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On the K-Theory Proof of the Index Theorem

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1. Introduction

This paper is an exposition of the K-theory proof of the Atiyah-Singer Index Theorem. I have tried to separate, as much as possible, the analytic parts of the proof from the topological calculations. For the topology I have taken advantage of the Chern isomorphism to work mostly within the world of ordinary cohomology. The analytic part of the proof is done within the framework of asymptotic morphisms [**6**] [**7**]. Depending on the reader's outlook this may or may not be simpler than the usual approach through pseudodifferential operators.

The approach we take is due, more or less, to Kasparov [**12**]. It differs a little from the argument in [**2**] and has the useful feature that embeddings into Euclidean space are not required. This will be used in the article [**4**] which deals with the equivariant index theorem for manifolds equipped with proper actions of discrete groups.

See [**8**] for another K-theoretic proof of the index theorem, based on ideas of P. Baum.

2. Elliptic Operators

Let M be a smooth closed manifold, let E and F be smooth complex vector bundles over M, and let

$$D: C^{\infty}(M,E) \to C^{\infty}(M,F)$$

be a linear elliptic operator on M, mapping sections of E to sections of F. For simplicity assume that D is a differential—as opposed to pseudodifferential—operator, and that it has order one. So choosing local coordinates on M, along

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with local frames for E and F, the operator D is of the form

$$D = \sum a_i \, \partial/\partial x_i + b,$$

where $a_i \colon E \to F$, $b \colon E \to F$ are smooth matrix valued functions.

The *symbol* of D is the function σ which associates to each cotangent vector $\xi \in T^*M_p$ a linear transformation $E_p \to F_p$, according to the formula

$$\sigma(p,\xi) = \sqrt{-1} \sum_i \xi_i a_i(p) \qquad\qquad (\xi = \sum_i \xi_i dx_i).$$

It does not depend on the choice of local coordinates. The definition of ellipticity asserts that if $\xi \neq 0$ then the linear transformation $\sigma(p,\xi) \colon E_p \to F_p$ is *invertible.*

The rudiments of the theory of elliptic operators imply that the kernel and cokernel of D are finite dimensional complex vector spaces, and our objective is to calculate the quantity

$$\operatorname{Index}(D) = \dim_{\mathbb{C}}(\operatorname{kernel} D) - \dim_{\mathbb{C}}(\operatorname{cokernel} D) \in \mathbb{Z}$$

in terms of the symbol of D and the algebraic topology of M. See [**15**].

3. K-Theory

We review a few facts about the K-theory of C^*-algebras. See [**5**] and [**7**] for details. In fact we shall scarcely go beyond the K-theory of commutative C^*-algebras, which amounts to the same thing as topological K-theory [**1**], but for one or two constructions it is convenient to adopt the C^*-algebra point of view.

Let A be a C^*-algebra. Recall that if A has a unit then $K(A)$ is the abelian group generated by homotopy classes of projections in matrix algebras over A, subject to the relation that addition of disjoint projections correspond to addition in $K(A)$.

A homomorphism $A \to B$ between C^*-algebras with unit determines a homomorphism of abelian groups $K(A) \to K(B)$, making $K(A)$ into a covariant functor.

If A does not have a unit then we define $K(A)$ by adjoining a unit to A, so as to obtain a C^*-algebra A^+, and setting

$$K(A) = \operatorname{kernel}\{K(A^+) \to K(A^+/A)\}.$$

Since any homomorphism of C^*-algebras $A \to B$ extends to a homomorphism $A^+ \to B^+$ we obtain a covariant functor on the category of all C^*-algebras and all C^*-algebra homomorphisms.

DEFINITION. Let A and B be C^*-algebras. An *asymptotic morphism* from A to B is a family of functions $T^\omega\colon A \to B$ ($\omega \in [1,\infty)$) such that

(1) $T^\omega(a)$ is jointly continuous in a and ω;
(2) $\limsup_{\omega\to\infty} \|T^\omega(a)\| < \infty$ for every $a \in A$; and
(3) we have

$$\lim_{\omega\to\infty} \|T^\omega(a) + \lambda T^\omega(a') - T^\omega(a + \lambda a')\| = 0,$$
$$\lim_{\omega\to\infty} \|T^\omega(a^*) - T^\omega(a)^*\| = 0,$$
$$\lim_{\omega\to\infty} \|T^\omega(a)T^\omega(a') - T^\omega(aa')\| = 0,$$

and the convergence is uniform on compact subsets of A.

This differs a little from the definition in [**6,7**], but not in any essential way. We remark that condition (2) is in fact a consequence of conditions (1) and (3).

An asymptotic morphism $T^\omega\colon A \to B$ determines a homomorphism of K-theory groups

$$T\colon K(A) \to K(B),$$

as follows. Suppose first that A has a unit. Let p be a projection in A, or in a matrix algebra over A (in which case we note that T^ω applied entrywise gives an asymptotic morphism from matrices over A to matrices over B). Consider the continuous family $T^\omega(p)$ of elements in B (or in a matrix algebra over B). It is uniformly bounded, and

$$\|T^\omega(p) - T^\omega(p)^2\| \to 0,$$

as $\omega \to \infty$, so that $T^\omega(p)$ is "asymptotically" a projection. It follows easily from the functional calculus that there is a continuous family of projections q^ω in B such that

$$\|T^\omega(p) - q^\omega\| \to 0$$

as $\omega \to \infty$. We define

$$T[p] = [q^1].$$

If A does not have a unit then note that T^ω extends to an asymptotic morphism $T^\omega\colon A^+ \to B^+$ (mapping one adjoined unit to the other). We obtain a map $K(A^+) \to K(B^+)$ which restricts to a map from $K(A)$ into $K(B)$, as required.

Let X be a compact Hausdorff space. As usual, denote by $C(X)$ the continuous, complex valued functions on X. The group $K(C(X))$ has the structure of a commutative ring, for if $p \in M_n(C(X))$ and $q \in M_{n'}(C(X))$ are projections then we may form

$$p \otimes q(x) = p(x) \otimes q(x) \in M_{nn'}(C(X)) \tag{3.1}$$

(here we view matrices of functions on X as matrix valued functions on X). The multiplicative unit of $C(X)$ determines a unit

$$1 = [1] \in K(C(X)).$$

Denote by $A(X)$ the C^*-algebra of continuous functions from X into a C^*-algebra A. Then the group $K(A(X))$ is a module over $K(X)$. If A has a unit the module structure is defined by a formula like (3.1). If A has no unit we observe that

$$K(A(X)) \cong \operatorname{kernel}\{K\left(A^+(X)\right) \to K\left(A^+/A\,(X)\right)\},$$

and reduce to the unital case.

