Index Theory and Noncommutative Geometry

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Introduction

I'm going to talk about developments in operator algebra theory, topology and representation theory that took place following the work of Atiyah and Singer in index theory, and certainly wouldn't have existed without it.

Alain Connes uses the term noncommutative geometry to refer to the application of Hilbert space techniques, especially spectral theory, to geometric problems.

I shall focus on the part of noncommutative geometry where the influence of index theory and K-theory has been the strongest.

The lead actor here has been Gennadi Kasparov, and the main theme has been the treatment of the space of irreducible unitary representations of a group as a geometric space (typically very singular space).

Where this story starts . . .

Here is a paper of Atiyah from 1969:

> **Global theory of elliptic operators**
>
> By M. F. Atiyah
>
> **§ 1. Fredholm operators and *K*-theory.**
>
> In recent years the problem of computing the index of an elliptic differential operator on a closed manifold (or on a manifold with boundary) has been successfully solved [3] by the use of a new branch of algebraic topology (K-theory). Further investigation has brought to light many fundamental connections between elliptic operators and K-theory and in this lecture I want to present a new view-point on these matters.
>
> The best place to start is with abstract functional analysis which provides the natural meeting ground of algebraic topology and partial differential equations...

Atiyah's paper was soon connected with work of Brown, Douglas and Fillmore in operator theory (discussed in Atiyah's *Unity of Mathematics* lecture), and expanded upon by Kasparov.

EXTENSIONS OF C^*-ALGEBRAS, OPERATORS WITH COMPACT SELF-COMMUTATORS, AND K-HOMOLOGY

BY L. G. BROWN, R. G. DOUGLAS, AND P. A. FILLMORE

Communicated by I. M. Singer, March 12, 1973

1. **Introduction.** The study of a certain class of extensions of C^*-algebras is suggested by recent developments in two diverse areas of mathematics. Starting from the classical results of Weyl and von Neumann on compact perturbations of selfadjoint operators, operator theorists have become increasingly interested in operators which are normal modulo the comp
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TOPOLOGICAL INVARIANTS OF ELLIPTIC OPERATORS. I: K-HOMOLOGY

G. G. KASPAROV

Abstract. In this paper the homological K-functor is defined on the category of involutory Banach algebras, and Bott periodicity is proved, along with a series of theorems corresponding to the Eilenberg-Steenrod axioms. As an application, a generalization of the Atiyah-Singer index theorem is obtained, and some problems connected with representation rings of discrete groups and higher signatures of smooth manifolds are discussed.

Brown, Douglas and Fillmore came upon K-theory and K-homology by surprise while solving a problem in operator theory.

Kasparov took Atiyah's ideas, resolved some open issues, and (eventually) applied Atiyah's construction to manifold topology.

Fredholm Operators

One Fredholm operator

A linear transformation $F\colon V \to W$ between vector spaces is a Fredholm operator if kernel(F) and cokernel(F) are finite-dimensional, in which case the Fredholm index is

$$\mathrm{Index}(F) = \dim\big(\mathrm{kernel}(F)\big) - \dim\big(\mathrm{cokernel}(F)\big)$$

In finite dimensions

$$\mathrm{Index}(F) = \dim\big(V\big) - \dim\big(W\big)$$

This is the rank-nullity theorem.

In infinite dimensions the index continues to have remarkable stability properties. For instance the index is a locally constant function on bounded Fredholm operators between Hilbert spaces.

Families of Fredholm operators

The Fredholm index tells us something about the topology of the space of all Fredholm operators, since distinguishes connected components of this space.

In fact $\pi_0(\mathrm{Fred}(H)) \cong \mathbb{Z}$ via the Fredholm index. Moreover

$$\pi_j(\mathrm{Fred}(H)) \cong \begin{cases} \mathbb{Z} & j \text{ even} \\ 0 & j \text{ odd} \end{cases}$$

This is a consequence of Bott periodicity and the Atiyah-Janich theorem:

Theorem (Atiyah and Janich)

If X is any compact space, then

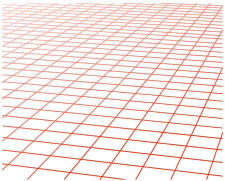
$$K(X) \cong [X, \mathrm{Fred}(H)]$$

A fundamental example

According to the Atiyah-Janich theorem, an element of K-theory is the same thing as a continuous family of Fredholm operators, up to homotopy.

Specific examples of Fredholm families come from a variety of places, and there are interesting and important families that involve only operators on finite-dimensional Hilbert spaces.

