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Chapter One

The Riemann-Roch Theorem

A *smooth manifold* is a Hausdorff topological space which is "smoothly locally modeled" on Euclidean space. We assume that the reader is already familiar with the precise definition of this concept (although we shall give a brief review below to help establish notation and terminology).

The definition of a manifold puts into prominence a central idea of modern mathematics, which is that objects having simple *local* structure may exhibit complicated *global* behavior. As is often the case, the origins of this idea can be found in the 19th-century study of holomorphic functions. The *local* definition of a holomorphic function—complex differentiability—could hardly be simpler; but when one studies a holomorphic function from a global point of view questions of quite a different character arise: counting zeroes and poles, the calculus of residues, branch points and Riemann surfaces, and so on. Complex differentiability can be expressed in terms of a partial differential equation (the Cauchy-Riemann equation), and function theory indicates how global *topological* questions naturally arise from studying such equations.

The Atiyah-Singer Index Theorem, which is the subject of this book, makes this sort of connection for a much more general class of partial differential equations. To understand the general theorem one needs to develop several different kinds of mathematical tools. First, one needs an understanding of the *local* structure of the solutions of the partial differential equations involved. This so-called "elliptic theory" is the analog, in a more general context, of facts like the C^∞-differentiability of holomorphic functions; it is the subject of Chapter 5 in this book. Second, one needs some topological machinery to keep track of the *global* aspects of these solutions: this is K-*theory*, which we will describe in Chapter 2.

Before getting involved in these generalities, though, we will take some time in this introductory chapter to motivate the index theorem by exploring the special case of holomorphic functions in more detail. In this case the index theorem reduces to the *Riemann-Roch theorem*, a classic result of 19th-century mathematics and the starting point for innumerable later developments. We are going to review the statement of this famous theorem in a way which brings out some key features of the index problem and suggests the line of development to be followed in the more general case.

1.1 MANIFOLDS

1.1 Definition. A *topological* n*-manifold* X is a Hausdorff topological space that is locally homeomorphic to $\mathbb{R}^n$; that is, for each $x \in X$ there are an open neighborhood U of x, an open subset V of $\mathbb{R}^n$, and a homeomorphism $\phi\colon U \to V$. The triple (ϕ, U, V) is called a *chart* near X.

For technical reasons it is usual also to require that a manifold be *paracompact*; all manifolds appearing in this book are easily seen to satisfy this condition.

1.2 Definition. Let X be a topological manifold. A collection $\{(\phi_\alpha, U_\alpha, , V_\alpha)\}_{\alpha \in \mathcal{A}}$ of charts whose domains U_α cover X is called an *atlas* for the manifold.

Associated to any atlas there is a collection of local homeomorphisms of $\mathbb{R}^n$ (homeomorphisms from one open subset of $\mathbb{R}^n$ to another), the so-called *transition functions* of the atlas. If $\phi_\alpha\colon U_\alpha \to V_\alpha$ and $\phi_\beta\colon U_\beta \to V_\beta$ are two charts whose domains have nonempty intersection, then the associated transition function is defined to be the homeomorphism

$$\phi_{\beta\alpha} = \phi_\beta \phi_\alpha^{-1}\colon V_\alpha \cap \phi_\alpha(U_\beta) \to \phi_\beta(U_\alpha) \cap V_\beta$$

between open subsets of $\mathbb{R}^n$.

Since the transition functions are maps defined on open subsets of $\mathbb{R}^n$, it makes sense to discuss their differentiability. One says that an atlas is *smooth* if its transition functions are all smooth (that is, infinitely differentiable). Similarly one says that an atlas is *holomorphic* if $n = 2k$ is even and the transition functions are *complex*-differentiable when we identify $\mathbb{R}^{2k} = \mathbb{C}^k$ in the usual way[1]. Complex-differentiable functions are smooth, so a holomorphic atlas is automatically also a smooth one.

We'll say that two smooth (or holomorphic) atlases are *equivalent* if their union is also a smooth (or holomorphic) atlas; it is easily checked that this is indeed an equivalence relation. A smooth (or holomorphic) *structure* on X is an equivalence class of atlases of the appropriate type, and X is said to be a *smooth manifold* (or *holomorphic manifold*) if it is provided with a smooth or holomorphic structure. These structures allow one to define various other kinds of smooth (or holomorphic) objects such as functions on the manifold, vector fields, differential forms, and so on; we will regularly make use of these objects and we will assume that the reader has some familiarity with them.

1.3 Definition. A *Riemann surface* is a holomorphic manifold of complex dimension 1.

A Riemann surface is in particular an oriented, real 2-manifold (hence the terminology "surface"), but this information does not completely determine the holomorphic structure. More refined invariants arise from complex analysis (for instance by considering the zeroes and poles of meromorphic functions) and the Riemann-Roch theorem is significant because it relates these refined invariants to ones coming from the ordinary topology of the surface.

[1] Which way *is* the usual way? For future reference, let us decide that we identify $\mathbb{R}^{2k}$ with $\mathbb{C}^k$ by sending $(x_1, x_2, \ldots, x_{2k}) \in \mathbb{R}^{2k}$ to $(x_1 + ix_2, x_3 + ix_4, \ldots) \in \mathbb{C}^k$.

1.2 MEROMORPHIC FUNCTIONS AND DIVISORS

The idea of a Riemann surface grew out of a long history of 19th-century work on function theory, one of whose major strands was the theory of so-called *elliptic functions*. Beginning from the classical problem of the *rectification of the ellipse* (determining the arc length as a function of an algebraic parameter), mathematicians were led to study a certain class of indefinite integrals (those involving the square root of a cubic polynomial in the integrand). These integrals can be resolved by introducing new special functions, in the same way that trigonometric functions can be used to resolve indefinite integrals involving the square root of a quadratic function. Eventually it became clear that the most important feature of these new special functions was their behavior in the complex domain. This gives rise to the following definition.

1.4 Definition. An *elliptic function* is a meromorphic function on $\mathbb{C}$ which is *doubly periodic*: that is, for which there exist two $\mathbb{R}$-linearly independent complex numbers τ_1 and τ_2 for which

$$f(z) = f(z + \tau_1) = f(z + \tau_2)$$

for all $z \in \mathbb{C}$.

It is natural to introduce the *period lattice*—the free abelian subgroup Γ of $\mathbb{C}$ generated by τ_1 and τ_2. Then an elliptic function f with the given periods is invariant under Γ, and so it can be thought of as defined on the quotient space

$$X = \mathbb{C}/\Gamma,$$

which is a compact Riemann surface (topologically equivalent to the 2-torus). Many of the fundamental questions in Riemann surface theory were first asked and answered for surfaces of this type, known as "elliptic curves". For example,

- **Question 1** Do there exist any non-constant holomorphic functions on X? (equivalently, do there exist doubly-periodic holomorphic functions on $\mathbb{C}$?)
- **Question 2** The same question for meromorphic functions: can we construct non-trivial examples of elliptic functions for any given periods?
- **Question 3** Assuming a positive answer to the second question, can we "count" how many elliptic functions there are satisfying appropriate conditions? How sensitively does this count depend on the conditions prescribed?

1.5 Remark. As topological spaces (or even as smooth manifolds), all elliptic curves are the same—the particular *period lattice* Γ makes only an inessential difference. However, this is *not* the case in the holomorphic category. In fact, it can be shown that there is a whole 1-parameter family of elliptic curves, classified by the complex parameter $\tau = \tau_2/\tau_1$ modulo the action of a certain countable group. This example illustrates a general phenomenon: holomorphic structures are much more "refined" or "sensitive" objects than smooth or topological ones. The power of the Riemann-Roch theorem comes from the bridge that it builds between these categories.

Let's begin by answering Question 1. Let Γ be a period lattice, and let us look for Γ-periodic holomorphic functions f on $\mathbb{C}$. Any such f comes from a function on the compact quotient space $X = \mathbb{C}/\Gamma$, and is therefore *bounded*. Thus f is a bounded entire function on $\mathbb{C}$, hence it is constant by Liouville's theorem. (In fact, Liouville's theorem was devised for this exact purpose.) There are therefore no nonconstant, holomorphic, elliptic functions.

Once the appropriate definitions have been made, the same result holds on any compact, connected Riemann surface X. One defines a *holomorphic function* $f\colon X \to \mathbb{C}$ on a Riemann surface X to be a function such that $f \circ \phi^{-1}$ is holomorphic on V for each chart $\phi\colon U \to V$ in a holomorphic atlas. If X is compact, $|f|$ attains its maximum at some point $p \in X$; by the usual maximum principle, f is constant on the domain of a chart containing p. An easy connectedness argument now shows that f is constant everywhere.

When we move on to Question 2 (the existence of meromorphic functions), things become more interesting. Explicit constructions of elliptic functions are associated with the names of Weierstrass and Jacobi. For example, let us sketch Weierstrass' construction. Let Γ be the period lattice and let $p \geq 2$. Consider the series

$$f(z) = \sum_{w \in \Gamma} \frac{1}{(z-w)^p}.$$

For $p \geq 3$ this series converges uniformly and absolutely on compact subsets of $\mathbb{C} \setminus \Gamma$ to a meromorphic function. For $p = 2$ the convergence is not absolute, but it is still possible to interpret the sum as a conditionally convergent series which yields a meromorphic function on $\mathbb{C}$. In either case the function obtained is doubly periodic and has poles (of order p) at each point of Γ. In terms of the Riemann surface $X = \mathbb{C}/\Gamma$, we have constructed a meromorphic function with a single pole on X of order p.

nosimpole

1.6 Remark. It is interesting to note that one *cannot* generalize this construction to $p = 1$; there is no elliptic function with just *one, simple* pole. To see this, we use the residue theorem. According to this result, the integral

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

is equal to the sum of the residues of the poles of f enclosed within the simple closed contour γ. Let us apply this result when γ traverses the boundary of a period parallelogram for the lattice Γ. If f is doubly-periodic, the integrals along opposite pairs of sides of the period parallelogram will cancel, and so the sum of the residues will be zero. In particular, the parallelogram cannot enclose just one, simple pole, as this would give rise to a nonzero residue.

On a general Riemann surface X one can define a meromorphic function as a function $f\colon X \to \mathbb{C} \cup \{\infty\}$, not identically equal to ∞, and such that $f \circ \phi^{-1}$ is a meromorphic function in the usual sense for each chart $\phi\colon U \to V$. (Notice that the meromorphic functions form a complex vector space, and indeed a field extension of $\mathbb{C}$.) As the example of elliptic curves may suggest, every Riemann surface admits non-constant meromorphic functions. But it is no longer so easy to produce them by explicit constructions. Instead, there existence is assured by a general result which "counts" the *number* of meromorphic functions that satisfy certain conditions about the location of their zeroes and poles. We therefore pass on to Question 3.

The Weierstrass construction of elliptic functions (and the theory that grows from it, which we have not described) make it clear that the *location of the poles and zeroes* of a meromorphic function is a key piece of information about it. To formalize this one introduces the notion of a *divisor*.

1.7 Definition. A *divisor* on a compact Riemann surface X is an element of the free abelian group $\mathrm{Div}(X)$ generated by the points of X. That is, it is a finite formal linear combination of points of X, with integer coefficients.

We will denote a divisor by $D = \sum_j n_j a_j$, $n_j \in \mathbb{Z}$, $a_j \in X$. The *degree* $\deg D$ is $\sum n_j$. The divisor D is *positive* (written $D \geq 0$) if each $n_j \geq 0$; this definition allows us to put a partial order on the group of divisors. Obviously, a positive divisor has positive degree.

Recall that if f is a meromorphic function defined near a point $a \in \mathbb{C}$, there is a unique integer k such that $f(z) = (z-a)^k g(z)$ where g is holomorphic and nonvanishing near a. This integer is called the *order* of f at a and is written $\mathrm{ord}(f,a)$. The set of points a having nonzero order is discrete; if $k > 0$ the point a is a *zero* of order k for f; if $k < 0$, then a is a *pole* of order $-k$. Using a chart, the notion of order can be defined also for meromorphic functions on Riemann surfaces.

div-mero-def **1.8 Definition.** Let f be a meromorphic function, not identically zero, on the compact Riemann surface X. The *divisor of* f, written $D(f)$, is the formal sum

$$D(f) = \sum_a \mathrm{ord}(f,a)a$$

extended over all the zeroes and poles of f.

divzero-lemma **1.9 Lemma.** *Let f be a meromorphic function, not identically zero, on the compact Riemann surface X. Then $\deg(D(f)) = 0$; in other words, f has the same number of zeroes as poles, counted with multiplicity.*

We will prove this lemma after we have looked at a few more examples.

linequiv-def **1.10 Remark.** It is easy to check that the process that assigns to each meromorphic function its divisor is a homomorphism: we have $D(fg) = D(f) + D(g)$ and $D(1/f) = -D(f)$. It follows that the collection of divisors of meromorphic functions is a *subgroup* of the group of all divisors. Two divisors in the same coset of this subgroup are said to be *linearly equivalent*.

If D is a divisor, we will say that a meromorphic function f is *subordinate to* D if $D(f) + D \geq 0$. Let $\mathcal{O}(D)$ denote the set of meromorphic functions subordinate to a given divisor D, together with the zero function. Informally, $\mathcal{O}(D)$ is the space of meromorphic functions whose singularities are "no worse" than those prescribed by the divisor D. For example, if $D = +2a$, then $f \in \mathcal{O}(D)$ can have at worst a double pole at a; if $D = -3b$, then f must have at least a triple zero at b. The Weierstrass construction with exponent p produces functions on the corresponding Riemann surface that are subordinate to $p \cdot o$, where o is the point on $X = \mathbb{C}/\Gamma$ that is the image of $0 \in \mathbb{C}$.

1.11 Lemma. *If D is a divisor on a compact connected Riemann surface, then $\mathcal{O}(D)$ is a finite-dimensional vector space.*

Proof. It is a standard (and easy) fact that $\operatorname{ord}(f+g, a) \geq \min\{\operatorname{ord}(f, a), \operatorname{ord}(g, a)\}$ and that $\operatorname{ord}(\lambda f, a) = \operatorname{ord}(f, a)$ if $\lambda \neq 0$. This shows that $\mathcal{O}(D)$ is a vector space. Now we estimate its dimension.

Let $D = \sum n_j a_j$. If $n_j > 0$ consider the linear map α_j from $\mathcal{O}(D)$ to an n_j-dimensional vector space defined by sending $f \in \mathcal{O}(D)$ to the coefficients of singular part of its Laurent series at a_j (in some fixed chart near a_j). The intersection of the kernels of all the α_j consists of holomorphic (hence constant) functions, so has dimension at most 1. $\square$

The problem addressed by the Riemann-Roch theorem is to compute the dimension of $\mathcal{O}(D)$. In particular, by showing that $\dim \mathcal{O}(D) > 1$ for suitable D, it will assure us of the existence of nonconstant meromorphic functions (the fundamental existence theorem for Riemann surfaces). Before looking at what the theorem says in general let us study some simple examples.

1.12 Example. Suppose that X is the "Riemann sphere" — the compact Riemann surface obtained by adjoining a single point at infinity to the complex plane. It is well-known that meromorphic functions on X are simply the rational functions of $z \in \mathbb{C}$. In particular, if we fix a point $a \in \mathbb{C}$ and consider the divisor $D = k.a$, the dimension of $\mathcal{O}(D)$ can easily be computed in terms of k as follows:

k	0	1	2	3	$\cdots$
$\dim \mathcal{O}(D)$	1	2	3	4	$\cdots$

1.13 Example. Now suppose that X is an elliptic curve, and once again let us study the dimension of $\mathcal{O}(D)$ when $D = k.a$ for some fixed point a. We have seen that when $k = 1$, there are no nonconstant function in $\mathcal{O}(D)$ (Remark 1.6). When $k = 2$, the Weierstrass construction produces an example of a nonconstant meromorphic function, and it can be shown that the Weierstrass function, together with a constant function, actually *spans* $\mathcal{O}(D)$. These remarks fill in the first few entries in the corresponding table for an elliptic curve:

k	0	1	2	3	$\cdots$
$\dim \mathcal{O}(D)$	1	1	2	3	$\cdots$

sens-ex **1.14 Example.** It is tempting to guess a general pattern from the above examples, but things are not quite so simple as one might expect. In particular, the holomorphic invariant $\dim \mathcal{O}(D)$ depends very sensitively on the specific form of D—it is not just a function of the total degree $\deg D$. One can find an example of this sensitivity even on an elliptic curve.

To understand the example, notice that an elliptic curve X is *itself* an abelian group (a quotient of the additive group $\mathbb{C}$). Since $\mathrm{Div}(X)$ is the abelian group freely generated by X, there is a canonical group homomorphism

$$\sigma\colon \mathrm{Div}(X) \to X.$$

1.15 Lemma. *If* f *is any nonconstant meromorphic function on the elliptic curve* X*, then* $\sigma(D(f)) = 0$. $\square$

The interested reader will find it easy to give a proof of this lemma by applying the residue theorem (as in Remark 1.6 above) to the function $zf'(z)/f(z)$. But consider its implications for the study of $\dim(\mathcal{O}(D))$. Suppose for example that D is the divisor of a nonconstant meromorphic function f, so that $\sigma(D) = 0$, $\deg(D) = 0$, and $\dim \mathcal{O}(D) \geq 1$. Now perturb D by moving just one of its points by an arbitrarily small amount, obtaining a new divisor D'. The new divisor has $\sigma(D') \neq 0$, so D' is not the divisor of any meromorphic function, and $\deg(D') = 0$, so by lemma 1.9 there are no meromorphic functions f with $D(f) + D' \gneqq 0$. Consequently, $\mathcal{O}(D') = 0$. In other words, the dimension of $\mathcal{O}(D)$ "jumps" downwards under an arbitrarily small perturbation of D.

Recall now that the fundamental topological invariant of a compact, oriented surface X is its *genus* g. Riemann defined the genus to be half the number of "cross-cuts" required to make the surface simply connected. For example, a 2-torus has genus 1 (cutting along one circle produces a cylinder, and a further cross-cut produces a disc). There are many equivalent definitions:

- Triangulate the surface and compute the *Euler characteristic* $\chi(X) =$ (Number of vertices) $-$ (Number of edges) $+$ (Number of faces). The genus and Euler characteristic are related by

$$\chi(X) = 2 - 2g.$$

- Let $\Gamma = \pi_1(X)$ be the fundamental group of X, and consider the abelianization $\Gamma/[\Gamma, \Gamma]$. This is a free abelian group of rank $2g$.

- Consider the cohomology group $H^1(X; \mathbb{R})$. It is a vector space of dimension $2g$.

Note that the Riemann sphere has genus 0, and an elliptic curve has genus 1.

In the 1850s, Riemann sketched a proof of the result

$$\dim \mathcal{O}(D) \geq \deg(D) - g + 1$$

now called *Riemann's inequality*. (Notice how the lower bound provided by this inequality comports with the explicit computations for $g = 0, 1$ appearing in the first two examples above.) In particular, if $\deg(D) > g$ then there exist nonconstant meromorphic functions subordinate to D, and this fundamental existence theorem makes it possible to begin to study Riemann surfaces by a variety of geometrical and algebraic means.

1.16 Example. Suppose that X is a Riemann surface of genus 0, and let $a \in X$. Then Riemann's inequality shows that there exists a non-constant meromorphic function subordinate to the divisor $1 \cdot a$. Such a function takes each value in $\mathbb{C} \cup \{\infty\}$ exactly once, by Lemma 1.9, so it gives a holomorphic bijection $X \to \mathbb{C} \cup \{\infty\}$. In other words, the Riemann sphere is the *only* Riemann surface of genus 0, even when classified by the more restrictive notion of holomorphic equivalence.

