K}_{\text{norm}}[0,1)$ denotes the norm continuous, compact operator valued functions which vanish at 1. By homotopy invariance, $K_*(\mathscr{K}_{\text{norm}}[0,1)) = 0$. As for $\mathscr{B}[0,1)$, this is an ideal in the C^*-algebra $\mathscr{B}[0,1]$ of all bounded, *-strongly continuous maps from $[0,1]$ to $\mathscr{B}(\mathscr{L})$, with quotient $\mathscr{B}$. A standard argument (see [5, 12.2.1]) shows that $K_*(\mathscr{B}[0,1]) = K_*(\mathscr{B}) = 0$, and so $K_*(\mathscr{B}[0,1)) = 0$ by the long exact sequence in K-theory. It follows from another application of the long exact sequence that $K_*(\mathscr{D}_k[0,1)) = 0$. QED

Proof of Theorem 3.5. Denote by Q' the manifold obtained from Q by deleting collars $(1/2, 1] \times P$ at the boundary manifolds $(\partial Q)_1, \dots, (\partial Q)_k$. The complete manifold M associated with Q is obtained from Q' by attaching cylinders $[1/2, \infty) \times P$, and considering M as decomposed in this new way we obtain a new $\mathbb{Z}/k$-structure on $L^2(M)$. It follows from part (i) of Lemma 3.6 (applied to the identity map on $L^2(M)$, considered as an isometry from the old to the new $\mathbb{Z}/k$-structure) that we may compute the index of D_E using the new $\mathbb{Z}/k$-structure, and so for the rest of the proof we shall work with this.

Denote by D the self-adjoint, grading degree one, Hilbert space operator on $L^2(Q)$ associated with the Atiyah-Patodi-Singer boundary value problem for D_E on Q. Thus

$$D = \begin{pmatrix} 0 & \mathscr{D}^* \\ \mathscr{D} & 0 \end{pmatrix},$$

where $\mathscr{D}$ is the operator descirbed in [3, Sec. 3]. The domain of D is a subspace of the Sobolev space $H^1(Q)$ (consisting of sections whose L^2-restrictions to ∂Q satisfy the pseudodifferential equation giving the boundary conditions), and so by the Rellich Lemma D has compact resolvent.

Choose smooth compactly supported, real valued functions ϕ_0 and ψ on M, which are invariant under interchanging the cylinders $(0, \infty) \times P$ in M, such that: (i) $\phi_0 = 1$ on Q'; (ii) $\psi = 1$ on $\operatorname{supp}(\phi_0)$; and (iii) $\psi = 0$ on each cylinder $[3/4, \infty) \times P$ (in particular, $\psi = 0$ in a neighborhood of ∂Q). For $1 \geq t \geq 0$ let

$$D_t = (\psi + t(1-\psi))D(\psi + t(1-\psi)). \tag{3.1}$$

Each of the operators D_t is self-adjoint and each of the resolvents $R_t = (D_t \pm i)^{-1}$, considered as an operator on $L^2(M)$ via the natural inclusion of $L^2(Q)$ into $L^2(M)$, is an element of $\mathscr{D}_k$. For $t > 0$ this is clear since $\operatorname{domain}(D_t) = \operatorname{domain}(D_1) \subset H^1(Q)$, and so R_t is compact; for $t = 0$ the assertion follows from the argument of Proposition 3.3. Thus we may form $\operatorname{Index}_k(D_t)$, and the main part of our proof will be to show that this index is independent of t. We shall accomplish this by showing that the projections P_t obtained from D_t, as in (1.5), satisfy the hypotheses of Lemma 3.6, part (iii). By the Stone-Weierstrass Theorem it suffices to show that the resolvents R_t satisfy the continuity requirements of the lemma, and to begin with, it is easily seen that R_t is a strongly continuous path. Let $\phi_1, \dots, \phi_k$ be smooth functions on M such that $\phi_j = 1$ on the jth cylinder $[1/2, \infty) \times P$ and $\phi_j = 0$ on the other cylinders, and let Γ_{ij} be the Hilbert space partial isometries given interchanging the cylinders $(0, \infty) \times P$ in M. Then $p_0 = p_0\phi_0$ and

$$[e_{ij}, R_t] = [\phi_i\Gamma_{ij}, R_t] + (p_i - \phi_i)\Gamma_{ij}\phi_0 R_t + R_t\phi_0(p_i - \phi_i)\Gamma_{ij},$$

from which we see that it suffices to show that the paths $R_t\phi_0$ and $[\phi_i\Gamma_{ij}, R_t]$ are norm continuous. Noting that $\phi_0 D_t = \phi_0 D_1$ for all t, we see that $R_t\phi_0 - \phi_0 R_1 = R_t[D_1, \phi_0]R_1$. The operator $[D_1, \phi_0]R_1$ is compact by Lemma 2.2, so that $R_t\phi_0 - \phi_0 R_1$ is the product of a bounded, strongly continuous path and a fixed compact operator, and is therefore norm continuous. Next, note that $[\phi_i\Gamma_{ij}, R_t] = R_t\Gamma_{ij}[D_t, \phi_j]R_t$ (see the proof of Lemma 2.3), and $[D_t, \phi_j] = \phi_0[D_1, \phi_j]\phi_0$ (since $[D, \phi_j]$ is Clifford multiplication by grad(ϕ_j)). Hence the continuity of $[\phi_i\Gamma_{ij}, R_t]$ follows from that of $R_t\phi_0$.