An asymptotic morphism $T^\omega\colon A \to B$ extends in the obvious way to an asymptotic morphism

$$T^\omega_X\colon A(X) \to B(X),$$

and so we obtain homomorphisms of K-theory groups

$$T_X\colon K(A(X)) \to K(B(X)).$$

Lemma 3.1. *The maps T_X are $K(C(X))$-module homomorphisms. In addition, if $f\colon X' \to X$ is any continuous map then the diagram*

$$\begin{tikzcd} K(A(X)) \arrow[r, "T_X"] \arrow[d, "f^*"'] & K(B(X)) \arrow[d, "f^*"] \\ K(A(X')) \arrow[r, "T_{X'}"'] & K(B(X')) \end{tikzcd}$$

commutes. □

Let $\mathcal{K}$ denote the C^*-algebra of compact operators on a separable Hilbert space. Fix a rank one projection e in $\mathcal{K}$, and map $C(X)$ into $\mathcal{K}(X)$ by sending a function f to the function $x \mapsto f(x)e$.

Lemma 3.2. *The induced map*

$$K(C(X)) \to K(\mathcal{K}(X))$$

(*which is a $K(C(X))$-module homomorphism*) *is an isomorphism.* □

Let Y be a locally compact space and let $C_0(Y)$ be the C^*-algebra of continuous complex valued functions on Y which vanish at infinity.

For the rest of the paper we shall write $K(Y)$ in place of $K(C_0(Y))$.

Note that $C_0(Y)^+ = C(Y^+)$, where Y^+ denotes the one point compactification of Y. Thus if p and q are projection valued functions on Y^+, which are equal at infinity, then the difference $[p] - [q]$ is an element of $K(Y)$.

Note also that the algebra of continuous functions from X into $C_0(Y)$ is equal to $C_0(X \times Y)$.

Using this we can summarize what we need of the discussion in this section as follows.

PROPOSITION 3.3. *An asymptotic morphism $T^\omega : C_0(Y) \to \mathcal{K}$ determines a family of $K(X)$-module maps*

$$T_X : K(X \times Y) \to K(X),$$

which are natural in X as in Lemma 3.1. $\square$

4. The Symbol Class

We shall define two sorts of K-theory classes, the first associated to an elliptic operator on a manifold, and the second associated to the manifold itself.

Let M be a smooth, closed manifold and let

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

be an elliptic operator with symbol

$$\sigma : \pi^* E \to \pi^* F$$

(π is the projection from the cotangent bundle T^*M to M). Endow the E and F with metrics and form the self-adjoint endomorphism

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix} : \pi^* E \oplus \pi^* F \to \pi^* E \oplus \pi^* F.$$

LEMMA 4.1. *The resolvent operators*

$$(\boldsymbol{\sigma} \pm i)^{-1} : \pi^* E \oplus \pi^* F \to \pi^* E \oplus \pi^* F$$

are endomorphisms which vanish at infinity (in the operator norm induced from the metrics on E and F).

PROOF. Ellipticity implies that $\boldsymbol{\sigma}$ is bounded below on the complement of any neighbourhood of the zero section in T^*M. Using the homogeneity $\boldsymbol{\sigma}(x, t\xi) = t\boldsymbol{\sigma}(x, \xi)$ we see that for any $C > 0$ there is a compact subset of T^*M outside of which $\boldsymbol{\sigma}$ is bounded below by C. The lemma follows from this. $\square$

Now form the Cayley transform

$$\begin{aligned} \boldsymbol{u} &= (\boldsymbol{\sigma} + i)(\boldsymbol{\sigma} - i)^{-1} \\ &= 1 + 2i(\boldsymbol{\sigma} - i)^{-1}. \end{aligned}$$

Embed E and F into trivial bundles $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ over M, and extend the automorphism $\boldsymbol{u}$ of $\pi^* E \oplus \pi^* F$ to the trivial bundle $\mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}$ over T^*M by setting it equal to the identity on the complement of $\pi^* E \oplus \pi^* F$. By Lemma 4.1 $\boldsymbol{u}$ extends continuously to $(T^*M)^+$ upon setting $\boldsymbol{u}(\infty) = I$.

Let

$$\boldsymbol{\varepsilon} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

viewed as an automorphism of the trivial bundle $\mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}$ over $(T^*M)^+$. Then of course $\boldsymbol{\varepsilon}^2 = 1$, but in addition

$$(\boldsymbol{u}\boldsymbol{\varepsilon})^2 = 1.$$

This is a simple consequence of the fact that $\boldsymbol{\varepsilon}$ anticommutes with the endomorphism $\boldsymbol{\sigma}$.

Now to each involution $\boldsymbol{w}$ (meaning $\boldsymbol{w}^2 = 1$) there is associated a projection $\boldsymbol{p}(\boldsymbol{w})$ (meaning $\boldsymbol{p}(\boldsymbol{w})^2 = \boldsymbol{p}(\boldsymbol{w})$) according to the formula

$$\boldsymbol{p}(\boldsymbol{w}) = \frac{1}{2}(\boldsymbol{w} + 1).$$

In the case at hand we obtain two projection valued functions $\boldsymbol{p}(\boldsymbol{\varepsilon})$ and $\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{u})$ on $(T^*M)^+$ which are equal at infinity. Each defines an element of $K((T^*M)^+)$ and their difference defines an element

$$\sigma_D = [\boldsymbol{p}(\boldsymbol{\varepsilon})] - [\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{u})] \in K(T^*M).$$

This is the *symbol class* of the elliptic operator D. (Our construction of it, using the Cayley transform, is taken from [**16**].)