In the same way we will be able to see that the only Riemann surfaces of genus 1 are the elliptic curves that we have already discussed. This requires a discussion of holomorphic and meromorphic *differentials* on a Riemann surface, to which we now turn.

1.3 MEROMORPHIC DIFFERENTIALS AND RIEMANN-ROCH

Any student of complex analysis is familiar with the importance of integrals of the form $\int_\gamma f(z)\,dz$, where γ is a path in the complex plane. However, on a Riemann surface X there is no canonically-defined way to integrate a holomorphic or meromorphic *function* f along a path γ in the surface. This is because there is usually no canonically defined "dz" on a Riemann surface. Instead, one introduces a new class of objects—*differentials* (or *differential forms*)—which package the whole integrand "$f(z)dz$" in such a way that the integral along a path is well defined.

When we introduce the language of *line bundles* later in this chapter, we will be able to describe differentials as sections of a certain line bundle (the *canonical bundle*). We will also be able to relate the present discussion of differentials to the *de Rham complex* of X, an object that belongs to the realm of smooth (not holomorphic) geometry; we'll make this connection in Section 1.5. For now, however, let us take a more pedestrian and coordinate dependent approach. We are going to define a differential in terms of its representation in the charts of a holomorphic atlas.

diff-def **1.17 Definition.** Let X be a compact Riemann surface and suppose $(\phi_\alpha, U_\alpha, V_\alpha)$ is a holomorphic atlas for X. A *holomorphic differential* on X is defined by holomorphic functions $u_\alpha \colon V_\alpha \to \mathbb{C}$ which are related by the *transition relation*

$$u_\beta(\phi_{\beta\alpha}(z))\phi'_{\beta\alpha}(z)) = u_\alpha(z)$$

for all $z \in v_\alpha \cap \phi_\alpha(U_\beta)$. If we allow the u_α to be meromorphic functions we obtain the notion of a *meromorphic differential*.

The transition relation can be compactly expressed as

$$u_\beta(w)dw = u_\alpha(z)dz,$$

where z is the "coordinate" on V_α and $w = \phi_{\beta\alpha}(z)$ is the corresponding "coordinate" on V_β. For a full development we should also explain when two representations of this sort should be thought of as defining the same differential, but we will elide these details.

dfex **1.18 Example.** Let f be a meromorphic *function* on X and, for each α as above, let $f_\alpha = f \circ \phi_\alpha^{-1}$ be the function on V_α corresponding to f. Define $u_\alpha(z) = f'_\alpha(z)$. One checks immediately (via the chain rule) that the u_α above satisfy the transition relation for a meromorphic differential, which is naturally denoted df.

1.19 Example. Let η be a meromorphic differential defined by local meromorphic functions $\{u_\alpha\}$, as above, and let g be a meromorphic function on X. Then the functions

$$(g \circ \phi_\alpha^{-1})u_\alpha$$

also satisfy the transition relations for a meromorphic differential, which is naturally denoted $g\eta$. Thus the meromorphic differentials form a vector space over the field of meromorphic functions.

1.20 Example. Let X be an elliptic curve. Because of the representation $X = \mathbb{C}/\Gamma$, any sufficiently small open subset of X can be identified in a canonical way (up to a translation) with an open subset of $\mathbb{C}$; this gives rise to an atlas for X whose transition maps are all *translations* of $\mathbb{C}$, so have derivative 1. Setting all $u_\alpha = 1$ in the definition above then gives a canonically defined, nowhere vanishing holomorphic differential on X, which it is appropriate to denote by dz. As we will shortly see, elliptic curves are *characterized* among compact Riemann surfaces by the existence of such a differential.

Let α be a holomorphic or meromorphic differential on a Riemann surface X, and let $\gamma\colon [0,1] \to X$ be a piecewise smooth curve in X. If α is meromorphic, we assume that γ does not pass through any of its singularities. Then the integral

$$\int_\gamma \alpha$$

can be defined. (Here is one way to do this. By subdividing γ if necessary, we may reduce to the case where the range of γ is contained in the domain of a chart (ϕ, U, V). Represent α in this chart as $u(z)dz$, where u is a holomorphic function on V. Then we define the integral to equal

$$\int_{\phi\circ\gamma} u(z)\,dz;$$

the transition relation for meromorphic differentials ensures that this definition is independent of the particular chart chosen.) Cauchy's theorem holds for this notion of integration: if α is a holomorphic differential on X, the value of the integral $\int_\gamma \alpha$ depends only on the (endpoints-fixed) homotopy class of γ as a map $[0,1] \to X$. In the meromorphic case we have the same statement, but now the homotopy must take place within $X \setminus P$, where P is the set of poles of α.

It follows from Cauchy's theorem that if α is a meromorphic differential with a pole at p, the value of the integral $(2\pi i)^{-1}\oint \alpha$ taken around a small circle surrounding p does not depend on the particular choice of circular contour; this common value is called the *residue* $\mathrm{Res}(\alpha, p)$ of α at p. There are various versions of the Residue Theorem in the context of Riemann surfaces. We will only need the simplest.

res-thm **1.21 Proposition.** *Let α be a meromorphic differential on a compact Riemann surface, with poles $p_1, \ldots, p_n$. Then*

$$\sum_{i=1}^{n} \operatorname{Res}(\alpha, p_i) = 0;$$

the sum of all the residues of α is zero. □

Our first application of the Residue Theorem will be to complete the proof of Lemma 1.9, which states that the divisor $D(f)$ of any meromorphic function f has degree zero. Let $\alpha = f^{-1}\,df$, which is a meromorphic differential. The singularities of α are exactly the zeroes and poles of f, and the classical *argument principle* tells us that the residue of α at a is exactly $\operatorname{ord}(f, a)$. By the residue theorem, then,

$$\deg(D(f)) = \sum_a \operatorname{ord}(f, a) = \sum_a \operatorname{Res}(\alpha, a) = 0$$

as required. □

We have already made use of the fact that the *poles* of a meromorphic differential are invariantly defined. In fact, the notions of the *order* $\operatorname{ord}(\alpha, a)$ of a meromorphic differential at a point, and of the *divisor* $D(\alpha)$ of a meromorphic differential (compare Definition 1.8) make sense for meromorphic differentials just as they do for meromorphic functions.

1.22 Lemma. *If α and β are any two nonzero meromorphic differentials then $D(\alpha)$ and $D(\beta)$ are linearly equivalent (Remark 1.10).*

Proof. By considering local representatives one sees that "the ratio of two meromorphic differentials is a meromorphic function"; in other words, there is a meromorphic function f such that $\beta = f\alpha$. But then

$$D(\beta) = D(f) + D(\alpha)$$

and so $D(\beta)$ and $D(\alpha)$ are linearly equivalent. □

The *canonical divisor class* is the linear equivalence class of any non-zero meromorphic differential, and any divisor in this class is called a *canonical divisor* and denoted by K. For example, the canonical divisor class on an elliptic curve is 0 (the differential dz has neither zeroes nor poles). The canonical divisor class on the Riemann sphere (a curve of genus 0) is $-2 \cdot p$ for any point p on the sphere (all points are linearly equivalent and, if we regard the sphere as $\mathbb{C} \cup \{\infty\}$, the differential dz has a pole of order 2 at ∞).

We will later see that $\dim \mathcal{O}(K) = g$, the genus of the compact Riemann surface X, and that $\deg K = 2g - 2$. These observations relate the genus directly to the holomorphic structure. In fact, we can see straight away that $\dim \mathcal{O}(K)$ and $\deg K$ do not depend on the choice of K within the canonical class. This follows from:

1.23 Lemma. *Let D, D' be linearly equivalent divisors. Then* $\deg(D) = \deg(D')$ *and* $\dim \mathcal{O}(D) = \dim \mathcal{O}(D')$.

Proof. Suppose $D' = D + D(f)$. By Lemma 1.9, $\deg D(f) = 0$; this proves the first statement. As for the second, multiplication by f gives an *isomorphism* between the vector spaces $\mathcal{O}(D)$ and $\mathcal{O}(D')$. □

Let us now recall Riemann's inequality

$$\dim \mathcal{O}(D) \geq \deg(D) - g + 1.$$

By the previous lemma, both sides depend only on the linear equivalence class of D. But we have also seen (Example 1.14) that as soon as we move outside this class, the left side depends much more sensitively on D than the right side does.

Riemann's student Roch investigated the difference between the two sides of the Riemann inequality, publishing in 1865 a note entitled "On the number of arbitrary constants in algebraic functions". (Both Riemann and Roch died of tuberculosis the next year; Riemann was 39, Roch was 26.) Roch's result was that the difference between the two sides of the Riemann inequality is itself the dimension of a space of meromorphic functions: in fact it equals $\dim \mathcal{O}(K-D)$, where K is the canonical divisor. The final result, then, is the *Riemann-Roch Theorem*

riem-roch **1.24 Theorem.** *Let X be a compact Riemann surface of genus g, D a divisor on X. Then*

$$\dim \mathcal{O}(D) - \dim \mathcal{O}(K - D) = \deg(D) - g + 1,$$

where K denotes the canonical divisor.

Notice that the left side is now the *difference* of two term that depend "sensitively" on D, and it turns out that this difference is a much more stable invariant than either of the two terms considered individually. A similar phenomenon occurs for the general Atiyah-Singer Index Theorem, where the *index* is the difference of two quantities described by partial differential equations; each quantity taken by itself depends very sensitively on the data of the problem, but their difference is a topological invariant and is unchanged by small perturbations.

1.25 Example. Let X be a compact Riemann surface of genus 1. Applying the Riemann-Roch theorem to the divisor $D = 0$ we find that $\dim \mathcal{O}(K) = 1$. Suppose that K is defined by a meromorphic 1-form α, and let $f \in \mathcal{O}(K)$; then $\beta = f\alpha$ is a *holomorphic* 1-form.

As remarked above, $\deg K = 2g - 2 = 0$. Since $\deg D(f) = 0$ also, the condition $D(f) + K \geq 0$ for $f \in \mathcal{O}(K)$ can be satisfied only if $D(f) = -K$; thus the holomorphic 1-form β is nowhere vanishing.

Let $\tilde{X}$ be the universal covering surface of X, which is topologically a copy of $\mathbb{R}^2$. A point p of $\tilde{X}$ corresponds to a homotopy class $[\gamma_p]$ of curves in X. Define a map Φ from $\tilde{X}$ to $\mathbb{C}$ by integrating β along γ: that is,

$$\Phi\colon p \mapsto \int_{\gamma_p} \beta.$$

Cauchy's theorem shows that this map is well-defined, and it is clearly holomorphic and has nonzero derivative (because $\beta \neq 0$). It can be shown that Φ maps $\tilde{X}$ *onto* $\mathbb{C}$, so that $\tilde{X}$ is identified with $\mathbb{C}$ as a Riemann surface, and X is the quotient of

$\mathbb{C}$ by a free abelian group of rank 2 acting freely by holomorphic automorphisms. The only such actions are by translations, so X is the quotient of $\mathbb{C}$ by a group of translations, and is thus an elliptic curve.

The examples in this and the preceding section begin to show how the Riemann-Roch theorem can be used to analyze the structure of Riemann surfaces. Our object in this chapter, however, is not to develop the many applications of the Riemann-Roch theorem in more detail (a subject which is already addressed by many excellent texts), but to use the theorem as an example of the more general Index Theorem. To this end, the next couple of sections will be devoted to reformulating Riemann-Roch in the more modern language of line bundles and differential operators.

1.4 HOLOMORPHIC LINE BUNDLES

Let X be a compact Riemann surface. A *holomorphic line bundle* over X is a family L_x of one-dimensional complex vector spaces ("complex lines") that depend holomorphically on a point $x \in X$.

To give some substance to this idea, we take our cue from the theory of manifolds and express matters in "local coordinates". This leads to the following definition.

1.26 Definition. A *holomorphic line bundle* over a Riemann surface X is a 2-dimensional holomorphic manifold L provided with a holomorphic surjection $\pi\colon L \to X$, such that the following conditions hold.

(a) Each *fiber* $L_x := \pi^{-1}(\{x\})$ is provided with the structure of a 1-dimensional complex vector space.

(b) Each $x \in X$ has a neighborhood U for which there is a bijection
$$\psi\colon \pi^{-1}(U) \to U \times \mathbb{C}$$
such that

  (i) ψ is a holomorphic map, with holomorphic inverse,
  (ii) $\mathrm{pr}_1 \circ \psi = \pi$, where $\mathrm{pr}_1 \colon U \times \mathbb{C} \to U$ is the first coordinate projection, and
  (iii) For each $y \in U$ the restriction of ψ to a map $L_y \to \{y\} \times \mathbb{C}$ (which exists by (ii)) is an isomorphism of complex vector spaces.

The Riemann surface X is called the *base* of the line bundle. An *isomorphism* of holomorphic line bundles L, L' (with the same base) is a holomorphic bijection $L \to L'$, with holomorphic inverse, which makes the diagram

$$\begin{tikzcd} L \arrow[r] \arrow[dr] & L' \arrow[d] \\ & X \end{tikzcd}$$

commute and which restricts to a linear isomorphism on each fiber.

1.27 Definition. Let $\pi\colon L \to X$ be a holomorphic line bundle over X. A *holomorphic section* of L is a holomorphic map $s\colon X \to L$ such that $\pi \circ s$ is the identity map (equivalently, $s(x) \in L_x$ for all x). We use the notation $\mathcal{O}(L)$ for the vector space of holomorphic sections of L.

It amounts to the same thing to say that a holomorphic section is a map sending each $x \in X$ to a point of L_x, which, in a *local frame* $\psi\colon \pi^{-1}(U) \to U \times \mathbb{C}$, takes the form $\psi \circ s(x) = (x, f(x))$ with f a holomorphic function on U. This immediately suggests a generalization: we can define a *meromorphic section* of L to be a map sending each $x \in X$ to a point of $L_x \cup \{\infty\}$, which has a similar local representation with f a meromorphic function.

1.28 Example. The product $X \times \mathbb{C}$ is a line bundle in an obvious way, called the *trivial bundle*. A holomorphic (or meromorphic) section of this bundle is simply a holomorphic (or meromorphic) function on X.

1.29 Exercise. A line bundle is isomorphic to a trivial bundle if and only if it has a nowhere-vanishing holomorphic section.

Given two line bundles L' and L'' one can form a new space L equipped with a map to X, by taking the "fiberwise tensor product" of L' and L'', that is, $L_x = (L')_x \otimes (L'')_x$. Using local frames one easily sees that L is a line bundle, denoted $L_1 \otimes L_2$. Similarly one can take the fiberwise *dual* of a line bundle L, whose fiber at x is

$$(L^*)_x = \mathrm{Hom}(L_x, \mathbb{C}).$$

Notice that $L \otimes L^*$ is a *trivial* bundle because the identity map from L to L may be considered as a nowhere-vanishing section (for any finite-dimensional vector space V there is a canonical identification $V \otimes V^* = \mathrm{Hom}(V, V)$). This easily allows us to see that the isomorphism classes of holomorphic line bundles over X constitute an abelian *group*, with $\otimes$ as the group operation.

bun-cocyc-def

1.30 Remark. Again following the lead given by manifold theory, let us consider the transition between two local frames. Suppose that $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ and $\psi_\beta\colon \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{C}$ are two such frames whose domains U_α and U_β overlap. Then the composite

$$\psi_\beta \circ \psi_\alpha^{-1} = (\mathrm{id}, \psi_{\beta\alpha})\colon (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}.$$

Here, for each $x \in U_\alpha \cap U_\beta$, $\psi_{\beta\alpha}(x)$ is an isomorphism of vector spaces $\mathbb{C} \to \mathbb{C}$, that is, an element of the group $\mathbb{C}^*$ of nonzero complex numbers. In this way, given an *atlas* for the vector bundle (a covering of X by the domains of local frames) we obtain holomorphic *transition functions*

$$\psi_{\beta\alpha}\colon U_\beta \cap U_\alpha \to \mathbb{C}^*$$

which satisfy the *cocycle relation*

$$\psi_{\alpha\gamma}\psi_{\gamma\beta}\psi_{\beta\alpha} = 1.$$

Conversely, given an open cover $\{U_\alpha\}$ of X, and holomorphic transition functions $\psi_{\beta\alpha}$ defined on the intersections and satisfying the cocycle condition, one can construct a line bundle over X having these transition functions. The (holomorphic or meromorphic) sections of this line bundle are just families u_α of (holomorphic or meromorphic) functions on U_α that satisfy $u_\beta = \psi_{\beta\alpha} u_\alpha$.

coho-ex **1.31 Exercise.** Suppose that a non-vanishing holomorphic function u_α is defined on each U_α. Show that the transition functions $\psi_{\beta\alpha} = u_\beta / u_\alpha$ satisfy the cocycle condition, but that the resulting line bundle is (isomorphic to) a trivial bundle. (The underlying idea here can be developed to show that the group of isomorphism classes of holomorphic line bundles on X is just $H^1(X; \mathcal{O}^*)$, the first cohomology group of X with coefficients in the *sheaf* of non-vanishing holomorphic functions. Sheaf theory is vital to a deeper understanding of complex geometry, but we shall not require it in this book.)

We can extend the construction of Exercise 1.31 in an important way. Suppose that $D = \sum_j n_j a_j$ is a *divisor* on X. Choose a finite open cover U_α of X having the following properties:

(i) Each U_α is the domain of a coordinate chart for the holomorphic manifold X (so that it is identified with an open subset of $\mathbb{C}$).

(ii) Each of the (finitely many) points a_j belongs to exactly one U_α.

Using the local structure (i), build on each U_α a *meromorphic* function u_α having exactly the singularity prescribed by D; that is, if $a_j \in U_\alpha$ then u_α has a zero of order n_j (or pole of order $-n_j$) at a_j, and elsewhere u_α is holomorphic and non-vanishing. Because of (ii), the functions u_α are holomorphic and non-vanishing on all intersections $U_\alpha \cap U_\beta$ and therefore the formula

$$\psi_{\beta\alpha} = u_\beta / u_\alpha$$

defines a cocycle and hence a line bundle. Because the u_α are no longer holomorphic and non-vanishing *throughout* their domain, the *conclusion* of Exercise 1.31 need not hold: these bundles can be non-trivial.

1.32 Definition. The line bundle L_D so constructed is called the *line bundle of the divisor* D.

hombund **1.33 Exercise.** Show that the mapping $D \mapsto L_D$ is homomorphic, in the sense that $L_{D+D'} \cong L_D \otimes L_{D'}$ and $L_{(-D)} \cong L_D^*$.

The functions u_α that appear in the definition constitute a *meromorphic* section s of the line bundle L_D. Now let g be a meromorphic function on X. Then gs is also a meromorphic section of L_D, and the map $g \mapsto gs$ gives an *isomorphism* between the space of meromorphic functions on X and the space of meromorphic sections of L_D.

secto **1.34 Proposition.** *Under the isomorphism $g \mapsto gs$, the space $\mathcal{O}(D)$ of meromorphic functions subordinate to D corresponds exactly to the space $\mathcal{O}(L_D)$ of holomorphic sections of the line bundle L_D.*

Proof. Introduce the natural notion of the *divisor of a meromorphic section* of a holomorphic line bundle. By construction, $D(s) = D$. But then $D(gs) = D(g) + D(s)$, and gs is holomorphic precisely when its divisor is ≥ 0, that is, when $g \in \mathcal{O}(D)$. □

1.35 Remark. It is natural to ask whether this construction produces all the holomorphic line bundles that there are. In fact, it is not hard to prove the following facts: the isomorphism class of L_D only depends on the linear equivalence class of D, and, if L is a line bundle that has a nonzero meromorphic section at all, then the divisors D of all its meromorphic sections are linearly equivalent, and the bundle L_D for any one of these divisors is isomorphic to L. The only question left open, then, is whether every line bundle *has* a nonzero meromorphic section. The answer (which is "yes") is usually proved along with the Riemann-Roch theorem in a systematic development; for the purposes of this exposition, though, we shall just assume it when we need to.