It follows then that $\text{Index}_k(D_0) = \text{Index}_k(D_1)$. It is clear from Lemma 1.2, and the manner in which we identify $K_0(\mathcal{D}_k)$ with $\mathbb{Z}/k$, that $\text{Index}_k(D_1)$ is the congruence class, mod k, of the Atiyah-Patodi-Singer index. As for D_0, replacing D by D_E in the formula (3.1) we obtain a homotopy from D_0 to D_E, and so a repetition of the above argument shows that $\text{Index}_k(D_0) = \text{Index}_k(D_E)$. QED

We could replace the Atiyah-Patodi-Singer problem with any Fredholm boundary value problem for the Dirac operator (the domain of the corresponding self-adjoint Hilbert space operator should be invariant under multiplication by smooth functions which are locally constant near ∂Q). An analysis of the $\mathbb{Z}/k$-index from the point of view of boundary value problems is given in [11].

The techniques of this section provide for the construction of $\text{Index}_k(D_E)$ in a variety of more general circumstances. The simplest instance of this is that it is not necessary to assume Q is of product form near ∂Q; it would, for example, be sufficient to assume that there are isometric neighborhoods of the boundary pieces $(\partial Q)_1, \ldots, (\partial Q)_k$. (In fact our index is independent of the choice of metric on Q—this follows from the basic elliptic estimates; it will of course also follow from the index theorem.) Furthermore, instead of attaching cylinders we could attach any other ends to Q (as long as we attach isometric ends to the k parts of the boundary). The argument of the preceding theorem shows that Index (D_E) will be the same for all constructions.

4. The Topological $\mathbb{Z}/k$-Index

The construction of a topological index for a $\mathbb{Z}/k$-bundle E over a compact, even dimensional, spin^c-$\mathbb{Z}/k$-manifold Q is described by Freed in [6]. Since our definitions only differ from those of [6] in some minor respects, we shall be brief.

4.1. Definition. (i) We construct inside each even dimensional Euclidean space $\mathbb{R}^{2d}$ a $\mathbb{Z}/k$-manifold $\mathcal{I}_k$ by adjoining to the open half space $H = \{\mathbf{x} \in \mathbb{R}^{2d} : x_1 < 0\}$ k disjoint, relatively open, unit radius disks in the hyperplane $H_0 = \{\mathbf{x} \in \mathbb{R}^{2d} : x_1 = 0\}$ (we obtain collaring neighborhoods and diffeomorphisms $U_i \cong (0, 1] \times D$, where D is the unit disk in H_0, by translations in R^{2d}).

(ii) Denote by $\hat{\mathcal{I}}_k$ the locally compact space obtained by identifying the k disks in $\mathcal{I}_k$.

A computation very similar to the proof of Proposition 3.4 shows that:

4.2. Proposition. *The inclusion $H \hookrightarrow \hat{\mathscr{S}}_k$ induces an isomorphism $K^0(\hat{\mathscr{S}}_k) \cong K^0(H)/kK^0(H)$.*

The Bott Periodicity Theorem asserts that $K^0(H) \cong \mathbb{Z}$, and so a choice of generator for $K^0(H) \cong \mathbb{Z}$ determines an isomorphism $K^0(\hat{\mathscr{S}}_k) \cong \mathbb{Z}/k$. We shall choose the standard Bott generator for $K^0(H)$ (to be described in a moment).

We shall construct our topological index in $K^0(\hat{\mathscr{S}}_k)$, and for computations it will be useful to have a concrete description of this group in terms of cycles and relations among cycles.

By a *$\mathbb{Z}/k$-submanifold* of a $\mathbb{Z}/k$-manifold such as $\mathscr{S}_k$ we shall mean a submanifold in the ordinary sense such that the $\mathbb{Z}/k$-structure of $\mathscr{S}_k$ restricts to a $\mathbb{Z}/k$-structure for the submanifold.

4.3. Definition. A cycle for $K^0(\hat{\mathscr{S}}_k)$ is a triple $(\mathscr{U}, S, G)$, where:

(i) $\mathscr{U}$ is an open $\mathbb{Z}/k$-submanifold of $\mathscr{S}_k$;

(ii) S is a $\mathbb{Z}/2$-graded $\mathbb{Z}/k$-bundle over $\mathscr{U}$; and

(iii) G is a self-adjoint, grading degree 1, endomorphism of S such that $G^2 \geq 2$ outside of a compact subset of $\mathscr{U}$ (the inequality refers to comparison of positive operators).

Denote by $\hat{\mathscr{U}}$ the locally compact space obtained by identifying the k boundary pieces of $\mathscr{U}$. Because of all the identifications built into the definition, each cycle $(\mathscr{U}, S, G)$ determines a vector bundle $\hat{S}$ on $\hat{\mathscr{U}}$, together with an endomorphism $\hat{G}$ which is bounded below outside of a compact subset of $\hat{\mathscr{U}}$. Such an object determines an element of the group $K^0(\hat{\mathscr{U}})$ (see for example [4, Sec. 2]), and the inclusion $\hat{\mathscr{U}} \hookrightarrow \mathscr{S}_k$ induces a natural map $K^0(\hat{\mathscr{U}}) \to K^0(\hat{\mathscr{S}}_k)$. Therefore every cycle of the type described above does indeed determine an element of $K^0(\hat{\mathscr{S}}_k)$.

4.4. Definition. Put on the set of all cycles the equivalence relation generated by the following relations:

(i) *Isomorphism.* (The obvious notion for cycles defined over the same base $\mathscr{U}$.)

(ii) *Addition of trivial cycles.* A cycle is *trivial* if $G^2 \geq 2$ on all of $\mathscr{U}$. We deem that the direct sum of a trivial cycle over $\mathscr{U}$ with any other cycle $(\mathscr{U}, S, G)$ is equivalent to $(\mathscr{U}, S, G)$.

