The Baum-Connes Conjecture

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Abstract. The report below is a short account of past and recent work on a conjecture of P. Baum and A. Connes about the K-theory of group C^*-algebras.

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1. Introduction. Let G be a second countable, locally compact group. The Baum-Connes conjecture [6,7] proposes a means of calculating K-theory for the reduced C^*-algebra of G using group homology and the representation theory of compact subgroups. It originates in work of Kasparov [19] and Mishchenko [25] on the Novikov higher signature conjecture, ideas of Connes in foliation theory [10], and Baum's geometric description of K-homology theory [8]. The validity of the conjecture has implications in geometry and topology, most notably the Novikov conjecture [14] and the 'stable' Gromov-Lawson-Rosenberg conjecture [30] about positive scalar curvature manifolds. In addition there appear to be close connections to issues in harmonic analysis, for instance the problem of finding explicit realizations of discrete series representations [5]. Indeed a very striking feature of the conjecture is its generality, and the breadth of mathematics with which it makes contact.

The conjecture was first set forth in a 1982 article of Baum and Connes [6], which was unfortunately never published;[1] its current formulation was given in [7]. The last several years have seen a considerable clarification and development of the relationship between the Baum-Connes conjecture and topology, thanks largely to insights of Weinberger [33] linking index theory to surgery theory. A recent monograph of Roe [28] describes the current state of affairs here. Recent progress on the conjecture itself will be described in Section 7 below. A major obstacle to further progress is the lack of a full understanding of the relationship between harmonic analysis and the Baum-Connes conjecture. It seems likely that underlying the conjecture is an as yet unknown governing principle of harmonic analysis. But the conjecture has not drawn the attention of harmonic analysts the way it has the topologists, and this issue remains largely unexamined. This report ends in Section 8 with a few very tentative remarks on the problem.

[1]In fact it *will* be published in the near future, after a 16 year delay.

2. GROUP C*-ALGEBRAS. Denote by $L^1(G)$ the convolution algebra of integrable, complex-valued functions on the locally compact group G. There is a natural involution on $L^1(G)$, making it into a Banach $*$-algebra, and it is easy to see that the non-degenerate $*$-representations of $L^1(G)$ on Hilbert space are in one-to-one correspondence with the unitary representations of G. The group C^*-algebra of G, denoted $C^*(G)$, is the enveloping C^*-algebra of the $L^1(G)$. Its representations are also in one-to-one correspondence with the unitary representations of G, and so both $L^1(G)$ and $C^*(G)$ offer the possibility of a functional-analytic approach to the unitary representation theory and harmonic analysis of G. See [13, Chapter 13].

The group C^*-algebra is particularly well adapted to problems in which the unitary dual $\hat{G}$ [13] is viewed not as a set but as a topological space. Kazhdan's property T [23] offers a good illustration of this. It is equivalent to the assertion that there exists in $C^*(G)$ a projection p whose image in any unitary representation of G is the orthogonal projection onto the G-fixed vectors. In effect p is the continuous function on $\hat{G}$ which is 1 on the trivial representation and zero on its complement. It does not belong to $L^1(G)$ unless G is compact, so generally the disconnectedness of $\hat{G}$ is not simply reflected in the L^1-algebra.

If G is abelian then the Fourier transform provides an isomorphism from $C^*(G)$ to the commutative C^*-algebra of continuous functions on the Pontrjagin dual which vanish at infinity. So in this case $C^*(G)$ precisely captures the topological structure of $\hat{G}$. If G is non-abelian then the ordinary topological structure on the unitary dual is typically very poor (for instance if G is discrete then $\hat{G}$ is a T_0 space only when G is virtually abelian). But following a point of view emphasized by Connes [11] it is now standard in operator algebra theory to think of the noncommutative C^*-algebra $C^*(G)$ as an algebra of continuous functions on $\hat{G}$ which amplifies the classical topological structure of $\hat{G}$.