As Proposition 1.34 makes clear, the key concept in the Riemann-Roch theorem, the dimension on $\mathcal{O}(D)$, can be reformulated in terms of holomorphic sections of a suitable line bundle. We will now see how Roch's correction term can also be reformulated in this language. This involves interpreting differential forms in the language of line bundles.

Let X be a compact Riemann surface and let $(\phi_\alpha, U_\alpha, V_\alpha)$ be a holomorphic atlas for X, with transition functions $\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1}$. The *derivative* $\phi'_{\beta\alpha}$ belongs to $\mathbb{C}^*$, and if we define

$$\psi_{\beta\alpha} = \left(\phi'_{\beta\alpha} \circ \phi_\alpha\right)^{-1},$$

then the Chain Rule easily shows that $\psi_{\alpha\gamma}\psi_{\gamma\beta}\psi_{\beta\alpha} = 1$, in other words that ψ is a cocycle. (Taking the inverse here is a sign convention; the reason for this convention will become apparent in a moment.)

1.36 Definition. The line bundle associated to this cocycle is the *canonical bundle* K on the Riemann surface X.

canbund-def

1.37 Lemma. *Sections (holomorphic or meromorphic) of the canonical bundle are the same as (holomorphic or meromorphic) differentials on* X.

Proof. Compare the definition of a differential (Definition 1.17) with the definition of a section of a bundle defined by a cocycle (Remark 1.30). The "sign convention" is chosen to make these objects correspond exactly. □

Notice that we have used the symbol K in a double sense: for the canonical *bundle* and for the canonical *divisor class*. The lemma shows that these two uses are consistent under the standard correspondence between divisors and line bundles. Summarizing the discussion we have

1.38 Proposition. *Let* X *be a compact Riemann surface,* D *a divisor on* X*. The left hand side*

$$\dim \mathcal{O}(D) - \dim \mathcal{O}(K - D)$$

of the Riemann-Roch theorem 1.24 for D *can be expressed as*

$$\dim \mathcal{O}(\mathsf{L}) - \dim \mathcal{O}(\mathsf{K} \otimes \mathsf{L}^*)$$

where $\mathsf{L} = \mathsf{L}_\mathsf{D}$ *is the line bundle corresponding to the divisor* D*, and* K *is the canonical line bundle.*

Proof. By Proposition 1.34, $\dim \mathcal{O}(\mathsf{L}) = \dim \mathcal{O}(\mathsf{D})$. By the preceding lemma, the canonical *bundle* K is the line bundle associated to the canonical *divisor* K. By Exercise 1.33, $\mathsf{K} \otimes \mathsf{L}^*$ is therefore the bundle associated to the divisor $\mathsf{K} - \mathsf{D}$. Another application of Proposition 1.34 completes the proof. □

In the next section we shall relate this difference to an appropriate partial differential equation.

1.5 THE DOLBEAULT OPERATOR

dolbeault-sec

We have been emphasizing that the left hand side of the Riemann-Roch theorem is the difference of two quantities which depend "sensitively" on the data (the divisor D or line bundle L), but that nevertheless the difference is much more "stable" than the individual quantities that make it up.

Linear algebra provides a familiar example of this phenomenon.

1.39 Definition. Let $\mathsf{T}\colon V \to W$ be a linear mapping between two (perhaps infinite-dimensional) vector spaces. T is said to be *Fredholm* if the kernel and the cokernel of T are both finite-dimensional. (Recall that the cokernel of T is, by definition, equal to the quotient $W/\operatorname{image}(\mathsf{T})$.) If T is Fredholm its *index* is defined to be

$$\operatorname{Ind}(\mathsf{T}) = \dim(\ker(\mathsf{T})) - \dim(\operatorname{coker}(\mathsf{T})).$$

If V and W are finite dimensional, then the index of any linear $\mathsf{T}\colon V \to W$ is equal to $\dim(V) - \dim(W)$. In general, it is helpful to think of a Fredholm index as a 'regularization', using the operator T, of the difference $\infty - \infty$ of the dimensions of the domain and the codomain of T.

A key property of the index is that it is insensitive to "small" perturbations. In the present algebraic context this can be expressed as follows.

1.40 Lemma. *Let* $\mathsf{T}\colon V \to W$ *be a Fredholm operator and let* $\mathsf{F}\colon V \to W$ *be an operator with finite-dimensional range. Then* $\mathsf{T} + \mathsf{F}$ *is also a Fredholm operator, and* $\operatorname{Ind}(\mathsf{T} + \mathsf{F}) = \operatorname{Ind}(\mathsf{T})$. □

We won't prove this here, as we will give a more general result in Chapter 3. The point to notice, though, is that the index is insensitive to the finite-rank perturbation F, even though the kernel and cokernel dimensions can individually be changed by such a perturbation.

We are going to express the left-hand side of the Riemann-Roch theorem as the index of a suitable linear operator—actually a partial differential operator acting on sections of appropriate line bundles over X. Our starting point is the well-known fact that a smooth function $f = u + iv\colon \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if it satisfies the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

which express the *complex*-linearity of the derivative Df when considered as a map $\mathbb{R}^2 \to \mathbb{R}^2$. It amounts to the same thing to say that $\partial f/\partial \bar{z} = 0$, where we define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

to be the complex-linear and complex-antilinear parts of the derivative operator. We will usually abbreviate these operators by ∂ and $\bar{\partial}$ respectively.

1.41 Remark. It is a matter of linear algebra to express the *chain rule* in terms of these operators. Suppose that f is a function of w, which in turn is a function of z; then one has

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial f}{\partial \bar{w}}\overline{\left(\frac{\partial w}{\partial \bar{z}}\right)}, \quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial w}\frac{\partial w}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{w}}\overline{\left(\frac{\partial w}{\partial z}\right)}.$$

In particular if w is a *holomorphic* function of z,

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}, \quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{w}}\overline{\left(\frac{\partial w}{\partial z}\right)}.$$

Similar (but easier) calculations yield the *product rule* which we shall need in the form

$$\frac{\partial fg}{\partial z} = \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}, \quad \frac{\partial fg}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}}g + f\frac{\partial g}{\partial \bar{z}}.$$

In particular, the $\bar{\partial}$ operator *commutes* with multiplication by a holomorphic function g.

We are going to apply these operators to smooth *functions* on a Riemann surface X, and more generally to smooth *sections* of certain line bundles over X. The notions of *smooth line bundle*, *smooth section* (of a line bundle), and so on, may be modeled on the holomorphic definitions of the previous section; just replace the word "holomorphic" by "smooth" wherever it occurs. We'll use the notation $C^\infty(L)$ for the space of smooth sections of the bundle L.

If one applies the operator ∂ to a function on a Riemann surface, the resulting expression is not well-defined as a function, but it is well-defined as a *differential*—a section of the canonical bundle K. (Compare Example 1.18.) Similarly, the result of applying $\bar{\partial}$ to a function is well-defined as a section of the *anticanonical bundle* $\overline{K}$ (the line bundle whose transition functions are the complex conjugates of the transition functions for K). More generally we have

dbar-holo-prop

1.42 Proposition. *Let L be any holomorphic line bundle on the Riemann surface X. There is a well-defined differential operator*

$$\bar{\partial}_L\colon C^\infty(L) \to C^\infty(L \otimes \overline{K})$$

between spaces of smooth sections, which in local coordinates reduces to the operator $\partial/\partial\bar{z}$ defined above. The kernel of this operator consists exactly of the holomorphic sections of L.

The operator so defined is called the *Dolbeault operator* for L.

Proof. Choose an atlas $\{\phi_\alpha, U_\alpha, V_\alpha\}$ (for both X and L simultaneously); let $\phi_{\beta\alpha}$ be the transition functions for X, $\psi_{\beta\alpha}$ the transition functions for L. A section of L is defined by smooth functions $u_\alpha\colon V_\alpha \to \mathbb{C}$ with

$$u_\beta(z) = \psi_{\beta\alpha}(\phi_\beta^{-1}(z))u_\alpha(\phi_{\alpha\beta}(z)).$$

Differentiating this and using the chain rule (remembering that $\psi_{\beta\alpha}$ is holomorphic),

$$\bar\partial u_\beta(z) = \psi_{\beta\alpha}(\phi_\beta^{-1}(z))\bar\partial u_\alpha(\phi_{\alpha\beta}(z)) \cdot \overline{\phi'_{\alpha\beta}(z)}.$$

Remembering that $\phi'_{\alpha\beta} = \left(\phi'_{\beta\alpha}\right)^{-1}$, we recognize that the $\bar\partial u_\beta$ satisfy the transition relations for a section of $L \otimes \overline{K}$.

The final statement follows from the Cauchy-Riemann equations. □

1.43 Remark. As a special case we have the operator $\bar\partial\colon C^\infty(X) \to C^\infty(\overline{K})$. Similarly we can construct the conjugate operator $\partial\colon C^\infty(X) \to C^\infty(K)$. The direct sum of these

$$d = \partial + \bar\partial\colon C^\infty(X) \to C^\infty(K) \oplus C^\infty(\overline{K})$$

is called the *exterior derivative*. (The reader who is familiar with these matters should be able to prove that $K \oplus \overline{K}$ is isomorphic to the complexification of the cotangent bundle T^*X, and that our definition of the exterior derivative matches the standard one. We will say more about this below.) In the previous section, we introduced the notation df for a *holomorphic* function f; for such a function $\bar\partial f = 0$ and thus $df = \partial f$ is defined as a section of K. This reconciles our current notation with that of the earlier section.

The main analytic result underlying the Riemann-Roch theorem is then

dbar-fredholm-prop

1.44 Proposition. *Let L be a holomorphic line bundle on a compact Riemann surface X. The Dolbeault operator*

$$\bar\partial_L\colon C^\infty(L) \to C^\infty(L \otimes \overline{K})$$

is a Fredholm linear map. Moreover, its index $\mathrm{Ind}(\bar\partial)$ *is exactly equal to the left-hand side of the Riemann-Roch theorem for L.* □

We will see that $\bar\partial$ is one of a general class of linear partial differential operators, called *elliptic* operators, which are automatically Fredholm. The Atiyah-Singer theorem computes the index of an operator of this type. The Riemann-Roch theorem becomes the special case of Atiyah-Singer applied to the $\bar\partial$ operator.

In the remainder of this section we are going to sketch some of the analysis which goes into the proof of proposition 1.44. This will help us know what properties to look for when we develop the general analysis of elliptic operators in Chapter 5. The techniques used are rather different from those described earlier in this chapter.

Begin by observing that according to Proposition 1.42, the *kernel* of $\bar\partial_L$ is exactly the space of holomorphic sections $\mathcal{O}(L)$. Thus, the *first* terms match up in the two expressions

$$\dim \mathcal{O}(L) - \dim \mathcal{O}(K \otimes L^*)$$

(left side of Riemann-Roch), and

$$\dim\ker(\bar\partial_L) - \dim\operatorname{coker}(\bar\partial_L)$$

(index of $\bar\partial_L$). What about the second terms?

The key here is to use a duality principle. Under appropriate circumstances, the dual of the cokernel of a linear operator $T\colon V \to W$ is exactly the kernel of the dual operator $T'\colon W' \to V'$ between the dual spaces. In particular these spaces have the same dimension. We will apply this line of thinking to the case where T is the $\bar\partial$ operator itself.

In order to do so, we will need to identify the relevant dual spaces and dual operators in concrete terms, as function spaces associated to X. We will also (and this is where the analysis will come in) need to give suitable topological vector space structures to V and W and produce sufficiently strong estimates on the operator T so that the "principle" above actually becomes a known theorem of functional analysis.

Let us introduce some more notation. Let L be a line bundle. We'll use the following notation for spaces of sections of various bundles associated to L:

Notation	$\Omega^{0,0}(L)$	$\Omega^{1,0}(L)$	$\Omega^{0,1}(L)$	$\Omega^{1,1}(L)$
Smooth sections of	L	$K\otimes L$	$\overline{K}\otimes L$	$K\otimes\overline{K}\otimes L$

We'll call the members of $\Omega^{i,j}(L)$ (i,j)-*forms* with values in L. If L is a *holomorphic* line bundle we have $\bar\partial$ operators $\bar\partial\colon \Omega^{i,0}(L) \to \Omega^{i,1}(L)$, $i = 0, 1$. If L is trivial we will leave it out of the notation, and write simply $\Omega^{i,j}$.

1.45 Lemma. *There is a canonically defined operation of* integration

$$\int\colon \Omega^{1,1} \to \mathbb{C}$$

such that

$$(\alpha,\beta) \mapsto \frac{i}{2}\int \alpha\overline{\beta}$$

is a positive definite inner product on $\Omega^{1,0}$. □

The notion of integration is a standard part of the general theory of differential forms on manifolds; the definition can be given, for example, by using a partition of unity to reduce to the case when $\alpha \in \Omega^{1,1}(X)$ is supported within the domain of a coordinate chart, then writing α in local coordinates as $f dz d\bar z$; we define the integral to be

$$\int \alpha = -2i \iint f(x,y)\,dx\,dy.$$

The key point to the definition is that the coordinate transition equations for $K\otimes\overline{K}$ introduce the Jacobian $|\partial w/\partial z|^2$ into the integrand; so the integral turns out to be invariantly defined under changes of coordinates.

Using this notion of integration we have dual pairings (denoted $\langle,\rangle$)

$$\Omega^{0,1}(L)\otimes\Omega^{1,0}(L^*) \to \mathbb{C}, \quad \Omega^{0,0}(L)\otimes\Omega^{1,1}(L^*) \to \mathbb{C}$$

both of which are given by the formula $(\alpha, \beta) \mapsto \int \alpha\beta$. Moreover with respect to these pairings we have by integration by parts[2]

$$\langle \bar{\partial} f, \alpha \rangle = -\langle f, \bar{\partial}\alpha \rangle$$

for $f \in \Omega^{0,0}(L)$, $\alpha \in \Omega^{1,0}(L)$. This makes it apparent that the dual operator of $\bar{\partial}$, with respect to this pairing, is simply $-\bar{\partial}\colon \Omega^{1,0}(L^*) \to \Omega^{1,1}(L^*)$, and our general principle therefore indicates that the rightmost term in the definition of the index, that is the dimension of the cokernel of $\bar{\partial}_L$, should equal the dimension of the kernel of $-\bar{\partial}_{K\otimes L^*}$, that is, the dimension of the space $\mathcal{O}(K \otimes L^*)$. This is in fact true, and the equality

$$\dim \operatorname{coker}(\bar{\partial}_L) = \dim \mathcal{O}(K \otimes L^*),$$

which completes the proof of Proposition 1.44, is (a special case of) the *Serre duality theorem*.

To make this argument a rigorous one it is necessary to complete the spaces involved to topological vector spaces in such a way that our pairings $\langle , \rangle$ actually become the canonical pairing between a topological vector space and its dual, and in such a way that the analytical conditions necessary for a rigorous formulation of our duality principle are satisfied. There are a number of different routes that can be followed here, each with its own advantages. Serre's original formulation made use of the theory of distributions. In this book, however, we shall approach index theory via Hilbert space; we shall complete the spaces described above via natural inner products to Hilbert spaces (essentially spaces of "square-summable sections" of appropriate vector bundles). The operators $\bar{\partial}$ now become *unbounded* Hilbert space operators. A classic approach to the study of such operators T, going back to von Neumann, proceeds via the (bounded) *resolvent operator* $(1 + T^*T)^{-1}$. In terms of this operator the key analytic fact about $\bar{\partial}$ (or about any elliptic operator on a compact manifold) is easily stated: the resolvent of $\bar{\partial}$ is a *compact* Hilbert space operator.

In Chapter 3 we shall draw out the general functional-analytic implications of the compact-resolvent property; in Chapter 5 we shall show that $\bar{\partial}$ and other elliptic operators on compact manifolds have the property in question.

1.6 NOTES

[2] Formally, we apply Stokes' theorem to see that $\int \bar{\partial}(f\alpha) = 0$.

Chapter Two

K-Theory

KChapter

The Atiyah-Singer index theorem links analysis and topology, and our approach to its proof will use techniques from both areas. The focus of this chapter is topology, specifically vector bundles and K-theory.

Vector bundles and K-theory arise in index theory in two different ways. ...

The most important result in Atiyah-Hirzebruch K-theory is the Bott periodicity theorem. Our proof of the index theorem for the Dolbeault and Dirac operators will not require periodicity. However proof for general operators in Chapter 9 relies very heavily on periodicity, or at least on a beautiful mechanism discovered by Atiyah to prove periodicity. We shall give a proof of Bott's theorem there.

2.1 VECTOR BUNDLES

2.1 Definition. Let M be a topological space. A *complex vector bundle* over M is a space V that is equipped with:

(a) a continuous map $\pi\colon V \to M$, and

(b) a complex vector space structure on each fiber $V_m = \pi^{-1}\{m\}$

It is required that each point of M is included in some neighborhood U for which there is a homeomorphism $\pi^{-1}[U] \cong U \times \mathbb{C}^n$ that restricts to a vector space isomorphism on each fiber (the right-hand map π is the coordinate projection) and makes the diagram

$$\begin{tikzcd} \pi^{-1}[U] \arrow[r, "\cong"] \arrow[d, "\pi"'] & U \times \mathbb{R}^n \arrow[d, "\pi"] \\ U \arrow[r, equal] & U \end{tikzcd}$$

commute.

2.2 Definition. Isomorphism of bundles ... more generally a homomorphism from one bundle to another

2.3 Example. The constant bundles $X \times \mathbb{C}^n$ ...

A vector bundle E over a topological space X is said to be *trivial* if it isomorphic to a product $X \times \mathbb{C}^n$

... restriction to a subset

2.4 Definition. ... subbundle

2.5 Example. Classifying bundle over a Grassmannian space.

2.6 Proposition. *quotient bundle* □

Real vector bundles are defined in the same way. In addition *smooth vector bundles* over a smooth manifold M are defined by requiring V to be a smooth manifold and all maps appearing in the above definition to be smooth maps.

2.7 Example. Tangent bundle.

2.2 CLASSIFICATION OF VECTOR BUNDLES

2.8 Definition. If $f\colon M' \to M$ is a continuous map, and if V is a vector bundle over M, then the *pull-back* of V along f is the vector bundle

$$V' = \left\{ (m', v) \in M' \times V \,:\, f(m') = \pi(v) \right\}.$$

It is a vector bundle over M' (and it is a smooth vector bundle, if f is a map of smooth manifolds).

2.9 Proposition. *Let $f_0, f_1 : X \to Y$ be a pair of homotopic maps between compact Hausdorff spaces. If E is any vector bundle on Y, then the vector bundles f_0^*E and f_1^*E are isomorphic.* □

2.10 Proposition. *Let E be a vector bundle over a compact Hausdorff space X. For sufficiently large n, E is isomorphic to a subbundle of the trivial bundle $X \times \mathbb{C}^n$.* □

2.11 Definition. ... of $G(n,k)$

..., its topology via open subsets of $\mathbb{C}^n$. It is

$$G(n,k) \cong U(n)\big/\big(U(k)\times U(n-k)\big).$$

2.12 Definition. ... Classifying bundle

2.13 Theorem. *Let X be a compact Hausdorff space. The operation of pulling back the classifying bundle on the Grassmannian space $G(n,k)$ induces an isomorphism*

$$\varinjlim_n\, [X, G(n,k)] \cong \mathrm{Vect}_k(X)$$

for every nonnegative integer k. □

2.3 OPERATIONS ON BUNDLES

Various operations produce new vector bundles from old. The most important is the operation of direct sum: if V and W are vector bundles over the same base X, then their direct sum is defined to be

$$V \oplus W = \{ (v, w) \in V \times W \,:\, \pi(v) = \pi(w) \},$$

which is also a vector bundle over X.