For instance, take X to be the complex plane and consider the family of operators



$$F_z \colon H \longrightarrow H$$
$$F_z \colon v \mapsto zv$$

from a one-dimensional Hilbert space H to itself.

Note that this family cannot be deformed to a family of invertible operators through any bounded deformation.

A bit more about this example

It is worthwhile understanding the above example a bit better, since it is central to much of what follows.

The parameter space $X = \mathbb{C}$ in the example is not compact, but this is easily fixed. Embed $\mathbb{C}$ into $\mathbb{CP}^1$ in the usual way,

$$\mathbb{C} \to \mathbb{CP}^1, \qquad z \mapsto [z : 1]$$

Identify the restriction of the tautological bundle L over $\mathbb{CP}^1$ to the complex plane with the trivial bundle with fiber H (the one-dimensional Hilbert space) by

$$L_{[z:1]} \ni (wz, w) \mapsto wv_0 \in H \qquad (v_0 \text{ a basis vector in } H)$$

Then $\{F_z\}$ above extends to a (Fredholm) family from the fibers of the tautological bundle L to the fibers of the trivial line bundle over $\mathbb{CP}^1$ with fiber H. In K-theory we get, via Atiyah-Janich,

$$[L] - [1] \in K(\mathbb{CP}^1)$$

which is the Bott generator of the K-theory of a 2-sphere.

Fredholm families from Kasparov's point of view

I'm going to explain how Kasparov defines continuous families of Fredholm operators, because his definition opens up an enormous range of possibilities . . .

Theorem (Atkinson)

A bounded Hilbert space operator is a Fredholm operator if and only if it is invertible modulo compact operators.

The compact operators are the norm-limits of finite-rank operators, or equivalently the norm limits of linear combinations of rank-one operators

$$v \mapsto v'\langle v'', v\rangle$$

with $v', v'' \in H$.

Because of Atkinson's theorem, the emphasis for Kasparov is on defining appropriate families of compact operators.

Continuous fields of Hilbert spaces and compact operators

A *continuous field of Hilbert spaces* over a locally compact space X is a collection of Hilbert spaces H_x indexed by the points of X, together with a vector space of sections, called continuous, that are required satisfy some simple axioms, the most important being

- The pointwise inner product of any two continuous sections is a continuous scalar function vanishing in norm at infinity.
- The sections constitute a module over $C_0(X)$, the continuous scalar functions vanishing at infinity.

Definition. The compact operators on the field are the norm-limits of linear combinations of operators

$$v_x \longmapsto s'_x \langle s''_x, v_x \rangle \qquad (x \in X)$$

where s', s'' are continuous sections as above.

Fredholm operators

Definition. A Fredholm operator on a continuous field is an (appropriately continuous) family of operators that is invertible modulo compact operators.

Example. Every family of invertible operators is homotopic, through families of invertible operators, to $0\colon 0 \to 0$. The homotopy (a family over $X \times [0,1]$) is

$$H_{(x,t)} = \begin{cases} H_x & t \neq 0 \\ 0 & t = 0 \end{cases} \qquad F_{(x,t)} = \begin{cases} F_x & t \neq 0 \\ 0 & t = 0 \end{cases}$$

Example. Every family of Fredholm operators over a compact space X is homotopic to an esentially unique family

$$0\colon E_x \to E'_x$$

associated to vector bundles E and E' over X.

Fredholm operators over (noncommutative) C*-algebras

The continuous sections of a continuous field of Hilbert spaces constitute a module over $C_0(X)$ and carry a $C_0(X)$-valued inner product (the pointwise inner product of sections).

Now replace $C_0(X)$ by any C^*-algebra A, and define a Hilbert A-module to be a (right) A-module with an A-valued inner product (satisying some axioms, e.g. completeness). And one has, ready made from the above, notions of compact operator, Fredholm operator and K-theory:

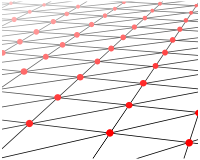
$$K(A) = \left\langle \begin{array}{c} \text{homotopy classes of Fredholm} \\ \text{operators on Hilbert } A\text{-modules} \end{array} \right\rangle$$

(Actually, there was a pre-existing notion of K-theory, and the above is Kasparov's version of the Atiyah-Janich theorem.)

Example. Poincaré duality and the signature

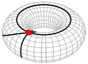
Let M be a smooth, closed, oriented, even dimensional manifold and let G be the fundamental group of M.