(iii) *Restriction.* Let $\mathscr{U}'$ be an open $\mathbb{Z}/k$-submanifold of $\mathscr{U}$. The cycle $(\mathscr{U}, S, G)$ is deemed to be equivalent to $(\mathscr{U}', S|_{\mathscr{U}'}, G|_{\mathscr{U}'})$ (assuming the latter is a cycle).

(iv) *Homotopy.* If $\{(\mathscr{U}, S, G_t)\}_{t \in [0,1]}$ are cycles, and if the family of endomorphisms $\{G_t\}_{t \in [0,1]}$ is uniformly continuous on compact sets, then $(\mathscr{U}, S, G_0)$ is equivalent to $(\mathscr{U}, S, G_1)$.

It is a simple matter to check that this is the correct relation (the task is left to the reader):

4.5. Lemma. *The quotient of the set of cycles by this equivalence relation is $K^0(\hat{\mathscr{S}}_k)$.*

Suppose now that Q is a compact, even dimensional, spin^c-$\mathbb{Z}/k$-manifold, and let E be a Hermitian $\mathbb{Z}/k$-bundle over Q. Since any compact manifold with boundary can be embedded in some half-space, it is easy to see that Q can be smoothly embedded into some $\mathscr{I}_k$ as a $\mathbb{Z}/k$-submanifold. Choose such an embedding and let νQ denote the normal bundle, which is a $\mathbb{Z}/k$-bundle over Q. Equip νQ with an orthogonal structure and the orientation induced from the orientations on Q and $\mathscr{I}_k$. The spin^c-structure on the tangent bundle TQ determines a spin^c-structure on νQ: we choose it so that the combined spin^c-structure on the direct sum $TQ \oplus \nu Q \cong Q \times \mathbb{R}^{2d}$ is the standard one for a trivial bundle. Denote by S the spinor bundle for this spin^c-structure.

Now, let N be the total space of the bundle νQ. Denote by S^* the complex conjugate of the spinor bundle for νQ and denote by $\pi^*(S^*)$ the pull back of this bundle to N. This is a $\mathbb{Z}/2$-graded, $\mathbb{Z}/k$-bundle over the $\mathbb{Z}/k$-manifold N. Define a self-adjoint, grading degree one endomorphism J of $\pi^*(S^*)$ by the formula $J(v) = \varepsilon v$, where v acts by Clifford multiplication and ε is the grading operator. In the usual way we embed into $\mathscr{I}_k$ an open neighborhood $\mathscr{N} \subset N$ of the zero set $Q \subset N$; let us scale the metric on νQ, if necessary, so that $\mathscr{N}$ consists of all vectors of length less than 3. Then by restricting to $\mathscr{N}$ we obtain a cycle $(\mathscr{N}, \pi^*(S^*), J)$ for $K^0(\hat{\mathscr{I}}_k)$.

This construction may be applied to the embedding of a single point (which is a degenerate $\mathbb{Z}/k$-manifold) into the interior of $\mathscr{I}_k$. The cycle obtained in this way is in fact a cycle for $K^0(H)$, and represents the Bott generator for that group. As indicated above, we use this generator to make the identification $K^0(\hat{\mathscr{I}}_k) \cong \mathbb{Z}/k$.

4.6. Definition. Let E be a $\mathbb{Z}/k$-bundle over Q. We define the *topological $\mathbb{Z}/k$-index* of E to be the class $\tau_k(E) \in K^0(\hat{\mathscr{I}}_k) \cong \mathbb{Z}/k$ of the cycle $(\mathscr{N}, \pi^*(S^* \otimes E), J)$.

We can now state the main theorem of the paper.

4.7. Theorem. *Let Q be an even-dimensional, spin^c-$\mathbb{Z}/k$-manifold, and let E be a $\mathbb{Z}/k$-bundle over Q. If D_E denotes the Dirac operator on Q with coefficients in E then* $\mathrm{Index}_k(D_E) = \tau_k(E)$.

Our proof of this index theorem will follow the argument of Atiyah and Singer in [4]. First, we shall construct a suitable operator D_N on N, and by analytic methods we shall show that $\mathrm{Index}_k(D_N) = \mathrm{Index}_k(D_E)$. To identify $\mathrm{Index}_k(D_N)$ with the topological index $\tau_k(E)$ we shall construct a homomorphism $\mathrm{Index}_k : K^0(\hat{\mathscr{I}}_k) \to \mathbb{Z}/k$, such that the K-theory class representing $\tau_k(E)$ is mapped to $\mathrm{Index}_k(D_N)$. Since by Bott Periodicity the group $K^0(\hat{\mathscr{I}}_k)$ is cyclic, we need then only check on any generator that the homomorphism Index_k is equal to the natural identification of $K^0(\hat{\mathscr{I}}_k)$ with $\mathbb{Z}/k$.

5. The Operator D_N

Let M be a complete, oriented manifold and let S be a Dirac bundle over M. Let V be a smooth, oriented, orthogonal vector bundle over M and denote by N the total space of V. We shall construct an operator D_N on N whose index theory is a model for the index theory of the Dirac operator on M (the construction is very close to that described in [4, Sec. 9]). At the end of the section we shall specialize to the situation relevant to the $\mathbb{Z}/k$-index theorem.

Denote by $\pi : N \to M$ the projection mapping, and choose a splitting

$$TN \cong \pi^*(TM) \oplus \pi^*(V) \tag{5.1}$$

of the tangent bundle into "horizontal" and "vertical" bundles. The isomorphism (5.1), together with the metrics on TM and V, determines a complete Riemannian metric on N.