3. C*-ALGEBRA K-THEORY AND INDEX THEORY. The K-theory groups of a C^*-algebra A are defined in such a way that if A is the C^*-algebra $C_0(X)$ of continuous, complex-valued functions, vanishing at infinity, on a locally compact space then $K_j(A)$ is the Atiyah-Hirzebruch K-theory group $K^{-j}(X)$. See [11] and [19] for overviews of the subject, and the references therein for more details. Following the principle that $C^*(G)$ substitutes for $\hat{G}$, operator algebraists view $K_j(C^*(G))$ as a substitute for $K^{-j}(\hat{G})$. Of course, if G is abelian then thanks to the Fourier isomorphism the formula $K_j(C^*(G)) \cong K^{-j}(\hat{G})$ is not only a point of view but actually a theorem.

There is a direct link between the K-theory of $\hat{G}$ and the index theory of elliptic operators. Suppose that M is a smooth closed manifold and that D is an elliptic partial differential operator on M. It has an integer-valued Fredhom index, but if $\pi_1(M)$ is provided with a homomorphism into a discrete group G then a more refined index, valued in $K_0(C^*(G))$, can be defined as follows. The quotient of $\tilde{M} \times C^*(G)$ by the diagonal action of $\pi_1(M)$ is a flat bundle over M whose fibers are finitely-generated projective modules over $C^*(G)$. If D_G denotes the canonical lifting of D to act on sections of this flat bundle then both kernel(D_G) and cokernel(D_G) are $C^*(G)$-modules. In favorable circumstances they are finitely generated and projective, and one defines

$$\mathrm{Index}_G(D) = [\mathrm{kernel}(D_G)] - [\mathrm{cokernel}(D_G)] \in K_0(C^*(G)).$$

In general kernel(D_G) and cokernel(D_G) may be perturbed so as to become finitely generated and projective, and $\mathrm{Index}_G(D)$ is defined by means of such a perturbation. For abelian groups this construction is due to Lusztig [24]. It was generalized to arbitrary G by Mischenko and, independently, Kasparov.

The ordinary Fredholm index of D can be recovered in an interesting way from $\mathrm{Index}_G(D)$ by noting first that any trace on a C^*-algebra A defines a functional on $K_0(A)$ [11], and then that $C^*(G)$ has a natural trace τ associated to the regular representation of G. It may be shown that $\tau[\mathrm{Index}_G(D)] = \mathrm{Index}(D)$; this is essentially a reformulation of Atiyah's index theorem for covering spaces [3]. A less interesting, but simpler, method is to note that the trivial representation of G also defines a trace τ_0 on $C^*(G)$. It is more or less a tautology that $\tau_0[\mathrm{Index}_G(D)] = \mathrm{Index}(D)$.

If G is a finite group then $\mathrm{Index}(D)$ is the only information within $\mathrm{Index}_G(D)$, but if G is infinite then $\mathrm{Index}_G(D)$ can contain a good deal more. The question of just what it contains is important for the following reason:

3.1. Proposition. [19, 29] *The G-index of the Dirac operator on a closed spin-manifold vanishes if the manifold has positive scalar curvature. The G-index of the signature operator on a oriented manifold is an oriented homotopy invariant.*

4. The Assembly Map. It is well known that a Dirac operator on a closed manifold M^n, combined with a map $M^n \to X$, determines a class in the K-homology group $K_n(X)$. This point was first emphasized by Atiyah [2], and an elegant development of it was given by Baum, who *realized* $K_n(X)$ as equivalence classes of triples (M, E, f), where M is a closed spin^c-n-manifold, E is a complex vector bundle on M, and $f\colon M \to X$ is a continuous map. Baum's equivalence relation involves a simple direct sum-disjoint union relation, bordism, and another relation related to the multiplicativity of the index of elliptic operators on fiber bundles. If $X = BG$ then a triple (M^{2n}, E, f) has an index in $K_0(C^*(G))$: form the Dirac operator on M with coefficients in E, and take its G-index along the map $\pi_1(M) \to G$ induced from f. The index depends only on the equivalence class of (M, E, f) and together with a related construction for odd-dimensional manifolds it defines a map

$$\mu\colon K_*(BG) \to K_*(C^*(G)).$$

This *assembly map,* so-called because of its connection with the assembly map of surgery theory [14], was first defined by Kasparov (*c.f.* [19]), although using Kasparov's own realization of K-homology rather than Baum's. He also formulated the following:

