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A NOTE ON THE COBORDISM INVARIANCE OF THE INDEX

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INTRODUCTION

THE purpose of this note is to give a simple proof of the cobordism invariance for the analytic index of Dirac type operators [5, Chapter 17, Theorem 3]. Our approach is based upon the analysis of operators on complete manifolds, and follows an argument due to J. Roe. In fact we shall prove rather more than the cobordism invariance of the index, namely Roe's index theorem for partitioned manifolds [6].

1. PRELIMINARIES

Let M be a complete, oriented, odd-dimensional manifold (without boundary), and let S be a smooth, Hermitian bundle over M, equipped with a Clifford action of TM and a compatible connection ∇ (see for example [7, Chapter 2]). Let D be the Dirac operator obtained from S. We wish to regard D as an operator on the Hilbert space $L^2(S)$, initially with domain the smooth, compactly supported sections. Our starting point is then the following result concerning solutions of the Dirac equation (see, for example, [1, Theorem 1.17]).

1.1. THEOREM. *The maximal and minimal domains of the operator D are equal. Thus if ξ, $\zeta \in L^2(S)$ and if $D\xi = \zeta$ in the sense of distributions, then there is a sequence $\{\xi_n\}$ of smooth, compactly supported sections such that $\|\xi_n - \xi\| \to 0$ and $\|D\xi_n - \zeta\| \to 0$.*

The operator D is formally self-adjoint, in the sense that if ξ and ζ are smooth, compactly supported sections then $(D\xi, \zeta) = (\xi, D\zeta)$ (see [7, Proposition 2.9]), and so it follows from Theorem 1.1 that D is an essentially self-adjoint Hilbert space operator [5], meaning that the closure of D is a self-adjoint operator on $L^2(S)$ (the domain of the extended operator is as described in the theorem). From now on we shall work with this extension.

If X and Y are bounded Hilbert space operators then we shall write $X \sim Y$ if X and Y differ by a compact operator.

1.2. THEOREM. *If θ is a compactly supported function on M then $\theta(D \pm i)^{-1} \sim 0$.*

This follows from the Rellich Lemma, together with the "basic elliptic estimate" $\|D\xi\| + \|\xi\| \geqslant \varepsilon_K \|\nabla\xi\|$ ($\varepsilon_K > 0$), for smooth sections ξ supported in a compact set K; see for example [7, Chapter 3].

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1.3. LEMMA. *Let ϕ be a smooth function on M which is locally constant outside of a compact set. Then $[(D \pm i)^{-1}, \phi] \sim 0$.*

Proof. Let us note that if ξ is a smooth section of S then

$$[D, \phi]\xi = D(\phi\xi) - \phi D\xi = \operatorname{grad}(\phi) \cdot \xi,$$

where "$\cdot$" denotes Clifford multiplication. Since $\operatorname{grad}(\phi)$ vanishes outside of a compact set, $[D, \phi]$ is compactly supported, and bounded as a Hilbert space operator. From the boundedness it follows that ϕ maps the domain of D into itself, and so we may write

$$[(D \pm i)^{-1}, \phi] = (D \pm i)^{-1}[\phi, D](D \pm i)^{-1}.$$

The Lemma thus follows from Theorem 1.2, upon choosing a compactly supported θ such that $\theta[\phi, D] = [\phi, D]$.

Suppose now that M is partitioned by some hypersurface N, so that $M = M_+ \cup M_-$, where M_+ and M_- are manifolds which are disjoint except for their common boundary N.

1.4. LEMMA. *Let $U = (D - i)(D + i)^{-1}$ be the Cayley transform of D, let ϕ_+ be a smooth function on M which is equal to the characteristic function of M_+ outside of a compact set, and let $\phi_- = 1 - \phi_+$. Then:*

(i) *the operators $U_+ = \phi_- + \phi_+ U$ and $U_- = \phi_+ + \phi_- U$ are Fredholm;*

(ii) $\operatorname{Index}(U_+) = -\operatorname{Index}(U_-)$; *and*

(iii) *the quantity* $\operatorname{Index}(U_+)$ *does not depend on the choice of ϕ, but only on the cobordism class of the partition determined by N (as explained below).*