Denote by $\Lambda^* V_{\mathbb{C}}$ the complexified exterior algebra bundle of V, equipped with the hermitian form induced from the inner product on V. Form the graded tensor product $S \hat{\otimes} \Lambda^* V_{\mathbb{C}}$ and let

$$S_N = \pi^*(S \hat{\otimes} \Lambda^* V_{\mathbb{C}})$$

(equipped with the hermitian form pulled back from S and $\Lambda^* V_{\mathbb{C}}$).

Let $\{\phi_\alpha^2\}$ be a smooth partition of unity for M which is subordinate to a locally finite cover of M by contractible open sets $\{U_\alpha\}$. For each α, choose an oriented, orthonormal frame $\{v_j\}_{j=1}^n$ for $V|_{U_\alpha}$, and define diffeomorphisms from $\pi^{-1}[U_\alpha]$ to $U_\alpha \times \mathbb{R}^n$ by

$$v = \Sigma_j y^j v_j \to (\pi(v), y^1, \ldots, y^n). \tag{5.2}$$

These diffeomorphisms lift to bundle isomorphisms from $S_N|_{\pi^{-1}[U_\alpha]}$ to the tensor products of $S|_{U_\alpha}$ with the trivial bundle over $\mathbb{R}^n$ with fiber $\Lambda^* \mathbb{C}^n$. Using them, we pull back the first order operators $D \otimes 1$ to $S_N|_{\pi^{-1}[U_\alpha]}$, and define a first order operator D_h ("h" for "horizontal") acting on S_N by

$$D_h = \Sigma_\alpha \bar{\phi}_\alpha (D \otimes 1) \bar{\phi}_\alpha.$$

Here $\bar{\phi}_\alpha$ denotes the pull-back of the function ϕ_α to N; we note that the family $\{\bar{\phi}_\alpha^2\}$ is a partition of unity for N.

If v is a section of V then denote by $d_v : \Lambda^* V_{\mathbb{C}} \to \Lambda^* V_{\mathbb{C}}$ the operator of exterior multiplication by v, and denote by δ_v its adjoint (interior multiplication). The map $v \to d_v - \delta_v$ is a Clifford action of V on $\Lambda^* V_{\mathbb{C}}$ (which bundle we regard as $\mathbb{Z}/2$-graded in the usual way). We pull this back to a Clifford action of $\pi^* V$ on $\pi^* \Lambda^* V_{\mathbb{C}}$, and (in the manner described in Sec. 2) pass to an action on $S_N \cong \pi^*(S) \hat{\otimes} \pi^*(\Lambda^* V_{\mathbb{C}})$. Define an operator D_v ("v" for "vertical") by

$$D_v = \Sigma_j (d_{v_j} - \delta_{v_j}) \partial / \partial y^j.$$

It is easily verified that the expression for D_v does not depend on the choice of frame $\{v_1, \ldots, v_n\}$ and so we obtain a globally well defined first order differential operator D_v acting on S_N.

Denote by W the endomorphism

$$W(v) = d_v + \delta_v$$

of $\pi^*(\Lambda^* V_{\mathbb{C}})$, and let W act on the graded tensor product $S_N \cong \pi^*(S) \hat{\otimes} \pi^*(\Lambda^* V_{\mathbb{C}})$ in the manner described in Sec. 2. Define D_N to be the operator

$$D_N = D_h + D_v + W.$$

5.1. Lemma. *The diffeomorphisms* (5.2) *preserve the Riemannian volume* (*we equip* $U_\alpha \times \mathbb{R}^n$ *with the product metric*).

Proof. Let $\{e_1, \ldots, e_m\}$ be an oriented, orthonormal frame for TU_α. Using this, together with the frame $\{v_1, \ldots, v_n\}$ and the isomorphism (5.1), build an oriented orthonormal frame $\{\bar{e}_1, \ldots, \bar{e}_m, \bar{v}_1, \ldots, \bar{v}_n\}$ for $T(\pi^{-1}[U_\alpha])$. The vector fields in this frame are mapped as follows under the derivative of (5.2):

$$\bar{e}_i \to e_i + (\text{vector field tangent to } \mathbb{R}^n)$$

$$\bar{v}_i \to \partial/\partial y^i.$$

Thus the volume of the parallelopiped spanned by the images of the frame vectors at any given point is 1. The lemma follows from this. QED

5.2. Lemma. *The operators D_h, D_v, D_N are essentially self-adjoint.*

Proof. By Lemma 5.1, the diffeomorphisms (5.2) induce *unitary* isomorphisms from $L^2(S_N|_{\pi^{-1}[U_\alpha]})$ to $L^2(S|_{U_\alpha}) \hat{\otimes} (L^2(\mathbb{R}^n) \otimes \Lambda^*\mathbb{C}^n)$ (without the introduction of any Radon-Nikodym derivative). Therefore D_h is formally self-adjoint (since D is), as are D_v and D_N. The same argument as used in the proof of Lemma 2.1 establishes essential self-adjointness (in fact the operator D_N is of the sort covered by Lemma 2.1). QED

It follows immediately from their definitions that the operators D_h and $D_v + W$ anticommute (considered as differential operators, acting on smooth sections), and so:

$$D_N^{\,2} = D_h^{\,2} + (D_v + W)^2.$$

Thus if ζ is a smooth, compactly supported section of S_N then

$$\|D_N\zeta\|^2 = \|D_h\zeta\|^2 + \|(D_v + W)\zeta\|^2. \tag{5.3}$$

In fact, by a simple approximation argument, the domain of D_N is precisely the intersection of the domains of D_h and $D_v + W$, and the identity (5.3) holds for all $\zeta \in \text{domain}(D_N)$.