4.1. Strong Novikov Conjecture. *The assembly map $\mu\colon K_*(BG) \to K_*(C^*(G))$ is rationally injective.*

Thanks to Proposition 3,1, the Strong Novikov Conjecture implies the Novikov higher signature conjecture (hence its name) [14], which asserts that the class in

$K_n(BG) \otimes \mathbb{Q}$ of the signature operator on a closed, oriented manifold M^n is an oriented homotopy invariant. The Strong Novikov Conjecture also implies the 'stable' Gromov-Lawson-Rosenberg conjecture [30] on positive scalar curvature.

5. THE BAUM-CONNES CONJECTURE. From the point of view of applications to geometry and topology the Strong Novikov conjecture is the most important issue in C^*-algebra K-theory, but the problem nonetheless remains to calculate the K-theory of group C^*-algebras. The Baum-Connes conjecture expresses the idea that every class in the K-theory of a group C^*-algebra is an index, and that the only relations among elements are the natural relations (like bordism, and so on) among index theory problems. Actually the conjecture concerns the quotient of $C^*(G)$ corresponding to the closed subset $\hat{G}_{\mathrm{red}} \subset \hat{G}$ comprised of those unitary representations which are weakly contained in the regular representation. This *reduced* C^*-algebra of G, denoted $C^*_{\mathrm{red}}(G)$, is the completion of $L^1(G)$ in its regular representation as bounded operators on $L^2(G)$. The C^*-algebra $C^*_{\mathrm{red}}(G)$ coincides with $C^*(G)$ if and only if G is amenable [13, Chapter 18].

The Baum-Connes conjecture is most easily formulated for discrete groups without torsion. It has already been noted that if G is a finite group then the assembly map is essentially trivial. Furthermore a finite, nontrivial subgroup H in any discrete group G contributes a projection $1/|H| \sum_{h \in H} [h]$ to $C^*(G)$ whose K-theory class is not in the image of the assembly map. So the restriction to torsion-free groups in the following is certainly necessary:

5.1. BAUM-CONNES CONJECTURE FOR TORSION-FREE GROUPS. *If G is a discrete and torsion-free group then the assembly map*

$$\mu_{\mathrm{red}} \colon K_*(BG) \to K_*(C^*_{\mathrm{red}}(G)),$$

obtained from the assembly map μ of 4.1 *using the regular representation $C^*(G) \to C^*_{\mathrm{red}}(G)$, is an isomorphism.*

It is usual to cite Kazhdan's property T as the reason why $C^*_{\mathrm{red}}(G)$ is used in place of $C^*(G)$: the Kazhdan projection $p \in C^*(G)$, if it exists, defines a class in $K_0(C^*(G))$ which is not in the image of the assembly map μ (if G is infinite), since the traces τ and τ_0 of the last section disagree on it, whereas $\tau(x) = \mathrm{Index}(D) = \tau_0(x)$ for every class $x \in K_0(C^*(G))$ which is the G-index of an elliptic operator D.

In fact if G is infinite and has property T then *every* finite-dimensional, unitary representation of G determines a Kazhdan-type projection in $C^*(G)$ and in this way the entire character ring of finite-dimensional unitary representations of G embeds as a direct summand of $K_0(C^*(G))$. This ring is typically very large and very complicated, and it is not within the range of the assembly map. So the idea that μ (as opposed to μ_{red}) is an isomorphism is not only incorrect, it is hopelessly wrong.