Proof. Writing $U = 1 - 2i(D + i)^{-1}$, we see from Lemma 1.3 that

$$\phi_+ U = \phi_+ - 2i\phi_+(D + i)^{-1} \sim \phi_+ - 2i(D + i)^{-1}\phi_+ = U\phi_+,$$

and similarly $\phi_- U \sim U\phi_-$. In addition, it follows from Theorem 1.2 that $\phi_+\phi_- U \sim \phi_+\phi_-$. Therefore $U_\pm U_\pm^* \sim 1$ and $U_\pm^* U_\pm \sim 1$, and so U_+ and U_- are Fredholm. Furthermore, $U_+ U_- \sim U$, so by the stability and additivity of the Fredholm index,

$$\operatorname{Index}(U_+) + \operatorname{Index}(U_-) = \operatorname{Index}(U_+ U_-) = \operatorname{Index}(U) = 0.$$

Choosing a different ϕ_+ will only alter U_+ or U_- by a compact operator, and this will not change the Fredholm index. Finally, suppose that N' is another hypersurface, partitioning M into say $M'_\pm$, as above, such that the symmetric differences $M_\pm \Delta M'_\pm$ are relatively compact. Then in the construction of U_+ and U_- for this new partition we may use the same ϕ_+, and so we will obtain the same operator and the same index.

We shall denote the index of U_+ by $\operatorname{Index}(D, N)$. Roe's index theorem for partitioned manifolds relates this quantity to the index of a Dirac operator D_N on N, which is constructed as follows (see [6, Section 3]). Let S_N be the restriction of S to N, equipped with a Clifford multiplication by restricting the Clifford multiplication of TM on S to $TN \hookrightarrow TM|_N$. Then choose any compatible connection on S_N (an adaptation of the standard argument shows that such a connection exists), and let D_N be the corresponding Dirac operator.

In order to state Roe's theorem we must establish some conventions concerning orientation. We shall make the minor simplifying assumption that if $\{e_0, \ldots, e_k\}$ is any oriented, local orthonormal frame for TM, then for all sections ξ of S,

$$i^{1+k/2} e_0 \cdot e_1 \cdot \ldots \cdot e_k \cdot \xi = \xi. \tag{1.1}$$

(In any case, the laws of Clifford algebra dictate that the left hand side of (1.1) defines a self-adjoint endomorphism σ of S with $\sigma^2 = 1$, and so we may in general write $D = D_+ \oplus - D_-$, where $D_\pm$ satisfy our hypothesis.) Let us orient N as the boundary of M_-, so that if n is the normal vector field on N pointing out of M_- and into M_+, and if $\{e_1, \ldots, e_k\}$ is a local, oriented, orthonormal frame for TN, then $\{n, e_1, \ldots, e_k\}$ is an oriented frame for TM on N. The formula

$$\varepsilon(\xi) = i^{k/2} e_1 \cdot e_2 \cdot \ldots \cdot e_k \cdot \xi \tag{1.2}$$

determines a self-adjoint endomorphism ε of S_N such that $\varepsilon^2 = 1$ and $\varepsilon D_N + D_N \varepsilon = 0$. Considered as an automorphism of $L^2(S_N)$, ε leaves kernel(D_N) invariant. Thus the kernel splits as a direct sum $K_+ \oplus K_-$ according to the ± 1 eigenvalues of ε, and we define $\mathrm{Index}(D_N) = \dim(K_+) - \dim(K_-)$. Roe's index theorem is the assertion

1.5. THEOREM ([6, Theorem 3.3].) *Index*(D_N) = *Index*(D, N).

Let V be a compact, oriented manifold with boundary N, and let D be a Dirac operator on V. By adding an infinite cylinder $M_+ = [0, \infty) \times N$ to the boundary of $M_- = V$ we obtain a complete manifold M to which D extends. Since M_- is compact, by applying Theorem 1.2 to ϕ_- we see that $U_- \sim 1$, and so $\mathrm{Index}(D, N) = -\mathrm{Index}(U_-) = 0$. Thus the cobordism invariance of the index follows from Theorem 1.5. For other interesting applications, the reader is referred to Roe's paper [6].

2. THE PRODUCT CASE

Our proof of Theorem 1.5 will follow the general lines of Roe's argument in [6]. In this section we shall analyze the special case where $M = \mathbb{R} \times N$, and in the next section we shall reduce the general case to this.