5.3. Lemma. $(D_v + W)^2 = -\Delta + \|v\|^2 + 2F - n$, *where Δ is the Laplacian on N in the vertical direction, and F is the "number operator" which multiplies a form in $\Lambda^* V_{\mathbb{C}}$ by its degree.*

Proof. Choosing $\{v_1, \ldots, v_n\}$, and hence $\{y^1, \ldots, y^n\}$ as in (5.2), we compute that

$$(D_v + W)^2 = \Sigma\partial^2/\partial y^{j2} + \Sigma y^{j2} + \Sigma d_{v_j}\delta_{v_j} - \delta_{v_j}d_{v_j}.$$

The operator $d_{v_j}\delta_{v_j} - \delta_{v_j}d_{v_j}$ multiplies a basic form $v_{j_1} \wedge \cdots \wedge v_{j_k}$ by 1 if it contains v_j, and multiplies it by -1 if it does not. Hence

$$\Sigma d_{v_j}\delta_{v_j} - \delta_{v_j}d_{v_j} = 2F - n. \qquad \text{QED}$$

The second order operator

$$H = \Delta + \|v\|^2 = \Sigma_j \partial^2/\partial y^{j2} + y^{j2}$$

appearing above (considered as an operator on scalar functions) is the harmonic oscillator, familiar from elementary quantum mechanics (see for example [7]). There is an orthonormal basis for $L^2(\mathbb{R}^n)$ consisting of eigenfunctions for H, which are in fact Schwartz class functions, and the eigenvalues are n, $n+1$, $n+2$, ... The lowest eigenvalue, n, has multiplicity 1, with corresponding eigenfunction

$$\psi(v) = \pi^{-n/4}\exp(-\|v\|^2/2). \tag{5.4}$$

Define a map U from the smooth, compactly supported sections of the bundle S (the original Dirac bundle on M) to the sections of S_N by the formula

$$(U\xi)(p) = \xi(\pi(v)) \otimes \psi(v)\mathbf{1}.$$

Here $\mathbf{1}$ denotes the unit zero form in $\Lambda^* V$. It follows from Lemma 5.1, and the fact that ψ is normalized, that U extends to an isometry from $L^2(S)$ into $L^2(S_N)$. By Lemma 5.3, if ξ is smooth and compactly supported then

$$(D_v + W)^2 U\xi = \xi \otimes (H - n)\psi\mathbf{1} = 0,$$

and so $(D_v + W)U\xi = 0$. Since the kernel of $D_v + W$ (considered as a self-adjoint operator) is closed, it follows that U maps $L^2(S)$ into kernel$(D_v + W)$. On the other hand, denote by $P : L^2(S) \to L^2(S_N)$ the projection onto $U[L^2(S_N)] \subset L^2(S_N)$, and let ζ be any smooth, compactly supported section of S_N. It follows from Lemma 5.1 that the restriction of $P\zeta$ to a fiber $V_p \subset N$ of V is the projection of the restriction of ζ onto the subspace generated by the sections of the form $\xi(p) \otimes \psi(\|v\|)\mathbf{1}$. Therefore, using Lemma 5.1 to compute inner products by integrating first in the fiber direction, and considering the fact that the next lowest eigenvalue of the harmonic oscillator above n is $n + 1$, we see that

$$\|(D_v + W)P^{\perp}\zeta\|^2 = \langle (D_v + W)^2 P^{\perp}\zeta, P^{\perp}\zeta\rangle \geq \|P^{\perp}\zeta\|^2. \tag{5.5}$$

By an approximation argument, this inequality holds true for all ζ in the domain of $D_v + W$. Therefore the kernel of $D_v + W$ is exactly the image of U, and $D_v + W$ is bounded from below by 1 on the orthogonal complement of its kernel.

5.4. Lemma. *Let D_1 and D_2 be self-adjoint operators on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and let $U : \mathscr{H}_1 \to \mathscr{H}_2$ be an isometry which maps a core H_1 for D_1 into* domain(D_2).

If $D_2U\xi = UD_1\xi$ for all $\xi \in H_1$ then D_2 commutes with the projection of $\mathscr{H}_2$ onto $U\mathscr{H}_1 \subset \mathscr{H}_2$ (in the sense of unbounded operators: the projection commutes with the resolvent of D_2), and the restriction of D_2 to $U\mathscr{H}_1$ is unitarily equivalent to D_1 via U.

Proof. If ζ is a vector in the subspace $K_1 = \{(D_1 \pm i)\xi | \xi \in H_1\}$ of $\mathscr{H}_1$ then $(D_2 \pm i)^{-1}U\zeta = U(D_1 \pm i)^{-1}\zeta$. But K_1 is dense in $\mathscr{H}_1$ (since H_1 is a core), and so $(D_2 \pm i)^{-1}U = U(D_1 \pm i)^{-1}$. The lemma follows easily from this identity. QED

Denote by D' the operator

$$D' = \Sigma_\alpha \phi_\alpha D \phi_\alpha = D + \Sigma_\alpha \phi_\alpha [\phi_\alpha, D] \tag{5.6}$$

acting on sections of S. This is essentially self-adjoint by Lemma 2.1, and if ξ is a smooth, compactly supported section of S then

$$D_h U\xi = D_h(\xi \otimes \psi \mathbf{1}) = \Sigma_\alpha \bar{\phi}_\alpha (D \otimes 1)\bar{\phi}_\alpha(\xi \otimes \psi \mathbf{1})$$

$$= \Sigma_\alpha (\phi_\alpha D \phi_\alpha \xi) \otimes \psi \mathbf{1} = UD'\xi.$$

Therefore, applying Lemma 5.4 to $D_1 = D'$ and $D_2 = D_N$, we conclude that D_N decomposes as a direct sum of self-adjoint operators,

$$D_N = UD'U^* \oplus D'', \tag{5.7}$$

where, by virtue of (5.3) and (5.5), the operator D'' is bounded below by 1.