The Kazhdan projection maps to zero in $C^*_{\mathrm{red}}(G)$ so these problems vanish for the reduced C^*-algebra and for μ_{red}. Of course this is not in itself a very powerful reason to believe in 5.1, which could fail for any number of reasons unrelated to property T. More compelling evidence will be presented in the next section.

The statement of the Baum-Connes conjecture for general (second-countable) locally compact groups uses Kasparov's equivariant KK-theory [21]. Associated to any G there is a proper G-space $\mathcal{E}G$, which is universal in the sense that any other proper G-space maps into it in a way which is unique up to equivariant homotopy [7]. Using Kasparov's KK-theory the equivariant K-homology $K_*^G(\mathcal{E}G)$ may be defined. If G is discrete and torsion free then $\mathcal{E}G$ is the universal principal space EG and $K_*^G(\mathcal{E}G) = K_*(BG)$. For general G there is an assembly map

$$\mu_{\text{red}}: K_*^G(\mathcal{E}G) \to K_*(C^*_{\text{red}}(G))$$

very similar to the one already considered: a cycle for $K_*^G(\mathcal{E}G)$ is an 'abstract' elliptic operator D on a proper G-space and μ_{red} associates to D its equivariant index.

5.2. Baum-Connes Conjecture. [7] *If G is any second countable, locally compact group then the assembly map μ_{red} is an isomorphism.*

6. Lie Groups. What is currently known about the Baum-Connes conjecture? Progress has been made in two ways: by representation-theoretic arguments which calculate both $K_*^G(\mathcal{E}G)$ and $K_*(C^*_{\text{red}}(G))$ explicitly as abelian groups, for certain G, and verify that μ_{red} is an isomorphism; and by K-theoretic arguments which construct an inverse to μ_{red} in certain other cases.

Perhaps the best evidence in favour of the Baum-Connes conjecture comes from the former method. If G is a connected Lie group and K is its maximal compact subgroup then the homogeneous space G/K is a universal proper G-space $\mathcal{E}G$. If, for simplicity, G/K is even-dimensional and admits a G-equivariant spin-structure then the Baum-Connes conjecture is equivalent to the assertion that the map

$$\tilde{\mu}_{\text{red}}: R(K) \to K_0(C^*_{\text{red}}(G)),$$

which associates to each representation $[V] \in R(K)$ the G-index of the twisted Dirac operator D_V on G/K, is an isomorphism of abelian groups, and that in addition, $K_1(C^*_{\text{red}}(G)) = 0$. See [7]. This is also known as the *Connes-Kasparov conjecture* for G [19,11].

If G is semisimple then a unitary representation of G is weakly contained in the regular representation if and only if it is tempered. If G is a *complex* semisimple group then there is a Morita equivalence $C^*_{\text{red}}(G) \sim C_0(\hat{G}_{\text{red}})$ [26] connecting $C^*_{\text{red}}(G)$ to the tempered dual $\hat{G}_{\text{red}}$, which in the complex case is a Hausdorff locally compact space. For general semisimple groups $\hat{G}_{\text{red}}$ is 'almost' Hausdorff and $C^*_{\text{red}}(G)$ is Morita equivalent to a C^*-algebra which is 'almost' $C_0(\hat{G}_{\text{red}})$. And while phenomena such as the reducibility of non-generic principal series representations and the non-vanishing of associated R-groups complicate matters, it is nonetheless possible (at least in the linear case) to explicitly compute $C^*_{\text{red}}(G)$ and the groups $K_*(C^*_{\text{red}}(G))$ [32]. It is further possible to calculate $\tilde{\mu}_{\text{red}}$ and verify that it is indeed an isomorphism. Each discrete series representation of G contributes a generator to K-theory, and following [5] these are accounted for as equivariant indices of twisted Dirac operators on G/K—in other words by elements in $R(K)$.