Let N be an even dimensional, oriented, closed Riemannian manifold, and Let S_N be a hermitian bundle over N equipped with a Clifford action of TN and a compatible connection. Denote by D_N the corresponding Dirac operator. Put the natural metric and orientation on $\mathbb{R}$, and equip $\mathbb{R} \times N$ with the product metric and orientation. Pull back S_N and its connection to $\mathbb{R} \times N$, and extend the Clifford action of TN to a Clifford action of TM by letting the unit tangent vector e_0 for $\mathbb{R}$ act as $-i\varepsilon$, with ε as in (1.2) (note that this choice is forced on us by our orientation convention (1.1)). The connection is compatible with this larger Clifford action, and we form the Dirac operator $D = \Sigma_{i=0}^k e_i \nabla_i = -i\varepsilon \mathrm{d}/\mathrm{d}t + D_N$.

Now, embed N into $M = \mathbb{R} \times N$ as $\{0\} \times N$, and let $M_- = (-\infty, 0] \times N$ and $M_+ = [0, \infty) \times N$. (This is consistent with our orientation conventions, and we note that the operator on N induced by D is the operator D_N we started with.) Choose ϕ_+ as in Lemma 1.4 which is a function only of $t \in \mathbb{R}$. Defining ψ to be $2\phi_+ - 1$, we note that

$$\begin{aligned} U_+ &= (D + i)(D + i)^{-1} - 2i\phi(D + i)^{-1} \\ &= (D - i\psi)(D + i)^{-1}. \end{aligned}$$

We see from this that the kernels of U_+ and U_+^* are isomorphic to the kernels of $D - i\psi$ and $(D - i\psi)^* = D + i\psi$, respectively, considering $(D \pm i\psi)$ as operators on the domain of D. In order to compute these we shall decompose the space $L^2(S)$ according to $K = \mathrm{kernel}(D_N)$. Using the obvious isomorphism $L^2(S) \cong L^2(\mathbb{R}) \otimes L^2(S_N)$, we define $E: L^2(S) \to L^2(S)$ to be the projection onto $L^2(\mathbb{R}) \otimes K$. The space K is finite dimensional and consists entirely of

smooth sections (see [7, Chapter 3]). Choosing an orthonormal basis $\xi_1, \ldots, \xi_d$, we have the formula

$$E\zeta(t, x) = \sum_{i=1}^{d} \xi_i(x) \int_N \langle \xi_i(y), \zeta(t, y) \rangle \, \mathrm{d}y,$$

from which we see that E maps the smooth, compactly supported sections of S to themselves. For smooth, compactly supported ζ we compute that

$$\begin{aligned} \|(D \pm i\psi)\zeta\|^2 &= ((D \mp i\psi)(D \pm i\psi)\zeta, \zeta) \\ &= (D_N^2\zeta, \zeta) + ((-i\varepsilon \mathrm{d}/\mathrm{d}t \pm i\psi)^*(-i\varepsilon \mathrm{d}/\mathrm{d}t \pm i\psi)\zeta, \zeta) \\ &\geq \|D_N\zeta\|^2 \\ &\geq \delta \|(1 - E)\zeta\|^2, \end{aligned}$$

where δ is the least positive eigenvalue of D_N (see [7, Chapter 3]). and also that

$$(D \pm i\psi)E\zeta = -i\varepsilon \mathrm{d}(E\xi)/\mathrm{d}t \pm i\psi E\zeta.$$

It follows from Theorem 1.1 and an approximation argument that the estimate $\|(D \pm i\psi)\zeta\|^2 \geq \delta\|(1 - E)\zeta\|^2$ holds for all $\zeta \in \mathrm{domain}(D)$, and so kernel $(D \pm i\psi) \subset L^2(\mathbb{R}) \otimes K$, whilst on $L^2(\mathbb{R}) \otimes K$ the operator $D \pm i\psi$ is equal to $-i\varepsilon \mathrm{d}/\mathrm{d}t \pm i\psi$, considered as a differential operator on vector valued functions, with its maximal domain. Now, the distributional solutions of $(-i\varepsilon \mathrm{d}/\mathrm{d}t \pm i\psi)f = 0$ are just the ordinary C^1 solutions [3, Corollary 3.1.5], namely

$$f(t) = \begin{cases} \exp\left(\pm \int_0^t \psi(s)\mathrm{d}s\right)v & (v \in K_+) \\ \exp\left(\mp \int_0^t \psi(s)\mathrm{d}s\right)v & (v \in K_-), \end{cases}$$

where we decompose K according to the eigenspaces of ε. Since only $\exp(-\int_0^t \psi(s)\mathrm{d}s)v$ is an L^2-function, we see that $\dim(\ker(D + i\psi)) = \dim(K_-)$ and $\dim(\ker(D - i\psi)) = \dim(K_+)$, which shows that $\mathrm{Index}(D, N) = \mathrm{Index}(D_N)$, as desired.