Now let V be a Euclidean vector bundle over a compact space X. We shall define a class

$$\lambda_V \in K(V \oplus V).$$

and from it, using the tangent bundle, we shall obtain a class

$$\lambda_M \in K(M \times T^*M).$$

Form the complexified exterior algebra bundle $\bigwedge_{\mathbb{C}}^* V$ and for $w \in V_x \otimes \mathbb{C}$ define

$$c(w) \colon \bigwedge\nolimits_{\mathbb{C}}^* V_x \to \bigwedge\nolimits_{\mathbb{C}}^* V_x$$
$$c(w)\eta = d_w \eta - \delta_w \eta,$$

where d_w denotes the operator of exterior multiplication by w, and δ_w denotes its adjoint. Define an endomorphism

$$\boldsymbol{c} \colon \pi^*(\bigwedge\nolimits_{\mathbb{C}}^* V) \to \pi^*(\bigwedge\nolimits_{\mathbb{C}}^* V)$$

of the vector bundle $\bigwedge_{\mathbb{C}}^* V$ pulled back to the space $V \oplus V$ by the formula

$$\boldsymbol{c}(v, v') = c(v) + c(\sqrt{-1}v').$$

It is self adjoint and

$$\boldsymbol{c}(v, v')^2 = \|v\|^2 + \|v'\|^2,$$

so that the resolvents $(\boldsymbol{c} \pm i)^{-1}$ vanish at infinity in $V \oplus V$. In addition, $\boldsymbol{c}$ anticommutes with the "grading operator" $\boldsymbol{\varepsilon}$ which multpilies a form by ± 1 according as the form is even or odd. Because of this we can follow the same procedure as above to define a K-theory class in $K(V \oplus V)$: we form the Cayley transform $\boldsymbol{v}$ of $\boldsymbol{c}$, and then define

$$\lambda_V = [\boldsymbol{p}(\boldsymbol{\varepsilon})] - [\boldsymbol{p}(\boldsymbol{\varepsilon v})] \in K(V \oplus V).$$

If V is a vector space (= vector bundle over a point) then λ_V is the "Bott element" familiar from the Periodicity Theorem.

Endow the smooth, closed manifold M with a Riemannian metric and define a map

$$\begin{gathered} \phi\colon TM \to M, \\ \phi(v) = \exp\left(\frac{\delta}{(1+\|v\|^2)^{1/2}} v\right). \end{gathered} \tag{4.1}$$

Here we use the exponential map from differential geometry, and $\delta > 0$ is chosen to be small enough so that the associated map

$$v \mapsto (\pi(v), \phi(v))$$

is a diffeomorphism from TM onto an open subset of $M \times M$ (see for example [**14**]). Define a diffeomorphism from $TM \oplus T^*M$ onto an open subset of $M \times T^*M$ as follows. For $m \in M$ the fibre of $TM \oplus T^*M$ over m may be identified with the cotangent bundle of TM_m. The map ϕ is a diffeomorphism from TM_m to an open subset of $W_m \subset M$, and so the transpose of the derivative of ϕ^{-1} is a diffeomorphism

$$\tilde{\phi}\colon T^*(TM_m) \to T^*W_m \subset T^*M.$$

We define

$$\begin{gathered} TM \oplus T^*M \to M \times T^*M \\ (v, \xi) \mapsto (\pi(v), \tilde{\phi}(v, \xi)). \end{gathered} \tag{4.2}$$

Identifying T^*M and TM using the metric, we define $\lambda_M \in K(M \times T^*M)$ to be the image of $\lambda_{TM} \in K(TM \oplus TM)$ under the map on K-theory groups induced from (4.2).

5. The Analytic Index

The K-theory proof of the Index Theorem is based on the following result of Atiyah and Singer.

THEOREM 5.1. (Atiyah and Singer [**4**]) *There are maps*

$$\mathrm{Ind}_X\colon K(X \times T^*M) \to K(X)$$

for each compact space X such that:

(1) Ind_X *is a $K(X)$-module homomorphism;*

(2) Ind *is a natural transformation, in the sense that for every continuous map* $f\colon X' \to X$ *the diagram*

$$\begin{tikzcd} K(X \times T^*M) \arrow[r, "\mathrm{Ind}_X"] \arrow[d, "f^*"'] & K(X) \arrow[d, "f^*"] \\ K(X' \times T^*M) \arrow[r, "\mathrm{Ind}_{X'}"'] & K(X') \end{tikzcd}$$

commutes;

(3) *if* D *is an elliptic operator on* M *then*

$$\mathrm{Ind}_{\mathrm{pt}}(\sigma_D) = \mathrm{Index}(D)$$

in $K(\mathrm{pt}) \cong \mathbb{Z}$*; and*

(4) $\mathrm{Ind}_M(\lambda_M) = 1 \in K(M)$.

We shall prove this by constructing in Section 8 an appropriate asymptotic morphism from $C_0(T^*M)$ into $\mathcal{K}(L^2(M))$ and applying the remarks made in Section 3. The verification of parts (3) and (4) will be done in Section 9.

6. Chern Character and Cohomology

Let Y be a locally compact space. Denote by $H^*(Y)$ the direct sum of the cohomology groups of Y with real coefficients and *compact supports.* Denote by $H^{ev}(Y)$ the direct sum of the even cohomology groups with real coefficients and compact supports.

For our purposes Y will always be a reasonable space, in fact a smooth manifold, so it is not necessary to specify a choice of cohomology theory.

Let X be a compact space. The cup product in cohomology makes $H^*(X)$ into a graded commutative ring, and $H^{ev}(X)$ is a subring. A continuous map $f\colon Y \to X$ provides $H^*(Y)$ with the structure of an $H^*(X)$-module. (If we are working with de Rham theory and if f is smooth then the module structure is given by pulling back forms from X to Y and taking wedge product.) We shall use the cup product symbol $a \smile b$ for the module action. It will be convenient to work with both left and right modules.

There is a *Chern character* homomorphism

$$\mathrm{ch}\colon K(Y) \to H^{ev}(Y)$$

(see [**11**]). It is a natural transformation which is multiplicative with respect to the ring and module structures on K-theory and cohomology described above and in Section 3.

As a consequence of the Bott Periodicity Theorem we have:

Chern Isomorphism Theorem. *The map*

$$\mathrm{ch} \otimes id_{\mathbb{R}} \colon K(Y) \otimes \mathbb{R} \to H^{ev}(Y)$$

is an isomorphism. □

7. Poincaré Duality and the Index Theorem

In this section we shall use Theorem 5.1 and the Chern isomorphism to obtain the Atiyah-Singer Index Theorem.

Given a smooth closed manifold M, orient the manifold T^*M as follows. Choose local coordinates $x_1, \ldots, x_n$ on M. Define functions $y_1, \ldots, y_n$ on T^*M by

$$y_i(\xi) = \langle \xi, \partial/\partial x_i \rangle$$

(the angle brackets denote the pairing between cotangent and tangent vectors). Then we deem $x_1, y_1, x_2, y_2, \ldots x_n, y_n$ to be an oriented system of local coordinates on T^*M.