The proof [32] that μ_{red} is an isomorphism is a careful extension of these discrete series arguments. It covers a great deal of the territory of tempered representation theory: as yet there is no simple, conceptual proof.[2]

For certain Lie groups G, for instance $G = SO(n,1)$, it is known by the other methods that the Baum-Connes conjecture is true not only for G but for any discrete subgroup (see the next section). Since the representation-theoretic analysis of $\tilde{\mu}_{\text{red}}$ reveals nothing special about $SO(n,1)$, an optimistic extrapolation suggests that if a counterexample to the Baum-Connes conjecture exists, it ought not to be found among the discrete subgroups of semisimple groups.

7. THE DIRAC-DUAL DIRAC ARGUMENT. For most groups it is impossible to determine $\hat{G}_{\text{red}}$, or to otherwise directly compute $K_*(C^*_{\text{red}}(G))$. Kasparov has however devised a beautiful, indirect route to the Baum-Connes conjecture using his equivariant KK-theory [21].

If A and B are G-C^*-algebras then the Kasparov's $KK^G(A,B)$ is the group of morphisms from A to B in additive category which broadens in a certain sense the homotopy category of G-C^*-algebras and equivariant $*$-homomorphisms. The composition law

$$KK^G(A,B) \otimes KK^G(B,C) \to KK^G(A,C)$$

is called the *Kasparov product.* Key to Kasparov's approach to the Baum-Connes conjecture is the notion of a *proper* G-C^*-algebra [21,15], which is a mildly noncommutative generalization of the notion of a proper G-space (the algebra of continuous functions, vanishing at infinity, on a locally compact, proper G-space is the prototypical example of a proper G-C^*-algebra).

7.1. PROPOSITION. *Let G be a second-countable, locally compact group. If the elementary G-C^*-algebra $\mathbb{C}$ is KK^G-equivalent to a proper G-C^*-algebra then the Baum-Connes assembly map for G is an isomorphism.*

This result of Tu [31], which condenses Kasparov's method to its essence, may be interpreted as follows.[3] Using KK-theory a generalized assembly map

$$\mu_{\text{red},A}\colon KK^G_*(\mathcal{E}G, A) \to K_*(C^*_{\text{red}}(G,A))$$

can be defined, involving a 'coefficient' G-C^*-algebra A (here $C^*_{\text{red}}(G,A)$ denotes the crossed product C^*-algebra). If $A = \mathbb{C}$ then $\mu_{\text{red},A}$ is the assembly map of 5.2. If on the other hand A is proper then $\mu_{\text{red},A}$ should be an isomorphism, because proper actions are locally modeled by actions of compact groups, while the Baum-Connes conjecture is readily verified for compact groups. But if $\mathbb{C}$ is KK^G-equivalent to a proper G-C^*-algebra then as far as the assembly map is concerned $\mathbb{C}$ *is* a proper G-C^*-algebra, and the proposition follows.

[2] Similar remarks apply to p-adic groups. The Baum-Connes conjecture has been checked for p-adic $GL(n)$ by a detailed determination of $\hat{G}_{\text{red}}$, of $C^*_{\text{red}}(G)$, of $K^G_*(\mathcal{E}G)$, and of the assembly map for $GL(n)$ [9]. But the verification does not offer much insight into what underlies the Baum-Connes isomorphism.

[3] Tu's proof is somewhat different, but see [15] for a similar result which is proved along these lines.

The approach to the Baum-Connes conjecture through 7.1 is hereditary, in the sense that it proves the Baum-Connes conjecture not only for G but for any closed subgroup of G at the same time. The method has been successfully applied to connected, amenable Lie groups [21], to $SO(n,1)$ [20], to $SU(n,1)$ [18], and most recently to groups which admit continuous, isometric *affine* actions on Hilbert space which are proper in the sense that $\|g \cdot v\| \to \infty$ as $g \to \infty$, for every vector v [16,17]. This last class includes all the previous ones, *all* (second countable) amenable groups, and all Coxeter groups. It seems to push Kasparov's method almost as far as it can go.