3. THE GENERAL CASE

We shall reduce the analysis of a general partitioned manifold M to the product case of the previous section using the following result.

3.1. Lemma. (Compare [2, Lemma 2.3].) *Let M_1 and M_2 be two partitioned manifolds, and let $\gamma: M_{2+} \to M_{1+}$ be an isometry which lifts to an isomorphism of Clifford structures. Then* $\mathrm{Index}(D_1, N_1) = \mathrm{Index}(D_2, N_2)$. *Similarly, if there is an isometry $\gamma: M_{2-} \to M_{1-}$ which lifts to an isomorphism of Clifford structures then* $\mathrm{Index}(D_1, N_1) = \mathrm{Index}(D_2, N_2)$.

Proof. Let ϕ_1 be a smooth function on M_1 such that $\phi_1 = 0$ in a neighborhood of M_{1-} and $\phi_1 = 1$ on all but a compact subset of M_{1+}, and let $\phi_2 = \phi_1 \circ \gamma$. Then

$$U_{1+} = 1 + 2i\phi_1(D_1 + i)^{-1}$$

and

$$U_{2+} \sim 1 + 2i(D_2 + i)^{-1}\phi_2.$$

Let $V: L^2(S_1) \to L^2(S_2)$ be any unitary operator obtained as an extension of the Hilbert space isometry $\Gamma: L^2(S_1|M_{1+}) \to L^2(S_2|M_{2+})$ induced by γ. Then:

$$\begin{aligned} VU_{1+} - U_{2+}V &\sim V(1 + 2i\phi_1(D_1 + i)^{-1}) - (1 + 2i(D_2 + i)^{-1}\phi_2)V \\ &= 2i(\Gamma\phi_1(D_1 + i)^{-1} - (D_2 + i)^{-1}\phi_2\Gamma) \\ &= 2i(D_2 + i)^{-1}((D_2 + i)\Gamma\phi_1 - \phi_2\Gamma(D_1 + i))(D_1 + i)^{-1}, \end{aligned}$$

(the last manipulation is legitimate because $\mathrm{domain}((D_2 + i)\Gamma\phi_1) \subset \mathrm{domain}(D_1)$). But $(D_2 + i)\Gamma\phi_1 = \Gamma(D_1 + i)\phi_1$ and $\phi_2\Gamma(D_1 + i) = \Gamma\phi_1(D_1 + i)$), and so the above expression reduces to

$$VU_{1+} - U_{2+}V \sim 2i(D_2 + i)^{-1}\Gamma[D_1, \phi_1](D_1 + i)^{-1}.$$

As we noted in the proof of Lemma 1.4, $[D_1, \phi_1](D_1 + i)^{-1} \sim 0$. Therefore $VU_{1+}V^* \sim U_{2+}$, and so $\mathrm{Index}(U_{1+}) = \mathrm{Index}(U_{2+})$.

The other assertion in the Lemma is proved in the same way, using the fact that $\mathrm{Index}(D_j, N_j) = -\mathrm{Index}(U_{j-})$ $(j = 1, 2)$.

Proof of Theorem 1.5. Choose a collaring neighborhood $(-1, 1) \times N$ of N in M. By Lemma 3.1, we may change M to be of product form $(-\infty, -1/2) \times N$ away from M_+ without altering $\mathrm{Index}(D, N)$. But then by Lemma 1.4, we can replace the partitioning hypersurface $N \cong \{0\} \times N$ with $\{-1/2\} \times N$ without changing the value of $\mathrm{Index}(D, N)$. Having done so, we can use Lemma 3.1 again to replace the part of M to the right of $\{-1/2\} \times N$ with the cylinder $(-1/2, \infty) \times N$, once again, without changing $\mathrm{Index}(D, N)$. We have now transformed M to the product $\mathbb{R} \times N$.

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