At this point let us return to $\mathbb{Z}/k$-manifolds. Let Q and E be as in Sec. 3, and fix an embedding of Q into the space φ_k as in Sec. 4. Attach cylinders at the boundaries of Q and φ_k to obtain an embedding $M \hookrightarrow \mathscr{M}$, and let V be the normal bundle for this embedding. Denote by S_h the spinor bundle for M; let $S = S_h \otimes E$; and let $D = D_E$ be the Dirac operator for this Dirac bundle.

Choose a partition of unity for M which respects the $\mathbb{Z}/k$-structure and which consists of finitely many functions, each one constant in the $[0, \infty)$ direction on each cylinder. Using this, together with choices of frames $\{v_1, \ldots, v_n\}$ which are compatible with the $\mathbb{Z}/k$-structure, construct an operator D_N on S_N as above. Note that the Hilbert space $L^2(S_N)$ has a natural $\mathbb{Z}/k$-structure.

5.5. Proposition. $\mathrm{Index}_k(D_N) = \mathrm{Index}_k(D_E)$.

Proof. Referring to Eq. (5.6), we see that D' is a perturbation of D_E by a bounded endomorphism of S. Following the arguments of Sec. 3, the resolvent of D' is an element of the C^*-algebra $\mathscr{D}_k$, and so $\mathrm{Index}_k(D')$ is defined. But the resolvents of the operators $tD_E + (1-t)D'$ ($t \in [0,1]$) vary continuously in norm, and so $\mathrm{Index}_k(D_E) = \mathrm{Index}(D')$ by the homotopy invariance of C^*-algebra K-theory. To see that we may form $\mathrm{Index}_k(D_N)$, note first that $U: L^2(S) \to L^2(S_N)$ is an isometry of $\mathbb{Z}/k$-Hilbert spaces, in the sense of Lemma 3.6(ii). Suppose that $h \in C_0(\mathbb{R})$ and $\mathrm{supp}(h) \subset [-1/2, 1/2]$. Then since D'' is bounded below by 1 it follows from (5.7) that $Uh(D')U^* = h(D_N)$. Therefore $h(D_N) \in \mathscr{D}_k(L^2(N))$ and $\mathrm{Index}_k(D_N) = \mathrm{Index}_k(D')$ by Lemma 3.6(ii). QED

Now, denote by S_v the spinor bundle for V (for the spin^c-structure described in Sec. 4). There are isomorphisms

$$S_v \hat{\otimes} S_v^* \cong \mathrm{End}(S_v) \cong \mathrm{Cliff}(V)_{\mathbb{C}} \cong \Lambda^* V_{\mathbb{C}}.$$

The first is the standard map from linear algebra (note that the inner product on $\mathrm{End}(S_v)$ is given in the usual way by the trace). The second isomorphism comes from the fact that the fibrewise action of $\mathrm{Cliff}(V)_{\mathbb{C}}$ is irreducible in the case of a spin^c-structure. The last isomorphism is the standard map $v_1 \cdot \ldots \cdot v_j \to v_1 \wedge \ldots \wedge v_j$ (see [14, Chapter 2], for example).

5.6. Lemma. *If v is a section of V then, via the above isomorphism, $d_v - \delta_v$ and $d_v + \delta_v$ act on $S_v \hat{\otimes} S_v^*$ as $v \hat{\otimes} 1$ and $1 \hat{\otimes} \varepsilon v$ respectively.*

Proof. This is a straightforward computation (compare [2]). QED

Note that the isomorphism (5.1) gives an irreducible Clifford action of TN on $\pi^*(S_h \hat{\otimes} S_v)$.

5.7. Lemma. *Equip the bundle $\pi^*(S_h \hat{\otimes} S_v)$ with a connection compatible with the Clifford action of TN and the Riemannian connection, and equip $\pi^*(S_v^* \otimes E)$ with any connection. If D is the Dirac operator for $S_N \cong \pi^*(S_h \hat{\otimes} S_v) \hat{\otimes} \pi^*(S_v^* \otimes E)$ then the difference $D - (D_h + D_v)$ is an endomorphism of S_N which is bounded on subsets of N which lie within a finite distance of the zero section $M \subset N$.*

Proof. If ϕ is any smooth function on N then both of the commutators $[D, \phi]$ and $[D_h + D_v, \phi]$ are equal to Clifford multiplication with $\mathrm{grad}(\phi)$. Therefore $D - (D_v + D_h)$ commutes with every ϕ, and so it is an endomorphism of S_N. Since the coefficients of all our operators are constant along the lengths of the cylinders attached to form N, the norm of $D - (D_v + D_h)$ on the set of all vectors in $V = N$ of length no more than some constant C is equal to the norm on the set of vectors over Q of length no more than C. Since this last set is compact, the norm is finite. QED

6. Proof of the Index Theorem

Fix a metric on $\mathcal{G}_k$ so as to make it a Riemannian $\mathbb{Z}/k$-manifold (it will be necessary to consider metrics other than the obvious one that $\mathcal{G}_k$ inherits as a subset of $\mathbb{R}^{2d}$). Equip $\mathcal{G}_k$ with its standard spin^c-structure and fix a Dirac $\mathbb{Z}/k$-bundle as in part (iii) of Definition 3.1.