For instance if V is a vector bundle on M, then there is a natural way to form the vector bundle V^*, whose fibers are the vector space duals of the fibers of V …(explain).

Similarly there is a natural way to form the vector bundles $\wedge^p V$ whose fibers are the exterior powers of the fibers of V, and so on.

Among the many other possibilities, let us mention the tensor product $V \otimes W$, whose fibers are the direct sums and tensor products of the fibers of V and W.

2.4 K-THEORY

2.14 Definition. Let X be a compact Hausdorff space. The K-*theory group* $K(X)$ is the abelian group generated by the set of isomorphism classes of complex vector bundles on X, subject to the relations that

$$[E] + [F] = [E \oplus F]$$

for all complex vector bundles E and F on X.

The operation of pullback of vector bundles makes $K(X)$ into a contravariant functor on the category of compact Hausdorff spaces. We shall denote by

$$f^* \colon K(Y) \longrightarrow K(X)$$

the map on K-theory groups induced by a continuous map $f \colon X \to Y$.

The first significant fact about the K-theory functor is that it is homotopy invariant:

2.15 Proposition. *If $f, g \colon X \to Y$ are homotopic maps, then the maps from $K(Y)$ to $K(X)$ induced from f and g are equal to one another.* $\square$

This is an immediate consequence of …

2.16 Definition. Relative groups and K-theory for locally compact spaces

2.17 Proposition. *Let Y be a closed subspace of a compact Hausdorff space X. The sequence*

$$K(X \setminus Y) \xrightarrow{\pi^*} K(X) \xrightarrow{\iota^*} K(Y)$$

is exact at $K(X)$. $\square$

2.18 Proposition. *Suppose that Y is a closed and contractible subspace of a locally compact Hausdorff space X. The projection $\pi \colon X \to X/Y$ induces an isomorphism from $K(X/Y)$ to $K(X)$.* $\square$

ntractible-quotient-prop

2.19 Proposition. *Suppose that Y is a closed subspace of a locally compact Hausdorff space X and that $X \setminus Y$ is homeomorphic to $(0, 1] \times Z$ for some locally compact Hausdorff space Z. The inclusion $\iota \colon Y \to X$ induces an isomorphism from $K(X)$ to $K(Y)$.*

nbd-vb-extension-lemma

2.20 Lemma. *Suppose that* Y *is a closed subspace of a compact Hausdorff space* X*, and that* E *is a vector bundle on* Y*. There is a neighborhood* U *of* Y *in* X *and a vector bundle on* U *whose restriction to* Y *is isomorphic to* E. □

Granted the classification theorem of . . . , this is very easy. The vector bundle E is determined by a map from Y into a Grassmannian space, and this map extends to a neighborhood of Y in X. The lemma may be proved directly by using a subset of the techniques used to prove the classification theorem.

Proof of Proposition 2.19. After replacing X and Y by their one-point compactifications, if necessary, we may assume that both spaces are compact.

Because K-theory is a homotopy functor, it follows from Proposition **??** that the map $\iota^*\colon K(X) \to K(Y)$ is one-to-one. To prove that the map is also onto it suffices to show that if E is a vector bundle on Y, then E is isomorphic to the restriction to Y of a vector bundle on X.

According to Lemma 2.20, there is a neighborhood U of Y in X and a vector bundle F on U whose restriction to Y is isomorphic to E. According to Exercise 2.21 there is a continuous map g from $(0,1] \times Z$ to itself that is the identity in a neighborhood of infinity and that has range contained in U. We may regard g as a map from X to itself that is the identity in a neighborhood of Y and has range contained in U. If we pull back the vector bundle F to X using g, then we obtain a vector bundle on X whose restriction to Y is isomorphic to E, as required. □

. . . discussion about computations made partially possible by the above tools. Another possibility is to compare to ordinary cohomology, which we shall do in the next chapter.

2.5 EXERCISES

point-set-ex

2.21 Exercise. Let Z be a locally compact Hausdorff space and form the product space $(0,1] \times Z$. Let U be a neighborhood of infinity in the one-point compactification of $(0,1] \times Z$.

(a) Show that U contains $(0,\varepsilon] \times Z$ for some $\varepsilon > 0$, as well as $(0,1] \times (Z \setminus K)$, for some compact subset $K \subseteq Z$.

(b) Using the same notation, show that for every open neighborhood W of K in Z there is a continuous function

$$f\colon (0,1] \times Z \longrightarrow [0,1]$$

such that $f(t,z) = \min\{t,\varepsilon\}$ if $z \in K$ and $f(t,z) = t$ if $z \notin W$.

(c) Show that if W is chosen to have compact closure in the locally compact space Z, then the formula $g(t,z) = (f(t,z),z)$ defines a continuous function

$$g\colon (0,1] \times Z \longrightarrow (0,1] \times Z$$

that is equal to the identity in a neighborhood of infinity, such that the range of g is contained in U.

2.22 Exercise. 1-cocycle form for vector bundles.

2.23 Exercise. ... map from $\mathrm{Vect}_k(S^n)$ to $[S^{n-1}, GL(k, \mathbb{C})]$. In fact this map is an isomorphism.

2.24 Exercise.

$$[E_0] - [E_1] = [X \times \mathbb{C}^n] - [F]$$

2.25 Exercise. Mayer-Vietoris and Bott generator (what it is for the $\mathbb{C}^n$, and that it generates an infinite-cyclic subgroup).

2.26 Exercise. On the basis of the previous exercises, show that

$$K(S^n) \cong \mathbb{Z} \oplus \varinjlim_k [S^{n-1}, GL(k, \mathbb{C})]$$

where the direct limit is formed ...

2.27 Exercise. (on sections ...)

A *(smooth) section* of a smooth vector bundle V over M is a smooth function from M to V that maps each point of M to an element of the fiber of V above that point. Smooth sections may be added together and muliplied by scalar functions on M using the vector space structures on the fibers of V.

Using sections, we can equip vector bundles with various sorts of additional structures. For example a *euclidean structure* on a vector bundle V is a family of inner products on the fibers of V that vary smoothly in the sense that the pointwise inner product of any two sections of V is a smooth function on V. A *hermitian structure* on a complex vector bundle is a family of hermitian inner products on the fibers for which the pointwise inner product of any two sections is a smooth function.

2.28 Exercise. Existence and essential uniqueness of euclidean or hermitian structures.

2.29 Exercise. Puppe sequence

2.30 Exercise. On the definition of $K^1(X)$.

2.6 NOTES

For an introduction to elementary K-theory the reader can refer to Atiyah's notes [**?**] (for spaces) or to the books of Blackadar [**?**] or Rørdam *et al.* [**?**] (for C^*-algebras). The main item missing from our brief account is a proof of Theorem **??** on excision. This is a purely algebraic issue, and the reader is referred to Milnor's notes on algebraic K-theory [**?**] for a proof—the excision theorem is equivalent to Milnor's Theorem XXX. We shall treat the Bott periodicity theorem in Chapter **??**.

Chapter Three

Fredholm Operators

CStarChapter

In this chapter we shall review some of the very basic analysis of Fredholm operators on Hilbert space, and then continue to families of Fredholm operators. Our main objective is the definition of the index of a family of Fredholm operators parametrized by a compact space X. This is a class in the Atiyah-Hirzebruch K-theory group $K(X)$.

3.1 THE OPERATOR NORM AND COMPACT OPERATORS

3.1 Definition. Let $T\colon H_0 \to H_1$ be a norm-continuous linear operator between two Hilbert spaces. Its *norm* is the quantity

$$\|T\| = \sup\{\, \|Tv\| \,:\, \|v\| \leq 1\,\}.$$

The norm is always finite, and indeed finiteness of the norm is equivalent to the continuity of T, which is why the term "continuous" is usually dropped in favor of "bounded."

The norm gives the set of all bounded linear operators on a single Hilbert space the structure of a Banach algebra. The most important consequence of this is that the set of invertible operators is an open subset of the set of all bounded linear operators. This stability property, that a sufficiently small (in norm) perturbation of an invertible operator is still invertible, is basic to everything that follows.

3.2 Definition. A linear operator between vector spaces is a *finite-rank operator* if its range is finite-dimensional.

A bounded finite-rank operator $F\colon H_0 \to H_1$ between Hilbert spaces can be written in the form

$$Tv = \sum_{j=1}^{n} \langle v_j, v\rangle w_j$$

for some n and some vectors $v_1, \ldots, v_n \in H_0$ and $w_1, \ldots, w_n \in H_1$ (where all the choices are independent of v, of course).

3.3 Definition. A bounded operator between Hilbert spaces is *compact* if it is a norm-limit of finite-rank operators.

Compact operators will play an important role throughout. They ought to be thought of as small perturbations of the zero operator—not small in norm, but small in the sense that their ranges are "approximately finite-dimensional."

If K is a compact operator and if T is a bounded operator, then both TK and KT are compact. The compact operators on a single Hilbert space therefore form a closed, two-sided ideal in the Banach algebra of all bounded operators.

3.2 BOUNDED FREDHOLM OPERATORS ON HILBERT SPACE

A linear operator $T\colon V_0 \to V_1$ between two vector spaces is a *Fredholm operator* if its kernel and cokernel are finite-dimensional vector spaces, in which case the *index* of T is defined to be

$$\operatorname{Ind}(T) = \dim(\ker(T)) - \dim(\operatorname{coker}(T)).$$

If the vector spaces V_0 and V_1 are themselves finite-dimensional, then obviously every linear operator between them is a Fredholm operator. Moreover the rank-nullity theorem of linear algebra says that in this case

$$\operatorname{Ind}(T) = \dim(V_0) - \dim(V_1).$$

This is an extreme stability property for the index: it is completely independent of the operator T. Here is a counterpart the infinite-dimensional context:

finite-perturb-lemma

3.4 Lemma. *If* $T\colon V_0 \to V_1$ *is a Fredholm operator, and if* $F\colon V_0 \to V_1$ *is a finite-rank operator, then* $T + F$ *is a Fredholm operator and has the same index as* T.

Proof. Suppose first that U_0 is a subspace of V_0 of finite codimension, and that the operator $S\colon U_0 \to V_1$ is the restriction of T to U_0. The exact sequences

$$0 \longrightarrow \ker(S) \longrightarrow \ker(T) \longrightarrow V_0/U_0 \xrightarrow{T} \operatorname{range}(T)/\operatorname{range}(S) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{range}(T)/\operatorname{range}(S) \longrightarrow \operatorname{coker}(S) \longrightarrow \operatorname{coker}(T) \longrightarrow 0$$

of finite-dimensional vector spaces show that

$$\operatorname{Ind}(S) = \operatorname{Ind}(T) + \dim(V_0/U_0)$$

(compare Exercise 3.73). Now take U_0 to be the kernel of F. Since the restrictions of the two operators T and $T + F$ to U_0 are equal to one another, the two operators obviously give rise to the same operator S, as above. So our formula for the index of S shows that the index of T is equal to the index of $T + F$. $\square$

Our aim is to investigate other stabilities properties that arise in the infinite-dimensional context, once linear algebra is supplemented with functional analysis.

3.5 Definition. A bounded Hilbert space operator $T\colon H_0 \to H_1$ is a *Fredholm operator* if it is Fredholm in the purely algebraic sense described above. Its *index* likewise defined as above.

While the definition makes no reference to Hilbert space theory, there is a simple but powerful interaction between the Fredholm property and the operator norm:

norm-perturb-lemma

3.6 Lemma. *The set of all Fredholm operators* $T\colon H_0 \to H_1$ *is an open subset of the set of all bounded operators from* H_0 *to* H_1. *Moreover the Fredholm index is a locally constant function on this set.*

Proof. The range of a Fredholm operator $T\colon H_0 \to H_1$ is always a closed subspace of H_1 (see Exercise 3.72). If P_1 is the orthogonal projection operator on H_1 whose range is the range of T, and if P_0 is the orthogonal projection operator on H_0 whose range is the orthogonal complement of the kernel of T, then T is an invertible operator from range(P_0) to range(P_1). Let $S\colon H_0 \to H_1$ be another bounded linear operator. Since

$$\|P_1SP_0 - P_1TP_0\| \leq \|S - T\|,$$

we see that if S is sufficiently close in norm to T, then $P_1SP_0\colon \text{range}(P_0) \to \text{range}(P_1)$ is invertible. It follows that P_1SP_0, considered as an operator from H_0 to H_1 has the same kernel and the same range as T, and hence the same index. But S is a finite-rank perturbation of P_1SP_0, so by Lemma 3.4 it has the same index as T too. $\square$

3.7 Remark. The space of all Fredholm operators from H_0 to H_1 is locally path-connected (since it is an open subset of the Banach space of all bounded linear operators). It is not difficult to show that two Fredholm operators lie in the same path component if and only if they have the same index.

Lemma 3.6 represents one way to strengthen the stability of the Fredholm index in the functional-analytic context. Here is another, which is very useful when it comes to verifying the Fredholm property:

compact-perturb-lemma

3.8 Lemma. *If* T *is a Fredholm operator and if* K *is a compact operator, then* $T+K$ *is a Fredholm operator too. Moreover* $\text{Ind}(T + K) = \text{Ind}(T)$.

Proof. We can write $K = K_1 + K_2$, where K_1 has sufficiently small norm that $T + K_1$ is Fredholm and has the same index as T, while K_2 has finite rank. The lemma now follows from Lemmas 3.4 and 3.6. $\square$

Lemma 3.8 can be repackaged, as follows:

3.9 Proposition. *A bounded linear operator* $T\colon H_0 \to H_1$ *is Fredholm if and only if there is a bounded operator* $R\colon H_1 \to H_0$ *such that the operators*

$$I - RT\colon H_0 \to H_0 \quad \textit{and} \quad I - TR\colon H_1 \to H_1$$

are compact.

Otherwise put, a bounded operator is Fredholm if and only if it is invertible modulo compact operators.

Proof. If T is Fredholm then an inverse modulo compact operators may be constructed by inverting the operator P_1TP_0 used in the proof of the previous lemma. Conversely, if R is an inverse modulo compact operators, then RT and TR are Fredholm operators, since they are compact perturbations of the identity. As a result, the kernel of T and the cokernel of T are finite-dimensional, as required. $\square$

3.10 Remark. Exercises 3.75 to 3.79 put the above results to use in the proof of a simple index theorem.

3.3 UNBOUNDED FREDHOLM OPERATORS

We aim to study the Fredholm theory of partial differential operators. Since differential operators are not bounded, or even everywhere defined, on the L^2-spaces that arise most naturally in Hilbert space theory, we must either work with different sorts of Hilbert spaces or work with Hilbert space operators that fail to be bounded. By and large we shall do the latter.

3.11 Definition. A *unbounded operator* from a Hilbert space H_0 to a Hilbert space H_1 is a pair consisting of a dense linear subspace of H_0, called the *domain* of the operator and a linear operator T from that subspace to H_1.

We shall write the domain as $\mathrm{dom}(T)$. Authors don't always require it to be a dense subspace, but we shall. Note that $\mathrm{dom}(T)$ might be all of H_0, and that T might in fact be bounded. So unbounded really means not necessarily bounded.

We shall write an unbounded operator as $T\colon H_0 \to H_1$ despite the fact that it may not be defined on all of H_0. It is important to bear in mind, however, that a specific choice of domain is part of the definition of unbounded operator.

3.12 Example. The simplest example relevant to us is the operator $T = d/dx$ mapping $L^2(\mathbb{R})$ to itself, with domain the smooth, compactly supported functions.

We could vary the example in a number of ways. For instance we could define $\mathrm{dom}(T)$ to be the Schwartz space, or the continuously differentiable compactly supported functions, or something more complicated (see Exercise **??**). But the possibilities are narrowed considerably, often to a single reasonable choice, if following requirement is imposed.

3.13 Definition. Let $T\colon H_0 \to H_1$ be an unbounded operator. The *graph* of T is the linear subspace

$$\mathrm{graph}(T) = \{\, (u, v) \in H_0 \oplus H_1 \mid u \in \mathrm{dom}(T) \quad \text{and} \quad Tu = v \,\}$$

of $H_0 \oplus H_1$. The operator is said to be *closed* if $\mathrm{graph}(T)$ is a closed subspace of $H \oplus H$.

Every bounded operator is obviously closed, and indeed the condition that T be closed should be taken as a weak substitute for boundedness. It is very important in this respect, since as we shall see it implies stability properties for the Fredholm index similar to those we witnessed in the previous section.

Proving that an unbounded operator is closed is often tricky (for an example see Exercise 3.80). Fortunately there is a simple technique for manufacturing closed operators that can be broadly applied.

3.14 Definition. An unbounded operator $T\colon H_0 \to H_1$ is *closeable* if the closure of its graph in $H_0 \oplus H_1$ is the graph of an unbounded operator $\overline{T}$, in which case this operator is called the *closure* of T.

In the examples to be studied later in the book, we shall in effect define T on some domain, then verify that T is closeable using the next lemma, then replace T by its closure.

closeable-lemma

3.15 Lemma. *Let* $T\colon H_0 \to H_1$ *be an unbounded operator. If there is an unbounded operator* $S\colon H_1 \to H_0$ *such that*

$$\langle Tv, w\rangle = \langle v, Sw\rangle$$

for all $v \in \operatorname{dom}(T)$ *and all* $w \in \operatorname{dom}(S)$*, then* T *is closeable.*

Proof. □

The analysis of closed operators can be reduced in some respects to the analysis of bounded operators, as follows:

3.16 Definition. …graph norm

$$\|v\|_T^2 = \|v\|^2 + \|Tv\|^2.$$

3.17 Lemma. *An unbounded operator* T *is closed if and only if* $\operatorname{dom}(T)$ *is a Hilbert space (that is, it is complete) in the graph norm.* □

3.18 Definition. An closed unbounded operator $T\colon H_0 \to H_1$ is a *Fredholm operator* if it is Fredholm as a bounded linear operator from $\operatorname{dom}(T)$ to H_1, which is to say, if and only if it is Fredholm as a linear operator from the vector space $\operatorname{dom}(T)$ to the vector space H_1.

3.19 Definition. Let $T\colon H_0 \to H_1$ be closed, unbounded operator. Its *graph projection* is the orthogonal projection operator P_T on $H_0 \oplus H_1$ whose range is $\operatorname{graph}(T)$.

3.20 Lemma. *A closed operator is Fredholm if and only if the bounded operator*

$$P_T\colon \operatorname{graph}(T) \longrightarrow H_1$$

is a Fredholm operator. In this case the Fredholm indexes of T *and* P_T *above are equal.*

3.21 Lemma. *If we equip the set of of closed operators* $T\colon H_0 \to H_1$ *with the metric*

$$\operatorname{dist}(T_1, T_2) = \|P_{T_1} - P_{T_2}\|,$$

then the set of unbounded Fredholm operators is an open subset and the Fredholm index is a locally constant function on this set.

3.4 CONTINUOUS FIELDS OF HILBERT SPACES

We shall need to study continuous families of Fredholm operators parametrized some topological space Y. As a prelude to doing so we shall present here the concept of a continuous field of Hilbert spaces.