Triangulate the universal cover equivariantly. The associated simplicial homology complex



$$\to C_p \xrightarrow{b} C_{p-1} \to$$

with complex coefficients is a complex of free, f.g. $\mathbb{C}[G]$-modules. From completion/base change we obtain free f.g. $C^*_r(G)$-modules.*



Poincaré duality gives a chain equivalence

$$S \colon C_p \to C_{n-p}, \qquad bS + Sb^* = 0$$

with $S = S^*$. The self-adjoint operator $b{+}b^*{+}S$ on $\oplus C_j$ is invertible, and the higher signature is the class in $K(C^*_r(G))$ of

$$0 \colon \operatorname{range}\bigl(P_+(b{+}b^*{+}S)\bigr) \to \operatorname{range}\bigl(P_-(b{+}b^*{+}S)\bigr)$$

between positive and negative spectral subspaces of $b{+}b^*{+}S$.

Example. Discrete series representations.

* Definition. If G is a discrete group or Lie group, then $C_r^*(G)$ is the C^*-algebra generated by the left convolution operators on $L^2(G)$ by compactly supported, smooth functions on G.

Now let G be a real reductive group (for instance $SL(2,\mathbb{R})$).

Let π be an irreducible, square-integrable representation of G, that is, a discrete series representation, as studied by Harish-Chandra.

The Hilbert space of π carries a $C_r^*(G)$-valued inner product,

$$\langle v, w\rangle_{C_r^*(G)} = \Big[g \mapsto \langle v, \pi(g)w\rangle_{H_\pi}\Big]$$

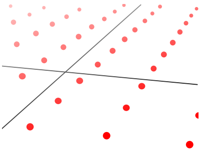
Once again, the zero operator is Fredholm in Kasparov's sense (or in other words, the identity operator on H_π is compact in Kasparov's sense). So we obtain

$$[H_\pi] \in K(C_r^*(G))$$

An example related to the Bott element

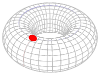
I shall describe an important construction of Kasparov, first in the special case of the torus $M = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}+i\mathbb{Z}$, and afterwards in more generality.

The inverse image in $\mathbb{C}$ of $p \in \mathbb{C}/\Lambda$ is the translated lattice $\Lambda+p$. The Hilbert spaces $\ell^2(\Lambda+p)$ constitute a continuous field of Hilbert spaces over $M=\mathbb{C}/\Lambda$, and the multiplication operators



$$M_z \colon \ell^2(\Lambda+p) \to \ell^2(\Lambda+p) \quad (p \in \mathbb{C}/\Lambda)$$

are a Fredholm family.



I shall explain the significance of this example later on.

Equivariant Index Theory

Equivariant index

So far I have avoided elliptic operators and index theory . . .

But suppose now that W is a smooth manifold equipped with a smooth, proper action of a (discrete or Lie) group G,

$$G \times W \longrightarrow W$$

Assume that the quotient space W/G is compact.

If D is a G-equivariant elliptic operator on W, then D has an equivariant index

$$\mathrm{Index}_G(D) \in K(C_r^*(G))$$

This is made by promoting $L^2(W)$ to a Hilbert $C_r^*(G)$-module, which involves defining a $C_r^*(G)$-valued inner product essentially as in the discrete series example.

The operator D becomes an (unbounded) Fredholm operator on this newly formed Hilbert $C_r^*(G)$-module.

The Baum-Connes conjecture

Some examples of the equivariant index:

- The symmetric signature of M is also the equivariant index of the signature operator on the universal cover of M.

- The discrete series classes are also equivariant indexes of Dirac-type operators on the symmetric space G/K.

As a matter of fact, Baum and Connes conjecture that for every locally compact group G, every K-theory class is an equivariant index, and moreover the only relations among indexes are those that come from geometric relations (e.g. cobordism) among operators.

This is made precise using an assembly map from (equivariant) K-homology, more or less as proposed by Atiyah in his 1969 paper, to the K-theory of $C_r^*(G)$, conjectured to be an isomorphism.

The full formulation of the conjecture requires a bit too much machinery for this lecture, but I shall explain some special cases . . .

Instances of the conjecture

- If M is a closed, even-dimensional, smooth manifold that is oriented in K-theory, and if G is the fundamental group of M, then the equivariant index gives a homomorphism

$$K(M) \longrightarrow K(C_r^*(G))$$

If M is aspherical (i.e. the universal cover is contractible, so $M = BG$) then this is the BC assembly map, conjectured to be an isomorphism.