The orientation gives a linear functional

$$p_* \colon H^*(T^*M) \to \mathbb{R} \tag{7.1}$$

(in de Rham theory, take the degree $2n$ component of an element in $H^*(T^*M)$, represent it as a compactly supported $2n$-form and integrate it over T^*M).

The projection $\pi \colon T^*M \to M$ gives $H^*(T^*M)$ the structure of an $H^*(M)$-module. For bookkeeping purposes take it to be a *right* module.

Poincaré Duality Theorem. *The pairing*

$$b \otimes a \mapsto p_*(b \smile a)$$

from $H^p(T^*M) \otimes H^{2n-p}(M)$ into $\mathbb{R}$ induces an isomorphism from $H^p(M)$ to the dual space of $H^{2n-p}(T^*M)$. □

This simple version of Poincaré Duality is easily proved using a Mayer-Vietoris argument, as is the following result.

Kunneth Formula. *View $H^*(X \times T^*M)$ as a left $H^*(X)$ module via the projection p of $X \times T^*M$ onto X. Denote by $q \colon X \times T^*M \to T^*M$ the other projection. Then the map $x \otimes y \mapsto x \smile q^*(y)$ is an isomorphism from $H^*(X) \otimes H^*(T^*M)$ to $H^*(X \times T^*M)$.* □

In view of the Kunneth Formula, the recipe

$$p_*(x \smile q^* y) = x \cdot p_* y,$$

where $x \in H^*(X)$ and $y \in H^*(T^*M)$, extends (7.1) above, giving maps

$$p_* \colon H^*(X \times T^*M) \to H^*(X).$$

They are $H^*(X)$-module homomorphisms, functorial in X.

These preliminaries dispensed with, we turn to an analysis of the maps

$$\mathrm{Ind}_X\colon K(X \times T^*M) \to K(X)$$

of Theorem 5.1. By the Chern isomorphism Theorem, there are homomorphisms

$$I_X^{ev}\colon H^{ev}(X \times T^*M) \to H^{ev}(X)$$

such that the diagrams

$$\begin{array}{ccc} K(X \times T^*M) & \xrightarrow{\mathrm{Ind}_X} & K(X) \\ \mathrm{ch}\downarrow & & \downarrow\mathrm{ch} \\ H^{ev}(X \times T^*M) & \xrightarrow[I_X^{ev}]{} & H^{ev}(X) \end{array}$$

commute. They are $H^{ev}(X)$-module homomorphisms, functorial with respect to maps $X' \to X$.

Replacing X with $X \times S^1$, it is easily checked that the I_X^{ev} extend to maps

$$I_X\colon H^*(X \times T^*M) \to H^*(X)$$

which are functorial $H^*(X)$-module homomorphisms. We shall work with these below.

Lemma 7.1. *View $H^*(X \times T^*M)$ as a right $H^*(M)$ module via the projection map*

$$X \times T^*M \to T^*M \to M.$$

There is a cohomology class $a_M \in H^*(M)$ such that

$$I_X(x) = p_*(x \smile a_M),$$

for every $x \in H^*(X \times T^*M)$. Thus if D is an elliptic operator on M then

$$\mathrm{Index}(D) = p_*(\mathrm{ch}(\sigma_D) \smile a_M).$$

Proof. Poincaré duality asserts that I_{pt} is given by multiplication with some element a_M of $H^*(M)$, followed by evaluation against the fundamental class. The formula for I_X follows from this in view of the Kunneth formula and the fact that I_X is natural and an $H^*(X)$-module homomorphism. □

We calculate a_M as follows. Observe that $H^*(M \times T^*M)$ is both a left $H^*(M)$-module, via the projection of $M \times T^*M$ onto the first factor, and a right $H^*(M)$-module, via the projection of $M \times T^*M$ onto M through the second factor.

LEMMA 7.2. *Let $\lambda_M \in K(M \times T^*M)$ be the class defined in Section 4. Then*

$$a \smile \mathrm{ch}(\lambda_M) = \mathrm{ch}(\lambda_M) \smile a$$

for all $a \in H^*(M)$.

PROOF. As in Section 4, regard $TM \oplus T^*M$ as an open subset of $M \times T^*M$. The projection of $M \times T^*M$ onto M via the second factor corresponds to the map $TM \oplus T^*M \to M$ given by the formula

$$(v, v') \mapsto \exp\left(\frac{\delta}{(1 + \|v\|^2)^{1/2}} v\right)$$

(compare (4.1)). This is homotopic to the standard projection $(v, v') \mapsto \pi(v)$ (which corresponds to the projection of $M \times T^*M$ onto the first factor) by contracting δ to zero. Therefore both maps induce the same $H^{ev}(M)$-module action on $H^*(TM \oplus T^*M)$ (note that multiplying $H^{odd}(M)$ against $H^{odd}(TM \oplus T^*M)$ on the right differs by a minus sign from multiplication on the left: this is why we consider only $H^{ev}(M)$). Since $\mathrm{ch}(\lambda_M)$ lies in the image of the map

$$H^{ev}(TM \oplus T^*M) \to H^{ev}(M \times T^*M)$$

given by (4.2) the result follows. $\square$

INDEX THEOREM, PRELIMINARY VERSION. *The class $p_*(\mathrm{ch}(\lambda_M))$ is a unit in the ring $H^{ev}(M)$, and for every $x \in H^*(X \times T^*M)$*

$$I_X(x) = p_*(x \smile p_*(\mathrm{ch}(\lambda_M))^{-1}).$$

In particular, if D is an elliptic operator on M then

$$\mathrm{Index}(D) = p_*(\mathrm{ch}(\sigma_D) \smile p_*(\mathrm{ch}(\lambda_M))^{-1}).$$

PROOF. Let $a_M \in H^*(M)$ be the class obtained in Lemma 7.1. Using Lemma 7.2 and the fact that p_* is a left $H^*(M)$-module homomorphism, we obtain

$$a_M \smile p_*(\mathrm{ch}(\lambda_M)) = p_*(a_M \smile \mathrm{ch}(\lambda_M)) = p_*(\mathrm{ch}(\lambda_M) \smile a_M) = I_M(\mathrm{ch}(\lambda_M)).$$

But according to part (4) of Theorem 5.1 and the definition of I_M,

$$I_M(\mathrm{ch}(\lambda_M)) = \mathrm{ch}(\mathrm{Ind}_M(\lambda_M)) = 1. \qquad \square$$

The customary formulation of the index theorem is obtained from the preliminary version above by using some further ideas in algebraic topology. What follows below is a rapid summary of this. For further details see, for example, [**3**] or [**13**].