Let Y be a set and suppose we are given a family of Hilbert spaces H_y indexed by the elements of Y. By a *section* of the family we shall mean any function that assigns to each point $y \in Y$ a vector in H_y. In other words, a section is an element of the direct product $\prod_{y\in Y} H_y$.

3.22 Definition. Let Y be a topological space. A *continuous field of Hilbert spaces* over Y consists of a family $H = \{H_y\}_{y \in Y}$ of Hilbert spaces indexed by the elements of Y and a set $\Gamma(H)$ of sections such that:

(a) The set $\Gamma(H)$ is a vector space under the operations of pointwise addition and scalar multiplication.

(b) For every $y \in Y$, the set $\{ s(y) : s \in \Gamma(H) \}$ is a dense linear subspace of H_y.

(c) If $s \in \Gamma(H)$, then the function $y \mapsto \|s(y)\|$ is a continuous real-valued function.

(d) If r is any section, and if for every $y \in Y$ and every $\varepsilon > 0$ there is a section $s \in \Gamma(H)$ and a neighborhood U of y such that

$$\sup\{ \|s(u) - r(u)\| : u \in U \} \leq \varepsilon,$$

then $r \in \Gamma(H)$.

The members of $\Gamma(H)$ are called the *continuous sections* of H. The Hilbert spaces H_y are the *fibers* of H.

3.23 Remarks. Condition (c) implies that the pointwise inner product of any two continuous sections is a continuous function on Y. Condition (d) might be summarized by saying that any section that is locally uniformly the limit of continuous sections is itself continuous (note that this is a simple property of continuous functions). It implies that the product of a continuous section and a continuous function is again a continuous section. So $\Gamma(H)$ is a module over the ring of continuous functions on Y.

3.24 Definition. Isomorphism of fields and restriction of fields

3.25 Example. ...trivial field

$$s = \sum_{n=1}^{\infty} \langle s, s_n \rangle s_n$$

Each coefficient $\langle s, s_n \rangle$ is a continuous function on X, and the sum converges uniformly on compact subsets of X to the section s.

3.26 Lemma. ... *orthonormal basis implies isomorphic to a trivial field.* □

3.27 Remark. the essence is not to topologize the "total space" of the field. ...However there is a topology available, which is occasionally useful.

3.28 Example. ...Hermitian vector bundle

3.29 Lemma. *Let H be a continuous field of Hilbert spaces over a topological space X. For every $y \in Y$, $H_y = \{ s(y) : s \in \Gamma(H) \}$.*

Proof. Fix $y \in Y$. Notice that if s is any continuous section, then the section

$$s'(x) = \frac{\|s(y)\| + 1}{\|s(x)\| + 1} s(x)$$

is continuous, is equal to s at the point y, and is bounded by $\|s(y)\| + 1$. □

We discussed the pull-back operation on vector bundles in the last chapter. There is a similar operation for continuous fields of Hilbert spaces but it won't play any role for us (except for vector bundles). A much more useful operation will be the push-forward, which is defined as follows.

3.30 Definition. C_0-sections

3.31 Definition. …extension by zero

3.32 Remark. Given a family of Hilbert spaces $H = \{H_y\}_{y \in Y}$ and a family $\Gamma_0(H)$ of sections satisfying (a), (b) and (c), there is a unique enlargement $\Gamma(H)$ of $\Gamma_0(H)$ that additionally satisfies (d). It consists of all sections r with the property that for every $y \in Y$ and every $\varepsilon > 0$ there is a section $s \in \Gamma_0(H)$ and a neighborhood U of y such that

$$\sup\{\, \|s(u) - r(u)\| \, : \, u \in U \,\} \le \varepsilon.$$

The continuous field determined by $\Gamma(H)$ is called the continuous field *generated* by $\Gamma_0(H)$.

3.33 Definition. …a generating family of sections

3.34 Lemma. *If a family of sections generates, then every continuous section is a norm limit of sections of the form …* □

3.35 Definition. …Continuous field generated by a family of sections

3.36 Example. …submersion example

3.37 Definition. A *bounded operator* on $\mathcal{H}$ is a uniformly bounded family $T = \{T_y \colon \mathcal{H}_y \to \mathcal{H}_y\}_{y \in Y}$ of operators on the fibers of $\mathcal{H}$ that, along with its adjoint family, maps continuous sections to continuous sections.

3.38 Remark. …on adjoints and adjointable operators

3.39 Example. In the case of a constant field, bounded operators are bounded $*$-strongly continuous families of operators.

…countably generated (by a countable generating family of continuous sections)

3.40 Theorem. *Every countably generated continuous field of Hilbert spaces may be embedded as an orthogonal direct summand of a constant field with separable, infinite-dimensional fibers.*

Proof. Let H be a constant field of Hilbert spaces with separable, infinite-dimensional fibers, and let H' be any countably generated continuous field of Hilbert spaces. We shall prove that $H \oplus H'$ is isomorphic to H', which will obviously suffice.

Let $u_1, u_2, \dots$ be the sequence of constant sections of the constant field H associated to an orthonormal basis of the constant fiber. Let $u_1', u_2', \dots$ be a sequence of sections of H' for which the values at any point $y \in Y$ are dense in the fiber H_y'. Next, let $v_1', v_2', \dots$ be any sequence of sections of H' in which each u_n' is repeated infinitely often. Consider now the sequence of sections

$$v_n = v_n' \oplus 2^{-n} u_n$$

of $H' \oplus H$. Each $u_n' \oplus 0$ is a norm limit of sections in this sequence (because each u_n' appears infinitely often as some v_n'). It follows that the sequence $\{v_n\}$ generates the field $H' \oplus H$.

We shall now apply the Gramm-Schmidt procedure to the sequence $\{v_n\}$. First we construct

$$w_1 = \text{Normalization}(v_1),$$

where the normalization operation multiplies a section v by the function $\langle v, v\rangle^{-\frac{1}{2}}$ so as to obtain a section of pointwise norm one. The operation is well defined as long as the section to which it is applied is nowhere vanishing, as it is in the case of v_1. Then we successively construct the sections

$$w_n = \text{Normalization}\big(v_n - \langle v_n, w_1\rangle w_1 - \cdots - \langle v_n, w_{n-1}\rangle w_{n-1}\big).$$

The normalization is well defined since the projection of the section being normalized onto the nth basis direction in the constant field H is $2^{-n}u_n$, and so in particular the section is nonzero at every point of Y.

We have constructed an orthonormal basis $\{w_n\}$ for $H' \oplus H$, and this determines an isomorphism with a constant field. $\square$

3.41 Remark. Assume for a moment that H' may indeed be embedded into H as an orthogonal direct summand, so that

$$H' \oplus H'' \cong H$$

for some H''. From the isomorphisms

$$\begin{aligned} H' \oplus H \oplus H \oplus \cdots &\cong H' \oplus (H'' \oplus H') \oplus (H'' \oplus H') \oplus \cdots \\ &\cong (H' \oplus H'') \oplus (H' \oplus H'') \oplus \cdots \\ &\cong H \oplus H \oplus \cdots \end{aligned}$$

and from the fact that

$$H \cong H \oplus H \oplus \cdots$$

we find that $H' \oplus H \cong H$. With this in mind, we may as well try to establish the latter isomorphism.

3.5 FAMILIES OF FREDHOLM OPERATORS

Throughout this section we shall fix two continuous fields of Hilbert spaces, H_0 and H_1, over a locally compact Hausdorff space Y.

Our main criterion is that a Fredholm operator from H_0 to H_1 should have a Fredholm index, formed from the kernels and cokernels of the individual fiber operators, which is close enough to a vector bundle to define an element of the K-theory group of Y. This requires us to proceed with a little bit of care, as the following example shows.

3.42 Example. Let H_0 be the constant field over the unit interval $[0,1]$ with one-dimension fiber $\mathbb{C}$. Let H_1 be the push-forward to $[0,1]$ of the constant field on $(0,1]$ with the same fiber. Define $T\colon H_0 \to H_1$ by setting the fiber operator T_t to be multiplication by $t \in [0,1]$. Then T is a bounded and adjointable operator, and moreover each T_t is a Fredholm operator since it is an operator between finite-dimensional vector spaces. But $\operatorname{Ind}(T_0) = 1$ whereas $\operatorname{Ind}(T_t) = 0$ if $t > 0$. So the fiberwise index is not a locally constant function.

3.43 Definition. A bounded operator on H is a *compact operator* if it is a norm-limit of operators of the form

$$F\colon v \mapsto \sum_{j=1}^{n} \langle u_j, v\rangle w_j,$$

where u_j and w_j are C_0-sections of H_0 and H_1, respectively.

3.44 Example. …constant fields

3.45 Definition. A bounded operator $T\colon H_0 \to H_1$ from one continuous field to another is a *Fredholm operator* if there is a bounded operator $R\colon H_1 \to H_0$ such that

$$I - RT\colon H_0 \to H_0 \quad \text{and} \quad I - TR\colon H_1 \to H_1$$

are compact operators.

3.46 Lemma. *The pointwise index of a bounded Fredholm operator between continuous fields is a locally constant function on* Y.

3.47 Lemma. … *invertible at infinity*

3.48 Definition. An *closed unbounded operator* from the continuous field H_0 to the continuous field H_1 is a family of closed unbounded operators

$$T_y\colon H_{0y} \longrightarrow H_{1y} \qquad (y \in Y)$$

for which the family of graph projections

$$P_{T_y}\colon H_{0y} \oplus H_{1y} \to H_{0y} \oplus H_{1y} \qquad (y \in Y)$$

is an operator on the continuous field $H_0 \oplus H_1$.

3.49 Remark. …This is more than saying the graph is closed.

3.50 Definition. A unbounded operator $T\colon H_0 \to H_1$ between continuous fields is an *unbounded Fredholm operator* if the projection operator

$$P_1\colon \operatorname{graph}(T) \longrightarrow H_1$$

is a bounded Fredholm operator.

3.51 Lemma. *The pointwise index of a closed, unbounded Fredholm operator between continuous fields is a locally constant function on* Y.

3.52 Lemma. *Extension by zero and restriction to a closed subset both send Fredholms to Fredholms*

3.6 THE INDEX OF A FAMILY OF FREDHOLM OPERATORS

main-fred-index-thm

3.53 Theorem. *Let* $T\colon H_0 \to H_1$ *be a bounded Fredholm operator between trivial and countably generated continuous fields of Hilbert spaces over a compact Hausdorff space* X. *There is a compact operator* $C\colon H_0 \to H_1$ *such that* $\ker(T+C)$ *and* $\operatorname{range}(T+C)^\perp$ *are vector subbundles of* H_0 *and* H_1, *respectively. The difference*

$$[\ker(T+C)] - [\operatorname{range}(T+C)^\perp] \in K(X)$$

is independent of the choice of compact operator $C\colon H_0 \to H_1$.

Proof of the existence statement in Theorem 3.53. Let $s_1, s_2, \ldots$ be an orthonormal basis for H_1. Denote by P_n the orthogonal projection onto the subfield generated by $s_1, \ldots, s_n$. This is a compact operator, and if C is any compact operator on H_1, then

$$\lim_{n\to\infty} \|C - P_n C\| = 0,$$

as checked in Lemma **??**. Let $Q_n = I - P_n$, which is of course equal to the identity modulo compact operators.

The operator $Q_n T$ is a compact perturbation of T. We shall prove that for all large n both $\operatorname{range}(Q_n T)^\perp$ and $\ker(Q_n T)$ are vector bundles.

First, for all large n the range of the operator $Q_n T$ is equal (fiberwise) to the range of Q_n. To prove this, let $S\colon H_1 \to H_0$ be inverse to T, modulo compact operators. From the fact that $TS = I + C$, where C is compact, we find that

$$\lim_{n\to\infty} \|Q_n - Q_n TS\| = \lim_{n\to\infty} \|Q_n C\| = \lim_{n\to\infty} \|C - P_n C\| = 0.$$

It follows of course that

$$\lim_{n\to\infty} \|Q_n - Q_n TSQ_n\| = 0,$$

and so for all large n the operator $Q_n TSQ_n$, viewed as an operator on $Q_n H$, is invertible, and in particular surjective.

Next, we claim that the kernel of $Q_n T$ is a vector bundle for all large n. For large n the operator $Q_n TSQ_n$ is invertible (considered as an operator on $Q_n H$), since its inverse can be represented as a Neumann series:

$$(Q_n TSQ_n)^{-1} = \sum_{k=0}^{\infty} (Q_n - Q_n TSQ_n)^k.$$

Notice, incidentally, that since TS is the identity modulo compact operators, the operator $Q_n TSQ_n$ and its inverse are both equal to Q_n modulo compact operators. Consider then the operator

$$E = SQ_n(Q_n TSQ_n)^{-1} Q_n T.$$

It is an idempotent and it is equal to the identity modulo compact operators. Moreover its kernel is fiberwise equal to the kernel of $Q_n T$. So by Lemma **??**, the fiberwise kernel of $Q_n T$ is a vector bundle, as required. ☐

3.54 Definition. Let $T\colon H_0 \to H_1$ be a bounded Fredholm operator between countably generated continuous fields of Hilbert spaces over a compact Hausdorff space X. The *index* of T, denoted $\mathrm{Ind}(T) \in K(X)$, is the index of the direct sum

$$T \oplus I\colon H_0 \oplus H \longrightarrow H_1 \oplus H,$$

where I denotes the identity operator on any trivial field with infinite-dimensional fibers.

3.55 Proposition. *The index of a bounded Fredholm operator* $T\colon H_0 \to H_1$ *between countably generated continuous fields over a compact Hausdorff space has the following properties:*

(a) *If* H_0 *and* H_1 *are vector bundles, then*

$$\mathrm{Ind}(T) = [H_0] - [H_1] \in K(X).$$

(b) *If* T *decomposes as a direct sum* $T' \oplus T''$*, then*

$$\mathrm{Ind}(T) = \mathrm{Ind}(T') + \mathrm{Ind}(T'').$$

(c) *If* T *is invertible,* $\mathrm{Ind}(T) = 0$.

3.56 Proposition. *The above properties characterizes the index of a compact family.*

3.57 Proposition. *Let* Y *be an open subset of a compact Hausdorff space* X*. and if* D *is extended by zero to* Y*, then …*

…This further property characterizes the index in general

3.7 SELF-ADJOINT AND COMPACT RESOLVENT OPERATORS

For the moment we shall considering individual Hilbert spaces rather than continuous fields of Hilbert spaces.

3.58 Definition. An unbounded operator Hilbert space operator $D\colon H \to H$ is *symmetric* if

$$\langle Du, v\rangle = \langle u, Dv\rangle$$

for all $u, v \in \mathrm{dom}(D)$.

simple-symmetric-ex **3.59 Example.** A simple example of a symmetric operator is $D = i d/dx$ acting on the Hilbert space $L^2(\mathbb{R})$, with domain equal to the space of smooth, compactly supported functions. Integration by parts verifies the symmetry condition.

3.60 Example.

$$D = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$$

acting on the Hilbert space $H_0 \oplus H_1$ with domain equal to $\mathrm{dom}(T) \oplus \mathrm{dom}(S)$.

Section **??**, the Fredholm operators relevant to index theory will frequently arise in a *graded* situation.

grading-remark **3.61 Definition.** A *grading operator* on a Hilbert space H is a bounded self-adjoint operator

$$\varepsilon \colon H \to H$$

whose square is equal to the identity. It decomposes H as a direct sum $H_0 \oplus H_1$ of eigenspaces for eigenvalues ± 1. A bounded operator on H is *even* if it commutes with ε, *odd* if it anticommutes. An unbounded operator is ...

... discussion of gradings, odd operators

3.62 Definition. ... self-adjoint ... essentially self-adjoint

3.63 Lemma. *Every bounded (everywhere-defined) symmetric operator is self-adjoint.*

Every symmetric operator that is bounded is automatically self-adjoint in the above sense. To verify this, the first step is to note the identity

$$\begin{aligned} \|(T \pm iI)u\|^2 &= \langle (T \pm iI)u, (T \pm iI)u \rangle \\ &= \langle (T \mp iI)(T \pm iI)u, u \rangle \\ &= \langle (T^2 + I)u, u \rangle \\ &= \|Tu\|^2 + \|u\|^2. \end{aligned}$$

which shows that the operators $T \pm iI$ are injective maps from H to H and are bounded below. The orthogonal complement of the range of $T \pm iI$ is the kernel of $T \mp iI$, which is zero by the identity above.

3.64 Lemma. *Every self-adjoint operator is closed.*

The same argument shows that if D is odd-graded, as in Example **??**, then each of the operators S and T is closed.

3.65 Definition. Essentially self-adjoint.

3.66 Lemma. *If D is a self-adjoint operator, then the operators $(D + iI)^{-1}$ and $(D - iI)^{-1}$ commute with one another.*

...For example

$$(D^2 + I)^{-1} := (D + iI)^{-1}(D - iI)^{-1} = (D - iI)^{-1}(D + iI)^{-1}$$

3.67 Theorem. *Let* D *be a self-adjoint operator on a Hilbert space* H. *There is a unique homomorphism from the algebra of continuous, bounded, complex-valued functions on* $\mathbb{R}$ *into the algebra of bounded operators on* H *that maps the functions* $(x \pm i)^{-1}$ *to the operators* $(D \pm iI)^{-1}$. □

...uniqueness is easy ...the proof of existence uses some rudimentary C^*-algebra theory, or the equivalent, and is outlined in Exercise **??**.

3.68 Lemma. *Let* $H = H_0 \oplus H_1$ *be a* $\mathbb{Z}_2$*-graded Hilbert space with grading operator* ε, *and denote by* P_1 *the orthogonal projection from* H *onto* H_1. *If*

$$D = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} : H_0 \oplus H_1 \to H_0 \oplus H_1$$

is an odd-graded, essentially self-adjoint operator on H, *then the graph projection for* T *is given by the formula*

$$P_T - P_1 = D(I + D^2)^{-1} + \varepsilon(I + D^2)^{-1}.$$

3.69 Lemma. *Let* D *be an unbounded self-adjoint operator on a Hilbert space* H. *If one of the resolvent operators* $(D \pm iI)^{-1}$ *is compact, then so is the other.*

3.70 Proposition. *Let* $H = H_0 \oplus H_1$ *be a* $\mathbb{Z}_2$*-graded Hilbert space with grading operator* ε, *and denote by* P_1 *the orthogonal projection from* H *onto* $\mathcal{H}_1$. *If*

$$D = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$$

is an odd-graded, essentially self-adjoint operator on H, *and if* D *has compact resolvent, then the closure of* T *is a Fredholm operator from* H_0 *to* H_1.

3.71 Proposition. *Suppose that the continuous field* H *is* $\mathbb{Z}_2$*-graded, and that on each* H_y *there is given an unbounded, odd-graded, essentially self-adjoint operator*

$$D_y = \begin{pmatrix} 0 & S_y \\ T_y & 0 \end{pmatrix} : H_{0y} \oplus H_{1y} \to H_{0y} \oplus H_{1y}$$

such that:

(a) *each operator* D_y *has compact resolvent, and*

(b) *the resolvent*

$$(D + iI)^{-1} := \{(D_y + iI)^{-1} : H_y \to H_y\}_{y \in Y}$$

is a compact operator on the continuous field H.

Then the fiberwise closure of T *is an unbounded Fredholm operator from the continuous field* H_0 *to the continuous field* H_1.