So that we do not have to keep mentioning it, we state once and for all that all the structure (bundles, connections, functions, etc.) we consider below will be compatible with the $\mathbb{Z}/k$-structure of $\mathcal{G}_k$, etc, and then extended in the natural way to the manifolds $\mathcal{M}$, etc. obtained by adjoining cylinders at the boundary.

Let $(\mathcal{U}, S, G)$ be a cycle for $K^0(\hat{\mathcal{G}}_k)$, and construct an operator D_G from $(\mathcal{U}, S, G)$ as follows. Extend $\mathcal{U}$ to an open set $\mathcal{V}$ in $\mathcal{M}$ by adding cylinders at the boundary, and extend S and G to $\mathcal{V}$. Equip $\mathcal{V}$ with the spin^c-structure restricted from the standard spin^c-structure on $\mathcal{M}$, and denote the spinor bundle by $S_{\mathcal{V}}$ (which we equip with a suitable connection). Choose a connection for the $\mathbb{Z}/k$-bundle S which respects the $\mathbb{Z}/2$-grading, and choose a smooth "cut-off" function ψ on $\mathscr{V}$ such that:

(i) $\operatorname{supp}(\psi) \cap \mathscr{U}$ is compact; and

(ii) $\psi = 1$ wherever $G^2 \not\geq 2$.

Let D be the Dirac operator for $S_{\mathscr{V}} \hat{\otimes} S$ and let

$$D_G = \psi D \psi + G.$$

Here G is taken to act on the graded tensor product $S_{\mathscr{V}} \hat{\otimes} S$ in the manner described in Sec. 2. It follows that G anticommutes with the coefficients of D, and so the term $DG + GD$ in the expression

$$D_G{}^2 = (\psi D \psi)^2 + G^2 + \psi(DG + GD)\psi \tag{6.1}$$

is an endomorphism of $S_{\mathscr{U}} \hat{\otimes} S$. By a compactness argument it is in fact bounded on $\operatorname{supp}(\psi)$.

The operator D_G is essentially self-adjoint on $L^2(\mathscr{V})$, and $L^2(\mathscr{V})$ has a natural $\mathbb{Z}/k$-structure. Our first task is to show that $\operatorname{Index}_k(D_G)$ is defined.

6.1. Lemma. *(i) If $h \in C_0(\mathbb{R})$ and $\operatorname{supp}(h) \subset [-1/2, 1/2]$ then $h(D_G) \in \mathscr{D}_k(L^2(\mathscr{V}))$.*

(ii) The quantity $\operatorname{Index}_k(D_G)$ (which is defined in view of (i) and the remarks at the end of Sec. 1) does not depend on the choice of cut-off function ψ in its construction.

(iii) Let E be a $\mathbb{Z}/k$-bundle over Q and let $(\mathscr{N}, \pi^*(S_v{}^* \otimes E), J)$ be the cycle representing $\tau_k(E)$. If we choose a metric on $\mathscr{M}$ such that $\mathscr{N} \hookrightarrow \mathscr{M}$ is an isometry then $\operatorname{Index}_k(D_J) = \operatorname{Index}_k(D_N)$, where D_N is the operator constructed in Sec. 5.

Proof. Let ϕ_0 be a smooth function on $\mathscr{U}$ such that $\phi_0 = 1$ on $\mathscr{G}_k \cap \mathscr{U}$ and such that ϕ_0 is supported within a finite distance of $\mathscr{G}_k \cap \mathscr{U}$. We shall show first that $\phi_0 h(D_G) \sim 0$. This implies that $p_0 h(D_G) \sim 0$; the remainder of the proof of part (i) follows from the argument in the proof of Proposition 3.3 (we replace the resolvent function $(t \pm i)^{-1}$ with the function $h(t)$). By the remarks preceeding this lemma, for $\varepsilon > 0$ sufficiently small, $\|\psi(DG + GD)\psi\| \leq 1$ on the open set W_ε where $\psi < \varepsilon$. It then follows from (6.1) that $\|D_G \xi\| \geq \|\xi\|$ for all ξ supported in W_ε. Choose a smooth function ϕ on $\mathscr{U}$ such that $\phi = 1$ on $W_{\varepsilon/2}$ and $\operatorname{supp}(\phi) \subset W_\varepsilon$. Then $\phi\phi_0 h(D_G) \sim 0$ by Lemma 2.5, whilst $(1 - \phi)\phi_0 h(D_G) \sim 0$ by Lemma 2.2.

Let ψ' be another cut-off function and define

$$D_{G,t} = t\psi D \psi + G \qquad (1 \geq t > 0),$$

$$D'_{G,t} = t\psi' D \psi' + G \qquad (1 \geq t > 0).$$

Denote by P_t and P_t' the projections constructed from these operators according to the formula (1.6). We shall prove that $\|P_t - P_t'\| \to 0$ as $t \to 0$, and therefore that P_t and P_t' are unitarily equivalent for small t. Since it is easily verified that P_t and P_t' vary continuously in the norm (for $t > 0$), it will follow from this that P_1 and $P_1{}'$ are unitarily equivalent, and so we shall have proved part (ii) of the lemma. Let W be the open set where $G^2 > 1\frac{1}{2}$, and let ϕ be a smooth function supported in W such that $\phi = 1$ on the set where $G^2 \geq 1\frac{3}{4}$. It follows from (6.1) that $\|D_{G,t}\xi\|$, $\|D'_{G,t}\xi\| \geq \|\xi\|$ for sufficiently small t and all ξ supported in W. Therefore, since $\|[D_{G,t}, \phi]\|$, $\|[D'_{G,t}, \phi]\| \to 0$ as $t \to 0$, it follows from Lemma 2.5 that $\|\phi h(D_{G,t})\|$, $\|\phi h(D'_{G,t})\| \to 0$ as $t \to 0$ for every h supported in $[-/1/2, 1/2]$. Inspecting the formula for P_t, P'_t, we see that $\|\phi P_t - \phi P'_t\| \to 0$ as $t \to 0$. On the other hand, $D_{G,t} = D'_{G,t}$ near supp$(1 - \phi)$. Therefore, by Lemma 2.4, $\|(1 - \phi)h(D_{G,t}) - (1 - \phi)h(D_{G,t})\| \to 0$ as $t \to 0$, and so $\|(1 - \phi)P_t - (1 - \phi)P'_t\| \to 0$ as $t \to 0$. This completes the proof of part (ii).