Thanks to important work of Pimsner concerning group actions on trees [27] it is also known that the groups to which 7.1 applies are closed under operations like amalgamated free products [31]. As far as discrete groups are concerned, nothing more is known, and in particular the conjecture is known for no infinite, discrete, property T group. Indeed, a feature of Kasparov's method, as summarized in 7.1, is that it treats $C^*(G)$ and $C^*_{\mathrm{red}}(G)$ equally, and proves that $K_*(C^*(G)) \cong K_*(C^*_{\mathrm{red}}(G))$ (in the language of KK-theory, G is K-amenable).

The 'Dirac-dual Dirac' terminology comes from Kasparov's original work on connected Lie groups. If $M = G/K$ then the G-C^*-algebra $A = C_0(TM)$ is proper. The Dirac operator on TM defines an element $\alpha \in KK^G(A,\mathbb{C})$, while a class $\beta \in KK^G(\mathbb{C}, A)$ may be defined which is, in the case $G = \mathbb{R}^n$, closely related to the Fourier transform of the Dirac operator. Kasparov was able to show that, for any G, $\beta \circ \alpha = 1 \in KK^G(A,A)$ (the argument is closely related to Atiyah's elliptic operator proof of Bott periodicity [1]), but not in general that $\alpha \circ \beta = 1 \in KK^G(\mathbb{C},\mathbb{C})$. Indeed, thanks to property T the latter is false in general. But the former is enough to imply the split injectivity of the assembly map for G and its discrete subgroups, a result which was subsequently extended by Kasparov and Skandalis to p-adic groups [22], and others. Injectivity of μ_{red} for a discrete group G implies the Novikov higher signature conjecture.

8. Representation Rings. Let G be a compact group and denote by $R(G)$ the its character ring. Atiyah, Hirzebruch and Segal [4] defined and studied an interesting ring homomorphism

$$\nu \colon R(G) \to K(BG),$$

connecting $R(G)$ to the representable K-theory of the classifying space BG. In his work on the Novikov conjecture [19,21] Kasparov defined a similar homomorphism for any locally compact group. It is in some sense dual to the assembly map.

A (unitary) *Fredholm representation* of a locally compact group G consists of two separable Hilbert spaces H_0 and H_1, equipped with unitary representations of G, and a bounded Fredholm operator $F \colon H_0 \to H_1$ for which $g(F) - F$ is a compact operator-valued and norm-continuous function of $g \in G$. The *Kasparov representation ring* of G, denoted $R(G)$, is the abelian group of homotopy equivalence classes of Fredholm representations of G. As its name suggests, $R(G)$ is a ring. In fact $R(G) = KK_G(\mathbb{C},\mathbb{C})$ and the ring structure is a special case of the Kasparov product.

The ring of finite-dimensional unitary representations of G maps into Kasparov's $R(G)$, and if G is compact then this map is an isomorphism: its inverse is defined by averaging over G any Fredholm representation to make the operator F exactly equivariant, and then taking the G-index of F, which is a formal difference of finite-dimensional unitary representations.

Kasparov's generalized Atiyah-Hirzebruch-Segal map $\nu\colon R(G) \to K(BG)$ takes a Fredholm representation $F\colon H_0 \to H_1$ and associates to it an equivariant family of Fredolm operators $F_x\colon H_0 \to H_1$, parametrized by the universal space EG, characterized up to equivariant homotopy by the property that each F_x is a compact perturbation of F. It follows from Kuiper's theorem that $K(BG)$ may be described as equivariant homotopy classes of equivariant Fredholm families on EG. Actually, for the present purposes it is better to consider the equivariant K-theory group $K_G(\mathcal{E}G)$, defined from equivariant homotopy classes of equivariant Fredholm families on $\mathcal{E}G$. Kasparov's prescription then defines a map

$$\nu\colon R(G) \longrightarrow K_G(\mathcal{E}G).$$

If G is compact then, unlike the Atiyah-Hirzebruch-Segal map, this is tautologically an isomorphism; if G is a connected Lie group then ν identifies with the restriction map $R(G) \to R(K)$.