3.8 EXERCISES

closed-range-ex **3.72 Exercise.**

euler-characteristic-ex **3.73 Exercise.** Show that if

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

is a short exact sequence of vector spaces, then $\sum_{j=0}^{n}(-1)^j \dim(V_j) = 0$.

ind-mult-ex **3.74 Exercise.** Suppose that S and T are Fredholm operators on a Hilbert space H. Show that ST is also a Fredholm operator and that

$$\mathrm{Ind}(ST) = \mathrm{Ind}(S) + \mathrm{Ind}(T).$$

first-toeplitz-index-ex **3.75 Exercise.** The stability properties of the index that are summarized by Atkinson's theorem are powerful tools for computing Fredholm indices. We are going to illustrate this straight away by proving the *index theorem for Toeplitz operators* on S^1. Denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$ and denote by H the closed subspace of $L^2(\mathbb{T})$ generated by the functions z^n, for $n \geq 0$. This is the *Hardy subspace* of $L^2(\mathbb{T})$.

3.76 Definition. Let $f \in C(\mathbb{T})$. The *Toeplitz operator with symbol* f is the bounded linear operator $T_f \colon H \to H$ on the Hardy space given by pointwise multiplication by f, followed by orthogonal projection from $L^2(\mathbb{T})$ into H.

If $f, g \in C(\mathbb{T})$, then $T_f T_g$ is equal to T_{fg}, modulo compact operators.

This follows from the fact, which is easily checked on the dense subalgebra of trigonometric polynomials in $C(\mathbb{T})$, that the operator on $L^2(\mathbb{T})$ of pointwise multiplication by any f commutes with the orthogonal projection from $L^2(\mathbb{T})$ onto H, modulo compact operators.

3.77 Exercise. It follows from Atkinson's theorem that the Toeplitz operator T_f is Fredholm if and only if its symbol $f \in C(\mathbb{T})$ is nowhere vanishing, and hence invertible in $C(\mathbb{T})$.

toeplitz-ex **3.78 Exercise.** Show that the index of the Fredholm operator T_{z^n} is equal to $-n$.

Here is the Toeplitz index theorem:

last-toeplitz-index-ex **3.79 Exercise.** Let $T \in \mathfrak{T}$ be a Toeplitz operator whose symbol $f = \sigma(T) \in C(\mathbb{T})$ is nowhere vanishing. Then T is a Fredholm operator, and its index is equal to minus the winding number of f about the origin.

Proof. By Atkinson's theorem, the index of T depends only on its symbol $f = \sigma(T)$, and in fact only on the homotopy class of f as a map from $\mathbb{T}$ to $\mathbb{C} \setminus \{0\}$. The group of such homotopy classes is

$$\pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$$

with the isomorphism being given by the winding number. Since the group operation in $\pi_1(\mathbb{C} \setminus \{0\})$ may be represented by pointwise multiplication, Exercise 3.74 shows that the operation

$$f \mapsto \mathrm{Ind}(T_f), \quad \pi_1(\mathbb{C} \setminus \{0\}) \to \mathbb{Z}$$

is a group homomorphism. Now to prove the theorem it suffices to verify it on the generator $f(z) = z$ of $\pi_1(\mathbb{C} \setminus \{0\})$, and this is the content of Exercise 3.78 above. $\square$

abs-cont-ex **3.80 Exercise.** Let T be the unbounded operator d/dx on $L^2(\mathbb{R})$ whose domain is the space of square-integrable, absolutely continuous functions with square-integrable derivatives. Prove that T is closed. (This exercise is for those who know some function theory on the line, and in particular those who know what an absolutely continuous function is!)

3.81 Exercise.

3.82 Exercise. If $T\colon H_0 \to H_1$ is a closed unbounded Hilbert space operator, then

$$\mathrm{graph}(T^*) = \{\, (Tv, v) \in H_1 \oplus H_0 \,:\, v \in \mathrm{dom}(T)\,\}^{\perp}.$$

compact-resolv-exercise2 **3.83 Exercise.** Let T be an unbounded self-adjoint operator on H. Show that T has compact resolvent if and only if the inclusion of $\mathrm{dom}(T)$ (with the graph norm) into H is a compact operator.

3.84 Exercise. If T has compact resolvent, then the set $\{\, f(T) \,:\, f = f^* \in C_0(\mathbb{R})\,\}$ is a commuting set of self-adjoint, compact operators. It follows from the spectral theorem for compact operators that there is an orthonormal basis $\{u_j\}$ for H consisting of simultaneous eigenvectors for all the operators $f(T)$. These eigenvectors are also eigenvectors for T:

cpt-resolv-prop **3.85 Proposition.** *With T and $\{u_j\}$ as above, there are real scalars λ_j such that $\lim_{j\to\infty} |\lambda_j| = \infty$ and $f(T)u_j = f(\lambda_j)u_j$, for all j. The vectors u_j belong to $\mathrm{dom}(T)$ (or $\mathrm{dom}(\bar{T})$, if T is essentially self-adjoint), and $Tu_j = \lambda_j u_j$, for all j.* $\square$

3.86 Exercise. Suppose that T is an operator with compact resolvent and that B is any bounded operator. The computation

$$(T + B + iI)^{-1} = (T + iI)^{-1} - (T + B + iI)^{-1} B (T + iI)^{-1}$$

shows that $T + B$ has compact resolvent also. Since the path $t \mapsto T + tB$ is continuous in the topology of norm resolvent convergence, the unbounded version of Atkinson's theorem given above shows that $\mathrm{Ind}(T + B) = \mathrm{Ind}(T)$.

3.87 Exercise. If D^{op} denotes the operator on the field $\mathcal{H}^{op}$ obtained by reversing the grading on $\mathcal{H}$, then $\mathrm{Ind}(D^{op}) = -\mathrm{Ind}(D)$.

3.88 Exercise. if Z is a closed subset of Y, then $\mathrm{Ind}(D)$ maps to $\mathrm{Ind}(D\big|_Z)$ under the restriction map from $K(Y)$ to $K(Z)$.

3.89 Exercise. …on Hilbert module regular operators

safex **3.90 Exercise.** Show that the index of a *self-adjoint* Fredholm operator must be zero. (See Remark 3.61 for the role of a *grading* in this situation.)

3.91 Exercise. Show that, for the formally self-adjoint operator D of Example **??**, the range of $D \pm iI$ fails to be dense in $L^2[0,1]$. Trace through the discussion above for this operator. Show that the operator $D' = i d/dx$, with domain the smooth functions f for which $f(0) = f(1)$, *is* essentially self-adjoint.

esa-def **3.92 Definition.** An unbounded, formally self-adjoint operator T is *essentially self-adjoint* if the operators $T \pm iI$ map $\mathrm{dom}(T)$ onto dense subspaces of H. An unbounded, formally self-adjoint operator T is *self-adjoint* if the operators $T \pm iI$ map $\mathrm{dom}(T)$ onto the whole of H.

3.93 Exercise. FIberwise invertibility versus invertibility as a family.

mult-op-ex2 **3.94 Exercise.** Let f be a real-valued polynomial on $\mathbb{R}^n$. Show that the operator of multiplication by f, considered as an unbounded operator on $L^2(\mathbb{R}^n)$ with domain equal to the smooth compactly supported functions, or the Schwartz space, is essentially self-adjoint.

3.95 Exercise. $Y = \mathbb{C}$. trivial family. $T_z = z\colon H_z \to H_z$. Fredholm operator.

3.96 Exercise. What if $H_0 = 0$. Closed? Fredholm?

3.97 Exercise. Bott element on $\mathbb{C}$ and Hopf line bundle

hilbert-hodge-ex **3.98 Exercise.** Prove a Hilbert space version of the Hodge theorem: If T is a self-adjoint Fredholm operator on a Hilbert space H, then the kernel of T is finite-dimensional, the range of T is closed, and $\ker(T) \oplus \mathrm{image}(T) = H$.

3.99 Exercise. example of a non-selfadjoint operator

3.9 NOTES

One which is quite closely aligned with the perspective of these notes is the book of Douglas [**?**], which also develops a wealth of detailed and interesting material about Toeplitz operators. Other possibilities are the texts of Davidson [**?**] or Murphy [**?**].

Dixmier . . .

The book of Dunford and Schwarz [**?**] is the standard, comprehensive introduction to spectral theory, unbounded operators, and applications to classical partial differential equations. A less massive account can be found in Rudin [**?**]. Our discussion of unbounded Fredholm operators is based on Kato [**?**]. Another useful reference is the slender functional analysis text of Zimmer [**?**].

Chapter Four

Characteristic Classes

CharacteristicChapter

In this chapter we are going to study in more detail the *characteristic classes* of vector bundles that were mentioned briefly in the last chapter. At the end of the chapter we shall be able to formulate more precisely the signature theorem we stated in Chapter 1 (as Theorem **??**), and sketch an application to the construction of an exotic sphere.

We shall not attempt a comprehensive treatment of the theory of characteristic classes. Our plan is to rapidly summarize the information that we shall need; the reader is referred to one of the standard texts on the subject for complete details (see the notes at the end of the chapter).

4.1 CHARACTERISTIC CLASSES FOR LINE BUNDLES

4.1 Definition. A *characteristic class* for vector bundles (of a certain kind, for instance rank-k complex vector bundles) is a map c which assigns to each isomorphism class of vector bundles V (of the given kind) over a base M a cohomology class $c(V) \in H^*(M)$, in such a way that if $f\colon M' \to M$ is a map, then $f^*(c(V)) = c(f^*(V))$.

For the purposes of these notes there will be no loss of generality if we think of M as a compact manifold and the cohomology as de Rham cohomology. But sometimes it is important to understand that certain characteristic classes are *integral*, that is, they are elements of the cohomology groups $H^*(M;\mathbb{Z})$ with integer coefficients. We shall touch on this at the end of the chapter.

Recall from Chapter 1 that the *Grassmannian* $G_k(\mathbb{C}^n)$ is the space of rank k subspaces of $\mathbb{C}^n$. It is a compact manifold. There is a *canonical bundle* of k-dimensional vector spaces over $G_k(\mathbb{C}^n)$, and if M is any compact manifold, then for n sufficiently large, the isomorphism classes of (continuous or smooth) complex vector bundles on M of rank k correspond to the homotopy classes of maps from M to $G_k(\mathbb{C}^n)$ via the operation which assigns to any map the pullback of the canonical bundle.

In order to obviate the need to continually make n "sufficiently large" it is convenient to speak of the space

$$G_k(\mathbb{C}^\infty) = \lim_{n\to\infty} G_k(\mathbb{C}^n).$$

This is a legitimate topological space in its own right (when given the direct limit topology). But for our purposes we can think of a map from a compact manifold into $G_k(\mathbb{C}^\infty)$ as a compatible family of maps into the $G_k(\mathbb{C}^n)$, for all large enough

n, and we can think of a cohomology class on $G_k(\mathbb{C}^\infty)$ as a family of cohomology classes, one on each $G_k(\mathbb{C}^n)$ which are compatible with one another under the maps in cohomology induced from the inclusions $G_k(\mathbb{C}^n) \subseteq G_k(\mathbb{C}^{n+1})$ obtained by regarding $\mathbb{C}^n$ as a subspace of $\mathbb{C}^{n+1}$. Notice that the canonical bundles on the $G_k(\mathbb{C}^n)$, for different n, are compatible with one another under these inclusion maps. With these conventions, we shall speak of the cohomology ring of $G_k(\mathbb{C}^\infty)$, the canonical bundle E over $G_k(\mathbb{C}^\infty)$, and so on.

If c is a characteristic class of rank k complex vector bundles then we can form the class $c(E) \in H^*(G_k(\mathbb{C}^\infty))$ associated to the canonical bundle. It determines the characteristic class c completely since every bundle is a pullback of E along some classifying map $f\colon M \to G_k(\mathbb{C}^\infty)$, and $c(f^*E) = f^*c(E)$. Moreover we can use this same formula to *define* a characteristic class, starting with any class in $H^*(G_k(\mathbb{C}^\infty))$. These observations prove the following result:

basic-char-class-prop

4.2 Proposition. *Let k be a positive integer. There is a bijection between characteristic classes of rank k complex vector bundles on compact manifolds and classes in the cohomology ring*

$$H^*(G_k(\mathbb{C}^\infty)) = \prod_p H^p(G_k(\mathbb{C}^\infty))$$

which associates to a characteristic class its value in $H^*(G_k(\mathbb{C}^\infty))$ on the canonical bundle. □

Let us consider the case where $k = 1$. The space $G_1(\mathbb{C}^n)$ is none other than the projective space $\mathbb{C}P^{n-1}$ of lines in $\mathbb{C}^n$. So according to Proposition 4.2, in order to determine the characteristic classes of rank-one complex vector bundles—complex line bundles—we need to compute the cohomology ring of the complex projective space $\mathbb{C}P^\infty$. This we will do by computing the cohomology of the finite projective spaces $\mathbb{C}P^{n-1}$.

Recall from Remark **??** that associated to any oriented, rank-d *real* vector bundle V over a closed manifold M there is a *Thom class* u_V in the compactly supported de Rham cohomology group $H^d_c(V)$. Let us apply this to the real bundle underlying the canonical line bundle on $\mathbb{C}P^{n-1}$. This real bundle has rank 2 and is oriented as follows: if e is any non-zero local section then we deem the pair e, ie to be an oriented local frame (the orientation so defined does not depend on e).

4.3 Definition. If V is an oriented, rank d, real vector bundle over a compact manifold M, the *Euler class* of V is the image $e_V \in H^d(M)$ of the Thom class $u_V \in H^d_c(V)$ under the map $H^d_c(V) \to H^d(M)$ induced from the inclusion of M into V as the zero section.

4.4 Remark. The Euler class is a characteristic class of real, oriented vector bundles. The name is derived from the following beautiful theorem (which we shall not need, except to compute examples): if e_{TM} is the Euler class of the tangent bundle of an oriented, closed manifold, then $\int_M e_{TM}$ is equal to the Euler characteristic of M. This result is due to Hopf.

hopf-remark

CPn-prop **4.5 Proposition.** *The cohomology ring* $H^*(\mathbb{C}P^{n-1})$ *is the unital ring generated by the Euler class* $e \in H^2(\mathbb{C}P^{n-1})$ *of the canonical bundle, subject to the relation* $e^n = 0$*.*

To prove this we shall use the following important result, which will also figure in later computations.

4.6 Theorem (Thom Isomorphism Theorem)**.** *If* V *is an oriented, real vector bundle over a compact manifold* M*, then* $H^*_c(V)$ *is a free module over the ring* $H^*(M)$ *generated by the Thom class.*

Proof (sketch). If M is a point, then the result is just a restatement of the characteristic property of the Thom class, that its restriction to each fiber of V generates the cohomology of the fiber. Next, if M is a contractible open manifold, then the result follows from the case of a point, using the homotopy invariance of cohomology.[1] The general result follows by choosing a suitable open cover by contractible sets, and applying a Mayer-Vietoris argument. For more details, see [**?**] or [**?**]. □

4.7 Exercise. Use the Thom isomorphism theorem to prove the following uniqueness theorem for the Thom class: if V is an oriented, rank d real bundle over M, then there is a unique cohomology class $u_V \in H^d_c(V)$ such that $\int_{V_m} u_V = 1$ for every $m \in M$.

Proof of Proposition 4.5. Associated to any rank-d vector bundle over a compact M there is a long exact cohomology sequence

$$\cdots \longrightarrow H^p_c(V) \longrightarrow H^p(D(V)) \longrightarrow H^p(S(V)) \longrightarrow H^{p+1}_c(V) \longrightarrow \cdots$$

where $D(V)$ is the bundle of closed d-disks obtained from V by adding a sphere at infinity to each fiber of V, and $S(V)$ is the bundle of added $(d-1)$-spheres. (These spaces are smooth manifolds in a natural way: if we put an inner product on V then $D(V)$ is diffeomorphic to the closed unit ball bundle and $S(V)$ is diffeomorphic to the unit sphere bundle.) If V is oriented, we may incorporate the Thom isomorphism $H^p_c(V) \cong H^{p-d}(M)$ into this long exact sequence. Observing that $D(V)$ is homotopy equivalent to M, we obtain the *Gysin* long exact sequence

$$\cdots \longrightarrow H^{p-d}(M) \xrightarrow{e_V} H^p(M)$$
$$\longrightarrow H^p(S(V)) \longrightarrow H^{p-d+1}(M) \longrightarrow \cdots$$

in which the map labeled e_V is multiplication by the Euler class. When M is the complex projective space $\mathbb{C}P^n$ and V is the canonical line bundle, the space $S(V)$ may be identified with the unit sphere in $\mathbb{C}^n$. Knowing the cohomology of the unit sphere, it is now easy to deduce the result. □

Passing to the limit as $n \to \infty$ we obtain the following result:

[1] When dealing with non-compact manifolds we should use de Rham cohomology with compact supports only in the fiber direction of V.

4.8 Theorem. *The ring $H^*(G_1(\mathbb{C}^\infty))$ is isomorphic to the ring of formal power series in the Euler class of the canonical line bundle. As a result, the characteristic classes of complex line bundles are in one-to-one correspondence with formal power series.* □

To put it another way, the only characteristic classes of a complex line bundle are the Euler class of the underlying real, oriented plane bundle, and the other classes obtained from it by simple algebraic operations (squaring, cubing, etc, and linear combinations of these).

While this may seem disappointingly (or reassuringly) simple, there are nonetheless some interesting questions to be answered. For instance, the set of isomorphism classes of complex line bundles over M has the structure of an abelian group. The group operation is tensor product of line bundles, and the inverse of L is the class of the complex conjugate line bundle[2] $\overline{L}$. How is this group structure reflected in the theory of characteristic classes?

euler-tensor **4.9 Proposition.** *If L and L' are line bundles over M then $e_{L\otimes L'} = e_L + e_{L'}$. Moreover if L is any line bundle on M then $e_{\overline{L}} = -e_L$.*

Proof. We shall prove the first relation; the second follows from the easily verified fact that the Euler class of the trivial bundle is zero. Let us consider first the "universal" situation in which $M = G_1(\mathbb{C}^n)$ and L and L' are both the canonical line bundle. Construct over $M \times M$ the line bundle L'' whose fiber over a pair (m, m') is $L_m \otimes L'_{m'}$. What is its Euler class? The Kunneth formula in cohomology says that wedge product of forms sets up an isomorphism

$$H^r(M \times M) \cong \oplus_{p+q=r} H^p(M) \otimes H^q(M).$$

In our case we are interested in the formula

$$H^2(M \times M) \cong H^2(M) \otimes H^0(M) \;\oplus\; H^0(M) \otimes H^2(M)$$

(there are no $H^1(M)$ terms since $H^1(M)$ is zero for projective space). By restricting L'' to $M \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \times M$ we see that

$$e_{L''} = e_L \otimes 1 + 1 \otimes e_{L'}.$$

If we now restrict to the diagonal $M \subseteq M \times M$, over which L'' restricts to $L \otimes L'$, then—using once again functoriality of the Euler class—we obtain the formula $e_{L\otimes L'} = e_L + e_{L'}$. In the case of general M and general line bundles, we pull back the formula we have just proved via the product of classifying maps $M \times M \to G_1(\mathbb{C}^n) \times G_1(\mathbb{C}^n)$. □

4.2 CHARACTERISTIC CLASSES OF HIGHER RANK BUNDLES

We are going to approach the characteristic classes of higher rank complex bundles by looking first at those higher rank bundles which can be decomposed into direct sums of line bundles.

---

[2] A rank-d complex line bundle can be thought of as a rank-$2d$ real vector bundle equipped with an endomorphism J ('multiplication by i') such that $J^2 = -1$. The conjugate bundle has the same underlying real bundle, but J is replaced by $-J$.