- If G is a connected Lie group with maximal compact subgroup K, and if G/K is even-dimensional and equivariantly oriented in K-theory, then there is an index homomorphism

$$R(K) \longrightarrow K(C_r^*(G))$$

(since homotopy classes of elliptic operators on G/K correspond precisely to elements of the representation ring). This is the BC assembly map and it is proved to be an isomorphism.

Consequences of the Baum-Connes conjecture

- The assembly map in the first example, involving discrete groups, was introduced by Kasparov (and in related work by Mishchenko), even before there was a Baum-Connes conjecture.

  The higher signature in $K(C_r^*(G))$ is a homotopy invariant of M. If the assembly map is injective, then the preimage in $K(M)$ is homotopy invariant too. From this one can extract Novikov's higher signatures, which are therefore homotopy invariants.

- When G is a reductive Lie group, the K-theory classes $[H_\pi] \in K(C_r^*(G))$ of discrete series (and indeed all K-theory classes) must arise as indexes of Dirac-type operators.

  In fact, an argument reminiscent the proof of the Weyl character formula shows that each discrete series is the index of a Dirac operator coupled to an irreducible representation of K.

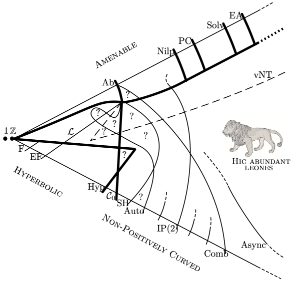
  This is the first step towards recovering Harish-Chandra's parametrization of the discrete series.

Resonances with the Baum-Connes conjecture

- Vanishing theorems for secondary invariants (e.g. the homotopy invariance of relative eta invariants).
- Borel's conjecture that homotopy equivalent closed, aspherical manifolds are homeomorphic, and more generally the Farell-Jones conjectures in surgery theory.
- The Gromov-Lawson-Rosenberg conjecture in positive scalar curvature (although Baum-Connes actually implies that e.g. a closed, aspherical spin manifold cannot admit a p.s.c. metric).
- Issues in the representation theory of p-adic groups (e.g. induction of supercuspidal representations from compact open subgroups).

Current status

- The injectivity part of the conjecture is in a reasonably satisfactory state, at least relatively speaking. For instance injectivity is proved for all linear groups, amenable groups, hyperbolic groups, many types of nonpositively curved groups and more.



- The full isomorphism conjecture is in a less satisfactory state. But it is proved for all amenable groups, all hyperbolic groups, some other discrete groups, and all connected Lie groups.

Alas, progress on the conjecture has slowed . . . stopped in fact, since the early 2000's.

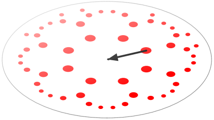
The Dual Dirac Operator

The dual Dirac operator

Now I want to discuss a crucial construction of Kasparov that is genuinely noncommutative . . .

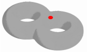
Let M be a closed, nonpositively curved, even-dimensional smooth manifold, oriented for K-theory. Then set

- W = universal cover
- S = spinor bundle
- for $p \in M$, $\Lambda_p = \pi^{-1}[p] \subseteq W$,



and define

$$C_p \colon \ell^2(\Lambda_p, S_+) \to \ell^2(\Lambda_p, S_-)$$



by fixing a basepoint $e \in W$ and setting

$$(C_p f)(q) = c(X_q) f(q) \quad \text{(Clifford multiplication)}$$

where $\exp(X_q) = e$ (a basepoint). This is a family of Fredholm operators over M. It has an index in $K(M)$ . . .

Abstract elliptic operators

Let G be the fundamental group of M.

Each operator C_p above is an (unbounded) abstract elliptic operator of the sort studied by Atiyah in his 1969 paper ... except that it is an abstract elliptic operator for $C_r^*(G)$, not for a space:

- C_p has compact resolvent.
- The Hilbert spaces $\ell^2(\Lambda_p, S_\pm)$ carry representations of $C_r^*(G)$.
- C_p commutes with the operators in the representation modulo "lower order" operators.

Atiyah's axioms (as adjusted by Kasparov) give

$$K(C_r^*(G)) \longrightarrow K(M)$$

Theorem (Kasparov)

This map is left-inverse to the Baum-Connes assembly map

$$K(M) \longrightarrow K(C_r^*(G))$$

Baum-Connes Duality?

Example. Free abelian groups

If $G = \mathbb{Z}^n$, then the dual Dirac method is a familiar construction from elsewhere, in disguise.