Let X be any compact space, let V be a Euclidean vector bundle over X, and let $\lambda_V \in K(V \oplus V)$ be the class defined in Section 4. Using the Thom isomorphism in cohomology,

$$\pi_*\colon H^*(V \oplus V) \to H^*(X),$$

we form the characteristic class

$$\tau(V) = \pi_*(\mathrm{ch}(\lambda_V)) \in H^*(X),$$

noting that if M is a smooth closed manifold then

$$\tau(TM) = p_*(\mathrm{ch}(\lambda_M)).$$

Using techniques of characteristic class theory one shows that

$$\tau(V) = (-1)^{\dim(V)}\,\mathrm{Todd}(V \otimes \mathbb{C})^{-1},$$

where $V \otimes \mathbb{C}$ is the complexification of V and $\mathrm{Todd}(V \otimes \mathbb{C})$ denotes its Todd class. Using a more suggestive notation for the functional $p_*\colon H^{ev}(T^*M) \to \mathbb{R}$ (borrowed from de Rham theory) we get:

Index Theorem.

$$\mathrm{Index}(D) = (-1)^{\dim(M)} \int_{T^*M} ch(\sigma_D) \smile \mathrm{Todd}(TM \otimes \mathbb{C}). \qquad \square$$

8. The Asymptotic Morphism

In this section we construct the asymptotic morphism

$$T^\omega\colon C_0(T^*M) \to \mathcal{K}(L^2(M))$$

used in the definition of the maps $\mathrm{Ind}_X\colon K(X \times T^*M) \to K(X)$.

Let U be an open subset of $\mathbb{R}^n$ and let $a(x,\xi)$ be a smooth, *compactly supported* function on T^*U. For $\omega \in [1,\infty)$ define an operator $T^\omega_a\colon L^2(U) \to L^2(U)$ by the formula

$$T^\omega_a f(x) = \int a(x,\omega^{-1}\xi)e^{ix\xi}\hat{f}(\xi)\,d\xi.$$

Thus

$$T^\omega_a f(x) = \int k^\omega_a(x,y) f(y)\,dy,$$

where

$$k^\omega_a(x,y) = \Big(\frac{\omega}{2\pi}\Big)^n \int a(x,\xi)e^{i\omega(x-y)\xi}\,d\xi.$$

Each T^ω_a is a compact operator.

We are interested in the asymptotic behaviour of the operators T^ω_a as $\omega \to \infty$ (compare [**17**]).

LEMMA 8.1. *The operators T_a^ω are uniformly bounded.*

PROOF. For $f, g \in L^2(U)$ the Cauchy-Schwarz inequality gives

$$\begin{aligned}
|(f, T_a^\omega g)|^2 &= |\iint \overline{f(x)} k_a^\omega(x,y) g(y)\, dxdy|^2 \\
&\leq \iint |f(x)|^2 |k_a^\omega(x,y)|\, dydx \cdot \iint |g(y)|^2 |k_a^\omega(x,y)|\, dxdy \\
&= \int |f(x)|^2 \left(\int |k_a^\omega(x,y)|\, dy \right) dx \cdot \int |g(y)|^2 \left(\int |k_a^\omega(x,y)|\, dx \right) dy.
\end{aligned}$$

It is easily verified that for every N,

$$|k_a^\omega(x,y)| \leq \text{constant} \cdot \omega^n / (1 + \omega|y - x|)^N.$$

Using polar coordinates and this estimate for $N = n + 1$ we get

$$\int |f(x)|^2 \left(\int |k_a^\omega(x,y)|\, dy \right) dx \leq \text{constant} \cdot \int |f(x)|^2 \left(\int_0^\infty \frac{r^{n-1}\omega^n}{(1+\omega r)^{n+1}}\, dr \right) dx,$$

where the term r^{n-1} comes from the change of variables formula. Substituting $\rho = \omega r$ we see that the integral is independent of ω (and of course finite). Treating the other iterated integral in a similar fashion we obtain

$$|(f, T_a^\omega g)|^2 \leq \text{constant} \cdot \|f\|_2^2 \|g\|_2^2. \qquad \square$$

The following lemmas are proved by the same method.

LEMMA 8.2. *Suppose that $A^\omega : L^2(U) \to L^2(U)$ are operators of the form*

$$A^\omega f(x) = \int k^\omega(x,y) f(x)\, dy,$$

where

$$|k^\omega(x,y)| \leq \text{constant} \cdot \omega^n / (1 + \omega|y - x|)^{n+1}.$$

For $L > 0$ let A_L^ω be the operator with kernel

$$k_L^\omega(x,y) = \begin{cases} k^\omega(x,y) & \textit{if } |x-y| < L\omega^{-1} \\ 0 & \textit{if } |x-y| \geq L\omega^{-1} \end{cases}.$$

Then $\|A^\omega - A_D^\omega\| \to 0$ as $D \to \infty$, uniformly in ω. $\square$

LEMMA 8.3. *Let A^ω be as above, but suppose that*

$$|k^\omega(x,y)| \leq \text{constant} \cdot \omega^{n-1} / (1 + \omega|y - x|)^{n+1}.$$

Then $\|A^\omega\| \to 0$ as $\omega \to \infty$. $\square$

PROPOSITION 8.4.
(1) If $b(x,\xi)$ is another smooth, compactly supported function on T^*U then

$$T_a^\omega T_b^\omega - T_{ab}^\omega \to 0,$$

in the operator norm, as $\omega \to \infty$.
(2) Denote by a^* the complex conjugate of a. Then

$$T_{a^*}^\omega - (T_a^\omega)^* \to 0,$$

in the operator norm, as $\omega \to \infty$.