To prove part (iii), regard $\mathcal{N}$ as an open submanifold of N, and so regard D_J as an operator on $L^2(N)$ (we extend the action of J to all of N; this will not effect the index). As a first step, note that by Lemma 5.6, the endomorphism J identifies with the endomorphism W discussed in Sec. 5. Therefore, by Lemma 5.7 and homotopy invariance, the indices of D_J and $\psi(D_h + D_v)\psi + J$ are equal. Applying the argument of part (ii) we see that the indices of $t\psi(D_h + D_v)\psi + J$ and $t(D_h + D_v) + J$ are equal for small enough $t > 0$ (it is necessary to observe that the anticommutator $J(D_h + D_v) + (D_h + D_v)J$ is bounded—but this follows from Lemma 5.3). Since it is once again easily verified that the index projections are continuous in t, this completes the proof.

QED

6.2. Proposition. *The map* $(\mathcal{U}, S, G) \to \mathrm{Index}_k(D_G)$ *passes to a homomorphism* $\mathrm{Index}_k : K^0(\hat{\mathcal{S}}_k) \to \mathbb{Z}/k$ *which is independent of the choice of metric on* $\mathcal{M}$. *It is in fact equal to the natural identification of* $K^0(\hat{\mathcal{S}}_k)$ *with* $\mathbb{Z}/k$.

Proof. Fix, for the moment, a choice of metric on $\mathcal{M}$. Let us note first that $\mathrm{Index}_k(D_G)$ is independent of the choice of connections on the spinor bundle and on the auxiliary bundle S. Indeed, the space of admissible connections is an affine space and a linear path between two connections gives rise to a path $D_{G,t}$ of operators which is norm continuous, in the sense that $\{D_{G,0} - D_{G,t}\}_{t \in [0,1]}$ is a norm continuous path of bounded operators. It follows that the resolvents of these operators form a norm continuous path, as do the index projections P_t defined by (1.6).

Let us consider next the four components of the equivalence relation of Definition 4.4.

It follows from Lemma 3.6, part (ii), that isomorphic cycles have the same index. If $(\mathcal{U}', S', G')$ is obtained from $(\mathcal{U}, S, G)$ by restriction, as in part (ii) of Definition 4.4, then by applying Lemma 3.6 (ii) to the inclusion $L^2(\mathcal{V}') \hookrightarrow L^2(\mathcal{V})$ we get that $\mathrm{Index}_k(D_{G'}) = \mathrm{Index}_k(D_G)$.

Given a homotopy $\{(\mathcal{U}, S, G_t)\}_{t \in [0,1]}$ as in part (iii) of Definition 4.4, by restricting to an open subset $\mathcal{U}'$ with compact closure in $\mathcal{U}$, we may assume that the homotopy is uniformly continuous. Choosing one cut-off function ψ for all the G_t, the resolvent operators $(D_{G_t} \pm i)^{-1}$, and hence the operators $h(D_{G_t})$ for any $h \in C_0(\mathbb{R})$, vary continuously in norm. It follows from the homotopy invariance of K-theory that $\mathrm{Index}_k(D_{G_0}) = \mathrm{Index}_k(D_{G_1})$.

Finally, if the cycle $(\mathcal{U}, E, G)$ is trivial then in the construction of D_G we can choose $\psi = 0$, and so $\mathrm{Index}_k(D_G) = 0$. Since the map $(\mathcal{U}, E, G) \to \mathrm{Index}_k(D_G)$ is obviously additive on cycles defined over the same base $\mathcal{U}$, it passes to a homomorphism on $K^0(\hat{\mathcal{S}}_k)$.

To show that this homomorphism does not depend on the choice of metric on $\mathcal{M}$, it suffices to check on a generator for $K^0(\hat{\mathscr{S}}_k)$, say a Bott generator $(\mathscr{B}, \pi^*(S^*), J)$ supported in a small ball in the interior of $\mathscr{S}_k$ (away from the collars at the boundary). Given two metrics on $\mathscr{M}$, we may deform the metric of one so that the two metrics are equal on some disjoint ball $\mathscr{B}'$, without changing $\mathrm{Index}_k(D_J)$, and hence without changing the index homomorphisms. But then by checking on a generating cycle supported on $\mathscr{B}'$, we see that the two index homomorphisms are the same.

Choosing the Euclidean metric, it follows from part (iii) of Lemma 6.1 that the index for this generating cycle is the index of the operator $D + J$ on $\mathbb{R}^d$. But this is simply the operator $D_v + W$ of Sec. 5, which has index 1. QED

The proof of the Index Theorem now follows from Proposition 5.5, Lemma 6.1, part (iii), and the above Proposition, which together give the equalities

$$\mathrm{Index}_k(D_E) = \mathrm{Index}_k(D_N) = \mathrm{Index}_k(D_J) = \tau_k(E).$$

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