- $M = \mathbb{R}^n/\mathbb{Z}^n$ and $C^*_r(G) = C(\widehat{M})$, where $\widehat{M} = \mathbb{R}^n/2\pi\mathbb{Z}^n$, by Fourier transform, and
- the dual Dirac map

  $$K(C^*_r(G)) \longrightarrow K(M)$$

  identifies in this way with the Fourier-Mukai transform

  $$K(\widehat{M}) \longrightarrow K(M)$$

One might speculate that underlying the conjectural Baum-Connes isomorphism in K-theory is a more fundamental correspondence between geometry and representations.

The evidence for this among discrete groups is sparse (so far at least), but the story for Lie groups is rather interesting ...

Reductive Groups

The L^2-index and representation theory

Atiyah and Schmid pointed out that the L^2-index on G/K can be expressed using the Plancherel measure as

$$\mathrm{Index}_{L^2}(D_S) = \int_{\widehat{G}} \dim_{\mathbb{C}}\big([H_\pi \otimes S]^K\big)\, d\mu(\pi)$$

This makes a link between the Atiyah-Singer index formula and Harish-Chandra's Plancherel theory.

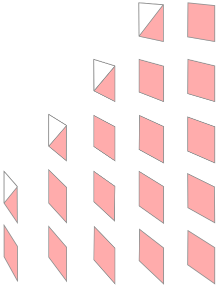
The link is seen, for instance, in Harish-Chandra's formula for the formal dimension of discrete series, and more deeply in the classification of the discrete series using index theory (Atiyah & Schmid) or the Baum-Connes conjecture (Lafforgue), or recent work of Bismut on orbital integrals.

But support of the Plancherel measure—Harish-Chandra's tempered dual—has geometric structure beyond the atomic points for the Plancherel measure. One might ask for an index theory that involves this structure, and not measure theory alone.

The tempered dual as a topological space

In effect, the equivariant index connects index theory to the geometry of the tempered dual. Surprisingly, perhaps, there are beautiful explicit formulas for the index at the level of cycles for K-theory . . . discovered by Penington, Plymen, Wassermann, . . .

Here's what the tempered dual looks like for $GL(2, \mathbb{C})$.



For each Dirac operator D_S, there is a unique component of the tempered dual, which is nonsingular, over which the family of operators

$$D_S\colon [H_\pi \otimes S]^K \to [H_\pi \otimes S]^K$$

is precisely the Bott element.

(Over the other components, either $[H_\pi \otimes S]^K = 0$ or D_S is invertible.)

Cartan motion group

Let G be a real reductive group (like say $SL(n,\mathbb{R})$) and let K be a maximal compact subgroup.

The Cartan motion group for G is the normal bundle for K in G, and a group in its own right: $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$.

Because it is a normal bundle, G_0 fits into a smooth, one-parameter family of groups $\{G_t\}_{t\in\mathbb{R}}$ with

$$G_t = \begin{cases} G & t \neq 0 \\ G_0 & t = 0 \end{cases}$$

This is the deformation to the normal cone from geometry.

How does the representation theory of G_t vary with t? The C^*-algebra? The K-theory?

A proposal of Mackey

The first question, about representations, was considered by George Mackey in the 1970's.

He proposed that the representation theories ought to be "analogous," leading to a correspondence between "almost all" of the irreducible unitary representations of G and G_0.

This was greeted with some skepticism in representation theory.

But the Mackey's idea was kept alive by Alain Connes, who noticed that the Baum-Connes theory implies that the groups $K(C_r^*(G_t))$ (they form the stalks of a sheaf over $\mathbb{R}$) are constant in t.

This topological statement reinforces Mackey's measure-theoretic idea, but there is also an obvious tension between the two ...

... and the tension is resolved most simply by guessing that the (tempered) duals of G and G_0 are actually the same.

Mackey bijection

- The representations in the tempered dual of G come in families (via parabolic induction), so it makes sense to scale a representation by a positive value, $\pi \to {}^{s}\pi$.

- Each irreducible representation of G includes a finite set of minimal K-types (after Vogan).

The following theorem not only gives a new parametrization of the tempered dual, it also quickly implies Baum-Connes for G.

Theorem (Afgoustidis)

There is a unique bijection $\alpha\colon \widehat{G}^{temp} \to \widehat{G_0}$ such that if $\pi \in \widehat{G}^{temp}$, and if v is a minimal K-isotypic vector for π, then the matrix coefficient function $\langle v, {}^{1/t}\pi(g)v\rangle$ for $G_t = G$ converges to a matrix coefficient function for the representation $\alpha(\pi)$ of G_0 as $t \to 0$.

This would not have been discovered without K-theory (even though in the end no K-theory is involved). And not without the influence of Michael Atiyah.

Thank you!

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