PROOF. It is easily checked that if the kernels of operators A^ω and B^ω satisfy the estimate of Lemma 8.2 then so do the kernels of $A^\omega B^\omega$. Because of this, along with Lemmas 8.3 and 8.4, it suffices to show that for any $L > 0$ the kernels of the operators $T_{ab}^\omega - T_a^\omega T_b^\omega$ are bounded by a multiple of ω^{n-1} on the set $|x-y| \le L/\omega$.
We have that

$$T_a^\omega T_b^\omega f(x) = \left(\frac{\omega}{2\pi}\right)^n \iint c_\omega(x,\xi) e^{i\omega(x-y)\xi} f(y)\, dy d\xi,$$

where

$$\begin{aligned} c_\omega(x,\xi) &= \left(\frac{\omega}{2\pi}\right)^n \iint a(x,\eta) b(z,\xi) e^{i\omega(x-z)(\eta-\xi)}\, dz d\eta \\ &= \left(\frac{\omega}{2\pi}\right)^n \int\int a(x,\xi+\eta) b(x+z,\xi) e^{-i\omega\eta z}\, dz d\eta. \end{aligned}$$

A simple special case of the stationary phase formula (see Lemma 7.7.3 of [**10**]) gives us

$$|c_\omega(x,\xi) - a(x,\xi)b(x,\xi)| \le \text{constant } \cdot \omega^{-1}. \tag{8.1}$$

Now, the kernel of $T_{ab}^\omega - T_a^\omega T_b^\omega$ is

$$\left(\frac{\omega}{2\pi}\right)^n \int \{a(x,\xi)b(x,\xi) - c_\omega(x,\xi)\} e^{i\omega(x-y)\xi}\, d\xi,$$

and by (8.1), together with the fact that $c_\omega(x,\xi)$ is uniformly compactly supported, this is bounded by a multiple of ω^{n-1}, as required.
Part (2) (which is much easier) is proved in a similar fashion by calculating the kernel of $T_{a^*}^\omega - (T_a^\omega)^*$ and applying Lemmas 8.2 and 8.3. $\square$

THEOREM 8.5. *There is an asymptotic morphism $T^\omega \colon C_0(T^*U) \to \mathcal{K}(L^2(U))$ such that*

$$\|T^\omega(a) - T_a^\omega\| \to 0,$$

as $\omega \to \infty$, for all $a \in C_c^\infty(T^*U)$. $\square$

PROOF. We start from the fact that a $*$-homomorphism from the algebra $C_c^\infty(T^*U)$ into any C^*-algebra is automatically continuous in the sup norm, and so extends to $C_0(T^*U)$ (this is left to the reader).

Form the quotient $\mathcal{K}_\infty/\mathcal{K}_0$ of the algebra of bounded continuous functions from $[1,\infty)$ to $\mathcal{K}(L^2(U))$ by the ideal of functions which vanish at infinity. It is a C^*-algebra, and by Lemma 8.1 and Proposition 8.4 the correspondence $a \to T_a^\omega$ gives a $*$-homomorphism from $C_c^\infty(T^*U)$ into $\mathcal{K}_\infty/\mathcal{K}_0$. Composing the extension to $C_0(T^*U)$ with a continuous (but not necessarily multiplicative, or even linear) section $\mathcal{K}_\infty/\mathcal{K}_0 \to \mathcal{K}_\infty$ we get the desired asymptotic morphism. □

These considerations are easily generalized from open sets U to arbitrary smooth manifolds M by means of a partition of unity argument and the following calculations.

LEMMA 8.6. *Let f be a smooth, compactly supported function on U and denote by $M_f\colon L^2(U) \to L^2(U)$ the operator of pointwise multiplication by f. Then*

$$M_f T_a^\omega - T_a^\omega M_f \to 0$$

in the operator norm, as $\omega \to \infty$.

PROOF. The kernel of $M_f T_a^\omega - T_a^\omega M_f$ is $(f(x) - f(y))k^\omega(x,y)$. By the Mean Value Theorem, $|f(x)-f(y)| \le \text{constant} \cdot L/\omega$ when $|x-y| \le L/\omega$. So the kernel is bounded by a multiple of ω^{n-1} on the set $|x-y| \le D/\omega$, and Lemmas 8.2 and 8.3 apply. □

LEMMA 8.7. *Suppose that U, W are open subsets of $\mathbb{R}^n$ and that $\phi\colon W \to U$ is a diffeomorphism. Denote by $\tilde{\phi}\colon T^*W \to T^*U$ the induced diffeomorphism of cotangent bundles and denote by $U_\phi\colon L^2(U) \to L^2(W)$ the induced unitary isomorphism of Hilbert spaces. Then*

$$T^\omega_{a\circ\tilde{\phi}} - U_\phi T_a^\omega U_\phi^{-1} \to 0$$

in the operator norm as $\omega \to \infty$.

To explain the notation, we define, as in (4.2),

$$\tilde{\phi}(x,\xi) = (\phi(x), (\phi_*^{-1})^t \xi),$$

where ϕ_* denotes the derivative of ϕ, mapping tangent vectors at x to tangent vectors at $\phi(x)$, and $(\phi_*^{-1})^t$ denotes the transpose of its inverse, mapping cotangent vectors at x to cotangent vectors at $\phi(x)$. Also, we define

$$U_\phi f(x) = f(\phi(x)) \cdot J^{1/2}(x),$$

where $J(x)$ denotes the absolute value of the Jacobian of ϕ at x.

PROOF. By reducing U and W to smaller sets, if necessary, we may suppose that the deriviatives of ϕ and its inverse are bounded. Then the operators $U_\phi T_a^\omega U_\phi^{-1}$ are of the sort considered in Lemaa 8.2. So it suffices to show that for each $L > 0$ the kernel of $T^\omega_{a^\phi} - U_\phi T_a^\omega U_\phi^{-1}$ is $O(\omega^{n-1})$ on the set $|x-y| \le L/\omega$.

The kernel of $U_\phi T^\omega_a U_\phi^{-1}$ is

$$\left(\frac{\omega}{2\pi}\right)^n J(x)^{1/2} J(y)^{1/2} \int a(\phi(x),\xi) e^{i\omega(\phi(x)-\phi(y))\xi}\, d\xi.$$

On the other hand the kernel of $T^\omega_{a^\phi}$ is

$$\begin{aligned} k^\omega_{a^\phi}(x,y) &= \left(\frac{\omega}{2\pi}\right)^n \int a(\phi(x),(\phi_*^{-1})^t\xi) e^{i\omega(x-y)\xi}\, d\xi \\ &= \left(\frac{\omega}{2\pi}\right)^n J(x) \int a(\phi(x),\xi) e^{i\omega(x-y)\phi_*^t\xi}\, d\xi \\ &= \left(\frac{\omega}{2\pi}\right)^n J(x) \int a(\phi(x),\xi) e^{i\omega\phi_*(x-y)\xi}\, d\xi. \end{aligned}$$

The required estimate follows from the approximation

$$\phi(x)-\phi(y) = \phi_*(x-y) + O(|x-y|^2)$$

as $|x-y|\to 0$. $\square$

THEOREM 8.8. *Let M be a smooth manifold without boundary, and fix a smooth measure on M. There is an asymptotic morphism $T^\omega\colon C_0(T^*M)\to \mathcal{K}(L^2(M))$ such that if $\phi\colon W\to U$ is a diffeomorphism from an open set in M to $\mathbb{R}^n$ then*

$$\|T^\omega(a\circ\tilde\phi) - U_\phi T^\omega_a U_\phi^{-1}\| \to 0$$

as $\omega\to\infty$, for all $a\in C_c^\infty(T^*U)$. $\square$

REMARK. In the definition of $U_\phi\colon L^2(U)\to L^2(W)$ we include the appropriate Radon-Nikodym derivative, so as to make U_ϕ a unitary operator.