There is a natural map

$$f_k \colon \underbrace{G_1(\mathbb{C}^n) \times G_1(\mathbb{C}^n) \times \cdots \times G_1(\mathbb{C}^n)}_{k \text{ times}} \to G_k(\mathbb{C}^{kn})$$

which associates to a list of k points in $G_1(\mathbb{C}^n)$ the direct sum of the k lines in $\mathbb{C}^n$ that the points represent. The direct sum can be considered as a k-dimensional subspace of $\mathbb{C}^{kn} = \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n$ and therefore as a point of $G_k(\mathbb{C}^{kn})$.

The pullback of the canonical bundle over $G_k(\mathbb{C}^{kn})$ along f_k is the direct sum of the canonical line bundles on the individual factor spaces $G_1(\mathbb{C}^n)$. In particular, the pullback is a direct sum of line bundles.

Let us consider the corresponding map on cohomology:

$$f_k^* \colon H^*(G_k(\mathbb{C}^{kn})) \to H^*(G_1(\mathbb{C}^n) \times \cdots \times G_1(\mathbb{C}^n)).$$

The symmetric group on k letters, Σ_k, acts on $G_1(\mathbb{C}^n) \times \cdots \times G_1(\mathbb{C}^n)$ by permutations, and if we compose f_k with any permutation, then we obtain a map which is homotopy equivalent to f_k. As a result, the map f_k^* takes $H^*(G_k(\mathbb{C}^{kn})$ into the permutation-invariant part of $H^*(G_1(\mathbb{C}^n) \times \cdots \times G_1(\mathbb{C}^n))$.

Passing to the limit as $n \to \infty$, we obtain a canonical homomorphism

$$f_k^* \colon H^*(G_k(\mathbb{C}^\infty)) \to [H^*(G_1(\mathbb{C}^\infty) \times \cdots \times G_1(\mathbb{C}^\infty))]^{\Sigma_k}.$$

Now by the Kunneth formula,

$$H^*(G_1(\mathbb{C}^\infty) \times \cdots \times G_1(\mathbb{C}^\infty)) \cong H^*(G_1(\mathbb{C}^\infty)) \otimes \cdots \otimes H^*(G_1(\mathbb{C}^\infty)),$$

and we saw in the last section that $H^*(G_1(\mathbb{C}^\infty))$ is a power series ring on one degree-two generator. So f_k^* maps the cohomology ring of $G_k(\mathbb{C}^\infty)$ into the ring of symmetric power series in k degree-two variables. In fact this map is an isomorphism:

split1 **4.10 Theorem.** *The ring homomorphism f_k^* identifies $H^*(G_k(\mathbb{C}^\infty))$ with the permutation-invariant elements in $H^*(G_1(\mathbb{C}^\infty) \times \cdots \times G_1(\mathbb{C}^\infty))$. Thus the ring $H^*(G_k(\mathbb{C}^\infty))$ is isomorphic to the ring of formal power series in degree 2 indeterminates $x_1, \ldots, x_k$ which are symmetric under permutation of the x_j.* □

We shall not prove this result here (but see Remark **??** in Section **??** for some comments on the how it can be done).

The theorem implies that characteristic classes of rank k complex vector bundles correspond to symmetric power series in k variables. How does this correspondence operate? If $F(x_1, \ldots, x_k)$ is a symmetric power series then according to the theorem there is a unique characteristic class c of rank k bundles such that if E is the k-fold direct sum of canonical line bundles on $G_1(\mathbb{C}^\infty) \times \cdots \times G_1(\mathbb{C}^\infty)$, then $c(E) = F(e_{E_1}, \ldots, e_{E_k})$, where e_{E_j} is the Euler class of the canonical line bundle over the jth copy of $G_1(\mathbb{C}^\infty)$. More generally, the same formula holds for any bundle which is a direct sum of line bundles. This prescription does not immediately tell us how the characteristic class c behaves on bundles which are not the direct sum of line bundles, but as we shall see in the next section and also later in the notes, it is frequently possible to reduce computations to the situation in which a vector bundle does split as a sum of line bundles. For this reason, Theorem 4.10 is sometimes called the "splitting principle."

4.11 Remark. Even though we used de Rham theory to define the Euler class, which was our starting point in characteristic class theory, it is not necessary to restrict to spaces M which are smooth manifolds when defining characteristic classes. Indeed, assuming that an extension of cohomology to general spaces has been provided, we can extend the definition of any characteristic class to bundles over general spaces X by the formula $c(E) = f^*c$, where $f\colon X \to G_k(\mathbb{C}^\infty)$ is the classifying map[3] for E and $c \in H^*(G_k(\mathbb{C}^\infty))$ is the cohomology class determined by the characteristic class according to Proposition 4.2.

4.3 THE CHERN CHARACTER

chern-def

4.12 Definition. The *Chern character* is the characteristic class of rank k vector bundles which corresponds to the symmetric formal power series

$$\exp(x_1) + \exp(x_2) + \cdots + \exp(x_k)$$

This definition applies to any positive integer k, so we have really defined a family of characteristic classes, one for each k. If E is any complex vector bundle over M we shall denote by $\mathrm{ch}(E) \in H^*(M)$ its Chern character, obtained by applying the formula in the definition for the appropriate k. According to the results of the previous section, the Chern character is characterized by the fact that if E is a direct sum of line bundles, $E = L_1 \oplus \cdots \oplus L_k$, then

$$\mathrm{ch}(E) = \exp(e_{L_1}) + \cdots + \exp(e_{L_k}),$$

where e_{L_j} is the Euler class of L_j.

The Chern character is important because it is a sort of "ring-homomorphism" from vector bundles to cohomology:

4.13 Proposition. *Let E and F be complex vector bundles over M. Then*

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$$

and

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F).$$

Proof. It suffices to prove these identities in the case where E and F are the pullbacks to the product $G_k(\mathbb{C}^\infty) \times G_\ell(\mathbb{C}^\infty)$ of the universal rank k and ℓ bundles on the two factors (compare the proof of Proposition 4.9). By the Kunneth formula and the theorem in the last section, the classifying map

$$f_{k,\ell}\colon \underbrace{G_1(\mathbb{C}^\infty) \times G_1(\mathbb{C}^\infty) \times \cdots \times G_1(\mathbb{C}^\infty)}_{k+\ell \text{ times}} \to G_k(\mathbb{C}^\infty) \times G_\ell(\mathbb{C}^\infty)$$

is injective on cohomology. It therefore suffices to verify the identities which appear in the statement of the proposition inside the cohomology of the product of the $G_1(\mathbb{C}^\infty)$. In particular, it suffices to prove the formula when E and F are

[3]The space X does need to be regular enough that vector bundles are classified by maps into $G_k(\mathbb{C}^\infty)$; if X is paracompact and Hausdorff this is the case.

complex vector bundles which decompose as direct sums of line bundles. But for a direct sum of line bundles, $\mathrm{ch}(L_1 \oplus \cdots \oplus L_p) = \exp(e_{L_1}) + \cdots + \exp(e_{L_p})$. This makes the relation $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$ obvious, and reduces the general relation $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$ to the special case of line bundles. In that special case the relation $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$ follows from Proposition 4.9 by exponentiation. □

4.4 MULTIPLICATIVE CHARACTERISTIC CLASSES

mult-class-def

4.14 Definition. A characteristic class $\mathcal{C}$ for complex vector bundles is *multiplicative* if

$$\mathcal{C}(E \oplus F) = \mathcal{C}(E) \cdot \mathcal{C}(F),$$

genus-def

for any vector bundles E and F.

Strictly speaking, a multiplicative characteristic class is, like the Chern character, a whole family of characteristic classes, one for each dimension of complex vector bundle. We could equally well have defined the notion of *additive* characteristic class by replacing the equation in Definition 4.14 with the equation $\mathcal{C}(E \oplus F) = \mathcal{C}(E) + \mathcal{C}(F)$. If we did so then the Chern character would be an example. However, the Chern character aside, we shall have more use for multiplicative than additive classes.

The great virtue of multiplicative (or additive) classes is that they may be determined by computation of a very limited set of examples, as the following proposition demonstrates.

4.15 Proposition. *Two multiplicative characteristic classes are equal if they are equal on the canonical line bundles over* $G_1(\mathbb{C}^\infty)$.

Proof. To show that two classes are equal, it suffices to show that they are equal on the universal bundles over $G_k(\mathbb{C}^\infty)$. But by the splitting principle, as illustrated in the last section, it then suffices to show they are equal for direct sums of line bundles. By multiplicativity, we can then reduce to single line bundles; and by universality it finally suffices to identify the two classes on the canonical line bundle over $G_1(\mathbb{C}^\infty)$. □

4.16 Proposition. *Let* $f(x)$ *be a formal power series in* x. *There is a unique multiplicative class* $\mathcal{C}_f$ *such that, on line bundles,*

$$\mathcal{C}_f(L) = f(e_L) \in H^*(M),$$

genus-classification

where e_L *is the Euler class.*

Proof. On rank k bundles, let $\mathcal{C}_f$ be the characteristic class associated to the symmetric formal power series

$$F(x_1, \ldots, x_k) = f(x_1)f(x_2)\cdots f(x_k) \in H^*(G_k(\mathbb{C}^\infty)).$$

By the splitting principle, as illustrated in the previous section, this defines a multiplicative characteristic class. □

4.17 Remark. Usually one confines attention to multiplicative classes corresponding to power series for which $F(1) = 1$ (these are sometimes called *genera*). These are the multiplicative classes which assign the value $1 \in H^0(M)$ to any trivial bundle. But later on we will encounter multiplicative classes which assign the value $(-1)^{\mathrm{rank}(E)} \in H^0(M)$ to any trivial bundle.

4.5 THE TODD CLASS

The most important multiplicative class in index theory is the *Todd class*, which is defined as follows.

4.18 Definition. The *Todd class* is the multiplicative characteristic class which is associated to the power series

$$\frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{720}x^4 + \cdots .$$

4.6 EXERCISES

4.19 Exercise. Let V be an oriented real vector bundle. Show that if $-V$ denotes the same bundle with the opposite orientation, then $e_{-V} = -e_V$. Deduce that the Euler class of an odd-rank, oriented real vector bundle is zero. (Hint: Consider the map $v \mapsto -v$.)

4.20 Exercise. If $\mathcal{C}$ is a multiplicative characteristic class for complex vector bundles, then so is the class $\overline{\mathcal{C}}$ defined by $\overline{\mathcal{C}}(E) = \mathcal{C}(\bar{E})$. If $\mathcal{C}$ corresponds to the formal power series $f(x)$, show that $\overline{\mathcal{C}}$ corresponds to the formal power series
conj-ex $f(-x)$.

4.21 Exercise. Show that no odd powers of x higher than the first appear in this expansion. (The coefficient of x^n for even n is $B_n/n!$, where B_n is the n'th Bernoulli number.)

As we shall see, it is a consequence of the Atiyah-Singer index theorem that if M is any compact complex manifold, then $\int_M \mathrm{Todd}(TM)$ is an integer. The following exercises verify this in the case where $M = \mathbb{C}P^n$.

4.22 Exercise. Show that if V and W are oriented, real vector bundles, then $e_{V \oplus W} = e_V e_W$.

extan **4.23 Exercise.** Show that there is an isomorphism of complex vector bundles $T\mathbb{C}P^n \cong \mathrm{Hom}(L, L^\perp)$, where $L^\perp$ denotes the orthogonal complement of L in the trivial bundle with fiber $\mathbb{C}^{n+1}$. Deduce that $T\mathbb{C}P^n \oplus \mathbb{C} \cong (n+1)\overline{L}$.

4.24 Exercise. Use the previous exercises and the Hopf Theorem (Remark 4.4) to prove that $\int_{\mathbb{C}P^n} e_L^n = (-1)^n$.

4.25 Exercise. Show that $\int_{\mathbb{C}P^n} \mathrm{Todd}(T\mathbb{C}P^n) = 1$. (Using Exercise 4.23, this boils down to showing that the coefficient of x^n in the Taylor series expansion of $(x/(1-e^{-x}))^{n+1}$ is equal to 1 — a property which actually characterizes the Todd power series. To do the computation, use Cauchy's theorem to write the coefficient of x^n as a contour integral, and evaluate the contour integral by an appropriate substitution.)

todd-proj

4.7 NOTES

The classic exposition of the theory of characteristic classes is [**?**]. Other texts that the reader might consult are Bott and Tu [**?**] and Hatcher [**?**].

The idea of a genus (a multiplicative class) was invented by Hirzebruch: see [**?**] and [**?**]. For exotic spheres see [**?**, **?**].

Chapter Five

Elliptic Partial Differential Operators

EllipticChapter

In this chapter we shall lay the foundations for index theory by developing the analysis of elliptic linear partial differential operators on manifolds. In particular, we shall complete the proof of the Hodge theorem (Theorem **??**), which we discussed in Chapter **??**.

The basic structure of the theory of (elliptic) differential operators resembles that of (Fredholm) Toeplitz operators (see Theorem **??**). As in the case of Toeplitz operators, we shall associate a "symbol" to each differential operator on a manifold. The key properties that the "symbol" should possess are these:

(a) The algebra of the symbols should be simpler than the algebra of the operators (ideally, the algebra of symbols should be *commutative* or at least closely related to a commutative algebra).

(b) The symbol of an operator should determine that operator "modulo lower order terms" (compact operators in the Toeplitz case).

The "elliptic" operators will then be those whose symbol is invertible. Property (b) will then imply that such operators are invertible "modulo lower order terms", and a suitable version of Atkinson's theorem translates "invertible modulo lower order" into "Fredholm".

There are several ways to present this theory and in keeping with our general philosophy we have chose one that maximizes the role of Hilbert spaces and C^*-algebras. At the end of the chapter we shall relate this to the classical approach involving smooth functions.

5.1 LINEAR PARTIAL DIFFERENTIAL OPERATORS AS HILBERT SPACE OPERATORS

We will use the theory of unbounded operators on Hilbert space to analyze partial differential operators on manifolds, such as the de Rham differential d which has already made its appearance in Chapter **??**.

Let M be a smooth manifold and let S be a smooth complex vector bundle over M. A complex-linear operator $D\colon C^\infty(M,S) \to C^\infty(M,S)$ acting on the space of smooth sections of S is a *linear partial differential operator* if:

(a) For every smooth section u of S and open set $V \subseteq M$, if u vanishes on V then Du vanishes on V also. (Informally, we can express this by saying that the restriction of Du to V depends only on the restriction of u to V.)

(b) In any coordinate neighborhood of M and local trivialization of S, where sections of S can be written as vector-valued functions of n real variables, the operator D has the form

$$Du = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha u$$

for some $k \geq 0$. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index composed of non-negative integers, ∂^α is shorthand for $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, where ∂_j is the partial derivative in the jth coordinate direction, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The quantities a_α are smooth, matrix-valued functions. (Note that the idea of a "local form" for D makes sense because of (a).)

We shall be mainly interested in *order one* linear partial differential operators—those for which we can take $k = 1$ in the local representation above. We shall concentrate on these in part because the analysis of order one operators is somewhat simpler than the analysis of higher order operators, and in part because most of the fundamental examples to which the index theorem applies are order one operators.

It will be crucial to consider operators which act on sections of vector bundles rather than on scalar functions, because except in low dimensions there are no operators acting on scalar functions which are interesting from the point of view of index theory. In contrast, we shall see several very interesting constructions of non-trivial operators on bundles, even when we restrict attention to operators of order one.

Suppose now that D is a linear partial differential operator acting on the sections of a (complex) vector bundle S over a smooth manifold M. Suppose also that S is provided with a hermitian structure and that M is provided with a smooth measure. In this case we can form the Hilbert space $L^2(M, S)$ of square-integrable sections of S by completing the space of smooth, compactly supported sections in the norm induced from the inner product

$$\langle u, v \rangle = \int_M \langle u(m), v(m) \rangle \, dm.$$

Let us regard D as an unbounded Hilbert space operator on $L^2(M, S)$ with domain the smooth, compactly supported sections of S.

Every linear partial differential operator D has a *formal adjoint* $D^\dagger$, a linear partial differential operator such that

$$\langle u, D^\dagger v \rangle = \langle Du, v \rangle,$$

for all smooth and compactly supported sections u and v. A linear partial differential operator is formally self-adjoint if and only if $D = D^\dagger$.

We shall say that D is compactly supported if there is an open set in M with compact complement such that D vanishes identically on sections supported in that open set. Having set this terminology, we can state the main theorem in this section:

self-adj-thm

5.1 Theorem. *A compactly supported, formally self-adjoint, order one, linear partial differential operator on a manifold is essentially self-adjoint.*

Let us begin by introducing some terminology concerning "generalized solutions" of linear partial differential equations. Let D be a linear partial differential operator on a bundle S over a manifold M. Let $u, v \in L^2(M, S)$. Then:

(i) We say that $Du = v$ in the *strong* sense (or that u is a *strong solution* of this equation) if there is a sequence $\{u_n\}$ of smooth, compactly supported sections such that $u_n \to u$ in $L^2(M, S)$ and $Du_n \to v$ in $L^2(M, S)$.

(ii) We say that $Du = v$ in the *weak* sense (or that u is a *weak* or *distributional* solution)) if, for every smooth, compactly supported section w, we have $\langle D^\dagger w, u\rangle = \langle w, v\rangle$. (The idea here is to think of $\langle D^\dagger w, u\rangle$ as a substitute for $\langle w, Du\rangle$, which is not necessarily well-defined since u may not belong to $\mathrm{dom}(D)$.)

Both of these notions allow us to give a meaning to the equation $Du = v$ for at least some non-smooth sections u and v.

5.2 Exercise. Show that every strong solution to $Du = v$ is a weak solution.

The relevance of these notions to the theorem that we want to prove is the following.

5.3 Lemma. *Let D be a formally self-adjoint linear partial differential operator on a manifold, as above. Consider D as an unbounded Hilbert space operator, with domain the space of compactly supported sections. Then D is essentially self-adjoint if and only if every weak solution of $Du = v$ is also a strong solution.*

Proof. Suppose that every weak solution is a strong solution. To show that D is essentially self-adjoint we must show that the operators $D \pm iI$ have dense range. However, the orthogonal complement of the range of $D \pm iI$ is, by definition, the space of weak solutions of the equation $Dv = \pm iv$. Such a solution must also be a strong solution, so there exists a sequence v_n of smooth, compactly supported sections with $v_n \to v$ and $Dv_n \to iv$. Now

$$0 = \lim_{n\to\infty} \|(D \mp iI)v_n\|^2 \geq \lim_{n\to\infty} \|\|v_n\|^2 = \|v\|^2.$$

Thus $v = 0$. It follows that the ranges of $D \pm iI$ are dense and so D is essentially self-adjoint, as required.

Since we won't need the converse direction of this lemma, we leave it as an exercise for the reader. □

The reader should review the discussion before Definition 3.92 in the light of this proof. It is now clear that Theorem 5.1 will follow from

weak=strong-lemma

5.4 Lemma. *Let D be an order one, linear partial differential operator and let u and v be compactly supported elements of $L^2(M, S)$. If $Du = v$ in the weak sense, then $Du = v$ in the strong sense.*

Proof. Let us suppose first that u and v are supported within a coordinate neighborhood U of M over which the bundle S is trivialized. By shrinking U slightly, we may identify U with an open set in $\mathbb{R}^n$ in such a way that the restriction of D to U identifies with the restriction to U of some compactly supported order one operator D_0 on $\mathbb{R}^n$, acting on vector-valued functions. We shall show that if $Du = v$ in the weak sense, then there are smooth, vector-valued functions u_n, compactly supported in U, such that $u_n \to u$ and $D_0 u_n \to v$.