If G is any group to which 7.1 applies then the map $\nu\colon R(G) \to K_G(\mathcal{E}G)$ is a ring isomorphism. If G is a Lie group and if $\gamma \in R(G)$ is the Kasparov product $\alpha \circ \beta \in KK^G(\mathbb{C}, \mathbb{C})$ considered in the last section then γ is an idempotent and Kasparov showed that the map ν takes $\gamma \cdot R(G)$ isomorphically to $K_G(\mathcal{E}G)$. So if ν is an isomorphism then $\gamma = 1$ and the Baum-Connes conjecture follows. On the other hand, if G has property T then certainly $\gamma \neq 1$ and so ν is not an isomorphism.

To explore this issue further it is worth contemplating a *non-unitary* Kasparov representation ring $R_{\text{n.u.}}(G)$, comprised of homotopy classes of *non-unitary* Fredholm representations (they are defined more or less as before, but the representations of G on H_0 and H_1 are required only to be continuous, not unitary). Considering non-unitary representations makes no change to $K_G(\mathcal{E}G)$, and one can proceed to construct and study the natural homomorphism $\nu\colon R_{\text{n.u.}}(G) \to K_G(\mathcal{E}G)$.

8.1. PROPOSITION. *If $G \subset SL(n, \mathbb{R})$ then the homomorphism $\nu\colon R_{\text{n.u.}}(G) \to K_G(\mathcal{E}G)$ is a ring isomorphism.*

The proof hinges on showing that the image of $\gamma \in R(G)$ in $R_{\text{n.u.}}(G)$ is 1. It can likely be generalized to any Lie group.

Suppose now that $C^*_{\text{red}}(G)$ is put to one side for a moment and in its place is considered a Banach, or Frechet, algebra tailored to the full (not necessarily unitary) representation theory of G. For instance for a finitely generated discrete group one might consider

$$\mathcal{S}(G) = \{\, f\colon G \to \mathbb{C} \;:\; \sum |f(g)| A^{|g|} < \infty,\ \forall A > 0 \,\},$$

where $|g|$ denotes the word length of $g \in G$. A Baum-Connes type assembly map

$$\mu_{\text{n.u.}}\colon K^G_*(\mathcal{E}G) \longrightarrow K_*(\mathcal{S}(G))$$

may be defined and it is reasonable to guess that 8.1 will imply that $\mu_{\mathrm{n.u.}}$ is an isomorphism (for Lie groups). The proof should be an adaptation of Kasparov's Dirac-dual Dirac argument to the context of Frechet algebras and non-unitary representations. The tools for carrying out such an argument—most important among them a serviceable KK-theory for non-C^*-algebras—are now coming into being [12], and it is likely that this guess about $\mu_{\mathrm{n.u.}}$ is not far from a theorem. Granted this, the Baum-Connes isomorphism for $C^*_{\mathrm{red}}(G)$ reduces to a sort of restriction isomorphism

$$K_*(\mathcal{S}(G)) \underset{\cong}{\longrightarrow} K_*(C^*_{\mathrm{red}}(G))$$

identifying K-theoretic invariants of the 'non-unitary dual' of G with the same for the reduced, or tempered, dual.

It might be thought that the inclusion of the Frechet algebra $\mathcal{S}(G)$ as a dense subalgebra of $C^*_{\mathrm{red}}(G)$ induces an isomorphism in K-theory by an elementary approximation argument like the one which shows that K-theory for manifolds, defined using smooth vector bundles, is the same as topological K-theory. Unfortunately the approximation arguments which are known do not apply: even at the cohomological level of K-theory a good deal of analysis $\mathcal{S}(G)$ seems to separate from $C^*_{\mathrm{red}}(G)$. On the other hand the idea that the non-unitary representation theory of G is describable, or parametrizable, in terms of the tempered dual is not foreign to harmonic analysis. At the present time a closer analysis of this point and its relation to K-theory seems to offer the best chance of broad progress on the Baum-Connes conjecture.

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