9. Completion of the Proof

Let M be a smooth closed manifold and let $D\colon C^\infty(M,E)\to C^\infty(M,F)$ be an elliptic operator with symbol $\sigma\colon \pi^*E\to\pi^*F$.

Put a smooth measure on M and metrics on E and F, and consider the formally self-adjoint operator

$$\boldsymbol{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \colon C^\infty(M,E\oplus F)\to C^\infty(E\oplus F).$$

Its symbol is the endomorphism

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix} \colon \pi^*(E\oplus F)\to \pi^*(E\oplus F)$$

considered in Section 4.

Basic elliptic operator theory tells us that there is a system of eigenvectors $\{u_n\}$ for the operator $\boldsymbol{D}$ on $C^\infty(M,E\oplus F)$ which constitute an orthonormal basis for $L^2(M,E\oplus F)$. The eigenvalues λ_n are real and converge to infinity, in absolute value, as $n\to\infty$.

Suppose now that $\alpha \in C_0(\mathbb{R})$. We may form the operators

$$\alpha(\omega^{-1}\boldsymbol{D})\colon L^2(M, E \oplus F) \to L^2(M, E \oplus F)$$

in the sense of spectral theory, so that

$$\alpha(\omega^{-1}\boldsymbol{D})u_n = \alpha(\omega^{-1}\lambda_n)u_n.$$

They are compact (since $|\lambda_n| \to \infty$).

On the other hand we may apply α to the symbol $\boldsymbol{\sigma}$ of $\boldsymbol{D}$. The endomorphism $\alpha(\boldsymbol{\sigma})$ so obtained vanishes at infinity (compare Lemma 4.1).

Viewing E and F as summands of trivial bundles $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, we may regard $\alpha(\boldsymbol{\sigma})$ as an endomorphism of the trivial bundle $\mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}$ on T^*M, by setting it to be zero on the complement of $\pi^*E \oplus \pi^*F$. So thought of it is a matrix valued function on T^*M, and we can apply T^ω to it.

Similarly, we may view $\alpha(\omega^{-1}\boldsymbol{D})$ as an operator on $L^2(M, \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2})$ by setting it to be zero on the complement of the subspace $L^2(M, E \oplus F)$ of $L^2(M, \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2})$.

The following key result links the spectral theory of $\boldsymbol{D}$ to the asymptotic morphism of the previous section. We shall only outline a proof.

Proposition 9.1. *If D is an elliptic operator on M with symbol σ then*

$$T^\omega(\alpha(\boldsymbol{\sigma})) - \alpha(\omega^{-1}\boldsymbol{D}) \to 0$$

as $\omega \to \infty$, for every $\alpha \in C_0(\mathbb{R})$.

Proof (sketch). ONe verifies that as $\omega \to \infty$ the operator $M_f\alpha(\omega^{-1}\boldsymbol{D})$ depends, asymptotically, only on the coefficients of $\boldsymbol{D}$ in a neighbouhood of $supp(f)$ (see [**9**], Lemma 2.4). Furthermore it follows from the basic elliptic estimates that $\alpha(\omega^{-1}\boldsymbol{D})$ varies continuously with the coefficients of $\boldsymbol{D}$. Using these facts and a partition of unity argument we reduce the Lemma to an analogous one for constant coefficient operators, for which $\alpha(\omega^{-1}\boldsymbol{D})$ may be computed explicitly using the Fourier transform. □

Theorem 9.2. $\mathrm{Ind}_{\mathrm{pt}}(\sigma_D) = \mathrm{Index}(D)$.

Proof. For $0 < \omega < \infty$ form the Cayley tranform

$$\begin{aligned}\boldsymbol{U}^\omega &= (\omega^{-1}\boldsymbol{D} + i)(\omega^{-1}\boldsymbol{D} - i)^{-1} \\ &= I + 2i(\omega^{-1}\boldsymbol{D} - i)^{-1}\end{aligned}$$

Extend it to a unitary operator on $L^2(M, \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2})$ by setting it equal to the identity on the complement of $L^2(M, E \oplus F)$. If $\boldsymbol{u}$ denotes the Cayley transform of $\boldsymbol{\sigma}$ then it follows from Proposition 9.1 that

$$\boldsymbol{U}^\omega - T^\omega(\boldsymbol{u}) \to 0,$$

as $\omega \to \infty$. Therefore

$$\mathrm{Ind}_{\mathrm{pt}}(\sigma_D) = [\boldsymbol{p}(\boldsymbol{\varepsilon})] - [\boldsymbol{p}(\boldsymbol{\varepsilon} \boldsymbol{U}^1)] \in K(\mathcal{K})$$

where $\boldsymbol{p}(\boldsymbol{\varepsilon})$ and $\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{U}^1)$ are the projections associated to the involutions $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}\boldsymbol{U}^1$.

We consider now what happens as $\omega \to 0$. The operator $\boldsymbol{U}^\omega$ converges in norm to minus the identity on the kernel of $\boldsymbol{D}$, and the identity on the complement. So the projection $\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{U}^\omega)$ converges to the projection

$$\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{U}^0) = \boldsymbol{P} - P_{ker(D)} + P_{ker(D^*)}$$

where the last two terms are the projections onto the kernels of D and D^*. Therefore

$$[\boldsymbol{p}(\boldsymbol{\varepsilon})] - [\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{U}^1)] = [P_{ker(D)}] - [P_{ker(D^*)}],$$

which proves the theorem. □

It remains to prove part (4) of Theorem 5.1.

Let V be a Euclidean vector space with basis $\{e_1, \ldots, e_n\}$ and correpsonding coordinates $x_i(v) = (v, e_i)$. Define operators

$$\boldsymbol{B}^\omega \colon \mathcal{S}(V, \textstyle\bigwedge^*_{\mathbb{C}} V) \to \mathcal{S}(V, \textstyle\bigwedge^*_{\mathbb{C}} V)$$

on Schwartz space by the formula

$$\boldsymbol{B}^\omega = \sum \frac{1}{\sqrt{-1}\omega} c(\sqrt{-1}e_j)\partial/\partial x_j + x_j c(e_j),$$

where $c(\sqrt{-1}e_j)$ and $c(e_j)$ are as in Section 4. The definition does not depend on the choice of basis.

The spectral theory of $\boldsymbol{B}^\omega$ is easily worked out:

LEMMA 9.3. *There is a system of eigenfunctions $\{u_n\}$ for $\boldsymbol{B}^\omega$ consistuting an orthonormal basis for $L^2(V, \bigwedge^*_{\mathbb{C}} V)$. The eigenvalues are real and converge to infinity in absolute value as $n \to \infty$. The kernel of $\boldsymbol{B}^\omega$ is one dimensional and is spanned by the 0-form $e^{-\omega|x|^2}$.*

PROOF. (See [**9**], Section 5.) Upon squaring $\boldsymbol{B}^\omega$ we obtain

$$(\boldsymbol{B}^\omega)^2 = -\omega^{-2}\Delta + |x|^2 + \omega^{-1}N,$$

where Δ is the Laplacian, $|x|^2$ denotes pointwise multiplication by the scalar function $|x|^2$, and N is the operator which multiplies a form of degree j by $2j - n$. So $(\boldsymbol{B}^\omega)^2$ is a direct sum of harmonic oscillators $-a\Delta + b|x|^2 + c$ whose spectral theory is well known from elementary quantum mechanics. □

Needless to say, our interest in $\boldsymbol{B}^\omega$ lies in its relation to the "symbol" $\boldsymbol{c}$ constructed in Section 4.

LEMMA 9.4. *For every* $\alpha \in C_0(\mathbb{R})$, $T^{\omega}_{\alpha(\boldsymbol{c})} - \alpha(\boldsymbol{B}^{\omega}) \to 0$ *as* $\omega \to \infty$. $\square$

This may be proved either by an approximation argument, as in Propsition 9.1, or by a direct calculation, based on Mehler's formula for the kernel of the operator $e^{-(\boldsymbol{B}^{\omega})^2}$.

THEOREM 9.5. $\operatorname{Ind}_M(\lambda_M) = 1$.

PROOF. For $m \in M$ let

$$\boldsymbol{U}_m \colon L^2(TM_m, \textstyle\bigwedge^*_{\mathbb{C}} TM_m) \to L^2(TM_m, \textstyle\bigwedge^*_{\mathbb{C}} TM_m)$$

be the Cayley transform of the operator $\boldsymbol{B} = \boldsymbol{B}^1$, and let

$$\varepsilon_m \colon L^2(TM_m, \textstyle\bigwedge^*_{\mathbb{C}} TM_m) \to L^2(TM_m, \textstyle\bigwedge^*_{\mathbb{C}} TM_m)$$

be the grading operator which multiplies a form by ± 1 according as its degree is even or odd. As usual, form the projections $\boldsymbol{p}(\boldsymbol{\varepsilon}_m)$ and $\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{U}_m)$.

Using the exponential map, identify TM_m with a neighbourhood W_m of m in M (compare (4.1)), and so view $L^2(TM_m, \bigwedge^*_{\mathbb{C}} TM_m)$ as a subspace of $L^2(M, \bigwedge^*_{\mathbb{C}} TM_m)$. By complementing the bundle TM over M we can view $L^2(M, \bigwedge^*_{\mathbb{C}} TM_m)$ as a subspace of the fixed Hilbert space

$$L^2(M, \mathbb{C}^N) = L^2(M) \oplus \dots L^2(M).$$

In this way, $\boldsymbol{p}(\boldsymbol{\varepsilon}_m)$ and $\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{U}_m)$ become projection valued functions from M to $M_N(\mathcal{K}^+)$

Using Lemma 9.4 and the invariance under diffeomorphism of T^{ω} (Lemma 8.7) we see that $\operatorname{Ind}_M(\lambda_M)$ is represented by the difference of projections

$$[\boldsymbol{p}(\boldsymbol{\varepsilon})] - [\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{U}] \in K(\mathcal{K}(M)).$$

In order to calculate this difference we use a homotopy similar to the one in Theorem 9.2, replacing $\boldsymbol{B}$ with $t^{-1}\boldsymbol{B}$ and letting $t \to 0$. Bearing in mind the calculation of the kernel of $\boldsymbol{B}$ we see that the Cayley transform of $\omega\boldsymbol{B}$ converges to the operator $\boldsymbol{U}^0$ which is is -1 on the 0-form $e^{-\|v\|^2}$ and $+1$ on its orthogonal complement. Therefore

$$\begin{aligned} \operatorname{Ind}_M(\lambda_M) &= [\boldsymbol{p}(\boldsymbol{\varepsilon})] - [\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{U}] \\ &= [\boldsymbol{p}(\boldsymbol{\varepsilon})] - [\boldsymbol{p}(\boldsymbol{\varepsilon}\boldsymbol{U}^0] \\ &= [p], \end{aligned}$$

where $p(m)$ is the projection onto the subspace spanned by the 0-form

$$e^{-|x|^2} \in L^2(TM_m, \textstyle\bigwedge^*_{\mathbb{C}} TM_m) \subset L^2(M, \mathbb{C}^N).$$

The "rotation"

$$\begin{pmatrix} \sin^2(\theta)p(m) & \sin(\theta)\cos(\theta)r(m) \\ \sin(\theta)\cos(\theta)r(m) & cos^2(\theta)e \end{pmatrix},$$

where e is the projection onto the subspace spanned by a fixed $v \in L^2(M)$ and $r(m)$ is the partial isometry mapping v to $e^{-|x|^2} \in L^2(TM_m, \bigwedge^*_{\mathbb{C}} TM_m)$ (appropriately normalized), shows that $[p] = [e]$ in K-theory. Bearing in mind the form of the isomorphism from $K(\mathcal{K}(M))$ to $K(M)$ we see that

$$\mathrm{Ind}_M(\lambda_M) = 1,$$

as required. $\square$

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