To do this, we use the existence of a family $\{K_\varepsilon\}_{\varepsilon>0}$ of bounded operators on $L^2(\mathbb{R}^n)$ with the following properties:

(i) For every $v \in L^2(\mathbb{R}^n)$ and every $\varepsilon > 0$, $K_\varepsilon v$ is a smooth function and $\operatorname{supp}(K_\varepsilon v)$ is contained within an ε-neighbourhood of $\operatorname{supp}(v)$.

(ii) $K_\varepsilon v \to v$ as $\varepsilon \to 0$, for every L^2-function v;

(iii) The commutator $[D_0, K_\varepsilon]$ extends to a bounded operator on $L^2(\mathbb{R}^n)$, for every $\varepsilon > 0$, and $[D_0, K_\varepsilon]v \to 0$, for every L^2-function v.

To construct such a family, let f be a smooth, compactly supported function on $\mathbb{R}^n$ with total integral 1, and for $\varepsilon > 0$ take K_ε to be the operator of convolution with $\varepsilon^{-n} f(\varepsilon^{-1}x)$. Since $D_0 u = v$ in the weak sense, it follows that $D_0 K_\varepsilon u = K_\varepsilon v + [D_0, K_\varepsilon]u$ in the honest sense. We see that $D_0 K_\varepsilon u \to v$, while $K_\varepsilon u \to u$, so we can set $u_n = K_{1/n} u$.

In the general case, assume $Du = v$ in the weak sense. Choose compactly supported smooth functions $\sigma_1, \ldots, \sigma_k$ on M such that $\sum \sigma_j = 1$ on the support of D and such that each σ_j is supported in a coordinate chart. Then $D\sigma_j u = \sigma_j v + [D, \sigma_j]u$ in the weak sense (note that the commutator $[D, \sigma_j]$ is a bounded operator), and hence also in the strong sense by the argument just given. Summing over j, and using the fact that $\sum [D, \sigma_j] = 0$, we see that $Du = v$ in the strong sense, as required. $\square$

5.2 CONSTANT COEFFICIENT OPERATORS AND THE SYMBOL

constant-coeff-sec

Let M be a smooth manifold and let S be a vector bundle over M. A (smooth) section of S can be multiplied by a (smooth) function on M, yielding another section of S; to put this another way, the space $C^\infty(S)$ of sections is a *module* over the algebra $C^\infty(M)$ of smooth functions. (Later, we shall see that if M is compact it is a *finite, projective* module.)

Now let D be an order one, linear partial differential operator on M acting on the smooth sections of the bundle S. Then $D\colon C^\infty(S) \to C^\infty(S)$ is a linear map, that is, a homomorphism of vector spaces. Is it also a homomorphism of *modules* over $C^\infty(M)$? — in other words, if f is a smooth function on M acting on sections of S by pointwise multiplication, do the operators Df and fD from $C^\infty(S)$ to itself agree? The product rule for differentiation shows that the answer is "no". But we do have

5.5 Proposition. *Let D be a first order linear partial differential operator on sections of a bundle S, as above. Let $\pi\colon T^*M \to M$ denote the projection of* *the cotangent bundle. There exists a unique section* σ_D *of* $\pi^* \operatorname{End}(S)$ *having the following properties*

(a) σ_D *is homogeneous of degree 1 in the cotangent vectors, that is,* $\sigma_D(x, t\xi) = t\sigma_D(x, \xi)$ *for all* $(x, \xi) \in T^*M$,

(b) *For any real-valued* $f \in C^\infty(M)$*, acting on sections of* S *by pointwise multiplication, any any section* s *of* S*, we have*

$$D(fs) - fDs = (1/i)\sigma_D(df)s.$$

In particular, the commutator $Df - fD$ *is an endomorphism of* S.

(c) *For all* (x, ξ) *we have*

$$\sigma_{D^\dagger}(x, \xi) = \sigma_D(x, \xi)^*$$

where $D^\dagger$ *denotes the formal adjoint of* D.

5.6 Definition. The section σ defined by the above proposition is called the *symbol* of D.

The appearance of $i = \sqrt{-1}$ in the definition of the symbol is purely conventional. It makes the symbol of a (formally) self-adjoint operator a self-adjoint endomorphism of S.

Proof. We compute in local coordinates. Write

$$D = \sum_{j=1}^{n} a_j \partial_j + b$$

where the a_j and b are matrix-valued functions. A short calculation using the product rule for derivatives yields

$$Df - fD = \sum_{j=1}^{n} a_j \partial f/\partial x_j = (1/i)\sigma(df)$$

where

$$\sigma(x, \xi) = \sum_{j=1}^{n} a_j \xi_j$$

which is of the form required. The formula $D(fs) - fDs = (1/i)\sigma_D(df)s$ shows that σ is globally well-defined (independent of the choice of coordinates). As for the adjoint formula, note that

$$(Df - fD)^\dagger = fD^\dagger - D^\dagger f$$

using the properties of adjoints and the fact that $f^\dagger = f$. □

5.7 Remark. More generally we may associate to a differential operator of order n a *principal symbol* which reflects the highest order behavior of the operator and is a section of $\pi^* \operatorname{End}(S)$ which is homogeneous of degree n. The principal symbol of a composite of differential operators is then the composite of their principal symbols. In this book, however, we shall largely restrict attention to operators of order one.

5.8 Definition. The first order operator D is *elliptic* if for every *nonzero* cotangent vector (x, ξ), the symbol $\sigma_D(x, \xi)$ is an *invertible* endomorphism of S. If this condition holds only for x belonging to some open subset $U \subseteq M$, we shall say that D is *elliptic on* U.

5.9 Example. Let us consider the operator $D = d + d^*$ that we introduced in the Chapter **??**. To compute its symbol we shall begin by computing the symbol of the de Rham differential d. According to the definition, if $df = \eta$ then

$$\sigma_d(\eta)\omega = i[d, f]\omega = i\eta \wedge \omega.$$

So the symbol of d is given in a very simple way by wedge product of forms. Since the symbol of the adjoint d^* is the adjoint of the symbol d, we find that the symbol of the operator $D = d + d^*$ is given by the formula

$$\sigma_D(\eta)\omega = i\eta \wedge \omega - i\eta \lrcorner \omega,$$

where the operator $\omega \mapsto \eta \lrcorner \omega$ is the adjoint of the map $\omega \mapsto \eta \wedge \omega$.

5.10 Lemma. *Let* V *be a finite-dimensional inner product space and let* $S = \wedge^* V$. *If* $v \in V$, *then the operator* $c\colon S \to S$ *given by the formula* $c(v)(x) = v \wedge x - v \lrcorner x$ *has the property that* $c(v)^2 = -\|v\|^2 I$.

signature-symbol-lemma

Proof. We can assume that v is a unit vector and the first vector in an orthonormal basis $v_1, \ldots, v_k$ for V. The products $v_{i_1} \wedge \cdots \wedge v_{i_p}$, where $i_1 < \cdots < i_p$, form an orthonormal basis for $S = \wedge^* V$, and in this orthonormal basis the operator $x \mapsto v \wedge x$ acts as

$$v_{i_1} \wedge \cdots \wedge v_{i_p} \mapsto \begin{cases} v \wedge v_{i_1} \wedge \cdots \wedge v_{i_p} & \text{if } i_1 \neq 1 \\ 0 & \text{if } i_1 = 1. \end{cases}$$

The operator is therefore a partial isometry, and its adjoint is therefore given by the formula

$$v_{i_1} \wedge \cdots \wedge v_{i_p} \mapsto \begin{cases} v_{i_2} \wedge \cdots \wedge v_{i_p} & \text{if } i_1 = 1 \\ 0 & \text{if } i_1 \neq 1. \end{cases}$$

The lemma follows immediately from these formulas. □

As a result of this computation, the square of the symbol of $D = d + d^*$ is $\|\eta\|^2 I$. Thus, the symbol is invertible for all $\eta \neq 0$—indeed up to a scalar multiple the symbol is its own inverse—and as a result D is elliptic.

We begin by looking at the simplest sorts of partial differential operators, the *constant coefficient* operators on $\mathbb{R}^n$. Unless otherwise specified the domain of such an operator will always be the space of smooth, compactly supported functions. It is important to consider operators that operate on *vector-valued* functions (so that our constant coefficients will be constant *matrices*). As usual we shall restrict attention to such operators of order one. These have the form

$$D = \sum_{j=1}^{n} a_j \partial_j + b$$

where the a_j and b are constant matrices.

The definitions of the preceding section now take the following form:

elliptic-constant

5.11 Definition. The *symbol* of a constant coefficient, order one partial differential operator $D = \sum_{j=1}^{n} a_j \partial_j + b$ is the matrix-valued function

$$\sigma(\xi) = i \sum a_j \xi_j .$$

The operator D is *elliptic* if, for all $\xi \neq 0$, the matrix $\sigma(\xi)$ is invertible.

5.12 Example. Consider the operator $\overline{\partial} = \partial_1 + i\partial_2$ on $\mathbb{R}^2$. The symbol of this operator is the (one by one) matrix $i\xi_1 - \xi_2$, which is a nonzero complex number for all nonzero ξ. Therefore $\overline{\partial}$ is elliptic.

5.13 Definition. We shall say that a bounded operator T on $L^2(\mathbb{R}^n)$ is *locally compact* if, for every function $g \in C_0(\mathbb{R}^n)$ (thought of as acting on $L^2(\mathbb{R}^n)$ as a multiplication operator), the operators $g \cdot T$ and $T \cdot g$ are compact.

5.3 ELLIPTIC CONSTANT COEFFICIENT OPERATORS

The following theorem is the main result of this section:

main-const-coeff-thm

5.14 Theorem. *Let D be a formally self-adjoint, constant coefficient, order one partial differential operator on $\mathbb{R}^n$. Then D is essentially self-adjoint. Moreover, if D is elliptic, then for every $f \in C_0(\mathbb{R})$ the operator $f(D)$ is locally compact.*

We shall prove the theorem using the Fourier transform. Recall that if u is a smooth function on $\mathbb{R}^n$, then the *Fourier transform* $\hat{u}$ of u is given by

$$\hat{u}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx.$$

The *Plancherel theorem* states that the Fourier transform extends to a unitary isomorphism from the Hilbert space of $L^2(\mathbb{R}^n)$ to itself. The Fourier transform also maps the space of Schwartz-class functions on $\mathbb{R}^n$ isomorphically onto itself. The Fourier transform converts differentiation (in a certain coordinate direction) into multiplication (by the corresponding coordinate function). Thus we may rewrite a constant coefficient operator $D = \sum_j a_j \partial_j + b$ in terms of the Fourier transform and the symbol of D, as follows:

$$(\widehat{Du})(\xi) = i \sum a_j \xi_j \hat{u}(\xi) = \sigma(\xi)\hat{u}(\xi) + b\hat{u}(\xi). \tag{5.1}$$ xdisp-eq

The Fourier transform also converts pointwise multiplication of functions into convolution of their Fourier transforms: if v is smooth and compactly supported, then

$$\widehat{uv}(\xi) = \hat{u} * \hat{v}(\xi) = (2\pi)^n \int_{\mathbb{R}^n} \hat{u}(\xi - \eta)\hat{v}(\eta)\, d\eta.$$

basic-compactness-lemma

5.15 Lemma. *The composite of the operator of convolution by a Schwartz-class function with the operator of pointwise multiplication by a C_0-function is a compact Hilbert space operator.*

Proof. By an approximation argument, it suffices to prove the lemma in the case where both the Schwartz-class function ϕ and the C_0-function ψ are smooth and compactly supported. In this case the composition is given by the integral operator

$$K\theta(\xi) = (2\pi)^n \int_{\mathbb{R}^n} \kappa(\xi,\eta)\theta(\eta)\, d\eta,$$

where $\kappa(\xi,\eta) = \psi(\xi)\phi(\xi-\eta)$. The kernel function $\kappa(\xi,\eta)$ is smooth and compactly supported; this is a well-known criterion for the compactness of K. □

Proof of Theorem 5.14. An approximation argument shows that the domain of the closure of D contains the Schwartz-class functions. To prove the first part of the theorem it therefore suffices to show that the operator D with domain the Schwartz-class functions is essentially self-adjoint. Using the Fourier transform, we see that D is unitarily equivalent to the operator of multiplication by a polynomial, with domain the Schwartz-class functions. It follows from Exercise 3.94 that D is essentially self-adjoint.

Assume now that D is elliptic and that $g \in C_c^\infty(\mathbb{R}^n)$. Using the Fourier transform, D is unitarily equivalent to the operator of multiplication by $\sigma + b$, where σ is the symbol of D (see Equation 5.1). Hence if $f \in C_0(\mathbb{R})$, then $f(D)$ is unitarily equivalent to the operator of multiplication by $f(\sigma+b)$. Under Fourier transform, the operator of pointwise multiplication by g corresponds to the operator of convolution by the Schwartz-class function $\hat{g}$. The hypothesis of ellipticity implies that the continuous function $f(\sigma+b)$ on $\mathbb{R}^n$ vanishes at infinity. The required compactness property therefore follows from Lemma 5.15. □

5.4 ELLIPTIC OPERATORS ON MANIFOLDS

ell-ops-mflds-sec

We are now going to consider formally self-adjoint partial differential operators $D = \sum a_j \partial_j + b$ on $\mathbb{R}^n$ with *variable* coefficients. For simplicity we shall assume that the smooth, matrix-valued functions a_j and b are *compactly supported*. By Theorem 5.1, this implies that D is essentially self-adjoint. We are going to prove the following result, which parallels Theorem 5.14:

var-coeff-thm

5.16 Theorem. *Let D be a formally self-adjoint, compactly supported, order one, linear partial differential operator on $\mathbb{R}^n$. Then D is essentially self-adjoint. Moreover, if D is elliptic on an open set U, then for every $f \in C_0(\mathbb{R})$ the operator $f(D)$ is locally compact on U.*

The conclusion means, more precisely, that $f(D)\cdot g$ and $g \cdot f(D)$ are compact for all $f \in C_0(U)$. We shall prove the theorem by the "method of freezing coefficients". That is, at each point $x \in U$ we shall compare D with the (elliptic) constant coefficient operator D_x on $\mathbb{R}^n$ whose coefficients agree with the coefficients of D at x. The main technical result that we need is as follows:

local-elliptic-lemma

5.17 Lemma. *Let $D = \sum a_j\partial_j + b$ be a formally self-adjoint, compactly supported, order one, linear partial differential operator on $\mathbb{R}^n$, and assume that D is elliptic over an open set U. Let $x \in U$ and let D_x be the constant coefficient elliptic* *operator* $\sum a_j(x)\partial_j + b(x)$. *For every* $\varepsilon > 0$ *there is a function* $g \in C_0(\mathbb{R}^n)$ *such that* $g(x) = 1$ *and*

$$\|(tD + iI)^{-1} \cdot g - g \cdot (tD_x + iI)^{-1}\| < \varepsilon,$$

for all sufficiently small $t > 0$.

We shall prove this in a moment. But first, we shall apply it to Theorem 5.16 using a C^*-algebraic localization principle which will also feature in a related computation in Chapter **??**. First we introduce some terminology. Let A be a unital C^*-algebra, let X be a locally compact Hausdorff space, and let $\alpha\colon C_0(X) \to A$ be a unital $*$-homomorphism. Let $a \in A$ be an element that commutes with $\alpha(C_0(X))$. We will say that a is *null at* $x \in X$ if for every $\varepsilon > 0$, there is a function $h \in C_0(X)$ such that $h(x) = 1$ and $\|\alpha(h)a\| < \varepsilon$.

cstar-lemma

5.18 Lemma. *With the above notation, suppose that* a *is null at every point of some open subset* $U \subseteq X$. *Then* $\alpha(g)a = 0$, *for every* $g \in C_0(U)$.

Proof. By replacing a with aa^* we can assume that a is positive. By replacing X with U we may assume $U = X$. Fix some $g \in C_0(X)$. The C^*-algebra B generated by $\alpha[C_0(X)]$ and a is commutative. There is therefore some $*$-homomorphism $\phi\colon B \to \mathbb{C}$ such that

$$\|\alpha(g)a\| = \phi(\alpha(g)a) = \phi(\alpha(g))\phi(a).$$

The composition $\phi \circ \alpha\colon C_0(X) \to B$ is evaluation at some $x \in X$ or zero. If it is zero, we are done. Otherwise, fix $\varepsilon > 0$ and choose h such that $h(x) = 1$ and $\|\alpha(h)a\| < \varepsilon$. We get

$$\begin{aligned}\|\alpha(g)a\| = \phi(\alpha(g))\phi(a) &= \phi(\alpha(h))\phi(\alpha(g))\phi(a)\\ &= \phi(g)\phi(\alpha(h)a)\\ &< \phi(g)\varepsilon,\end{aligned}$$

and hence $\alpha(g)a = 0$. $\square$

Proof of Theorem 5.16, assuming Lemma 5.17. Let $g \in C_0(U)$. We are going to prove that $g \cdot f(D)$ defines the zero element of the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$.

Let A be the quotient of the C^*-algebra of bounded, continuous maps from $(0, 1]$ to the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ by the ideal of continuous maps from $(0, 1]$ to $\mathcal{B}(H)/\mathcal{K}(H)$ which vanish at 0. Let $\alpha\colon C_0(\mathbb{R}^n) \to A$ be the $*$-homomorphism given by the natural action of functions on $\mathbb{R}^n$ as multiplication operators. Let $a \in A$ be the element defined by the function $t \mapsto (tD + iI)^{-1}$. We shall begin by verifying that $\alpha(g)a = 0$ in A.

The identity

$$[(tD + iI)^{-1}, h] = -t(tD + iI)^{-1}[D, h](tD + iI)^{-1}$$

shows that a commutes with the range of α. Now let $x \in U$ and let D_x be the constant coefficient operator defined, as in Lemma 5.17, by freezing the coefficients of D at x. Let $a_x \in A$ be defined by the function $t \mapsto (tD_x + iI)^{-1}$. By Theorem 5.14, the operators $(tD_x + iI)^{-1}$ are locally compact, and therefore $a_x\alpha(g) = 0 \in A$ for all $g \in C_0(\mathbb{R}^n)$.

Now we apply the yet-to-be-proved Lemma 5.17 to see that for each $x \in U$ and each $\varepsilon > 0$ there is $g_x \in C_0(\mathbb{R}^n)$ with $g_x(x) = 1$ and $\|\alpha(g_x)a - a_x\alpha(g_x)\| < \varepsilon$. The second term inside the norm here is zero in A, as we just observed. Thus $\|\alpha(g_x)a\| < \varepsilon$. Now we can apply Lemma 5.18, which proves that $\alpha(g)a = 0$ in A, as we claimed.

To complete the proof we note that

$$\begin{aligned} g \cdot f(D) &= -i \lim_{t\to 0} g \cdot f(D)(tD + iI)^{-1} \\ &= -i \lim_{t\to 0} g \cdot (tD + iI)^{-1} f(D). \end{aligned}$$

According to the result just proved, the last limit is zero in the Calkin algebra. Therefore $g \cdot f(D)$ is zero in the Calkin algebra too, and hence is a compact operator. □

It therefore remains to prove Lemma 5.17.

5.19 Definition. Let u be a smooth, compactly supported function on $\mathbb{R}^n$. The *Sobolev* 1*-norm* of u is the quantity $\|u\|_1$ defined by

$$\|u\|_1^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)|\hat{u}(\xi)|^2 \, d\xi,$$

where $\hat{u}$ denotes the Fourier transform. The *Sobolev space* $W^1(\mathbb{R}^n)$ is the completion of the smooth, compactly supported functions in the Sobolev norm.

By the Plancherel theorem, $\|u\|_1$ is greater than the L^2-norm of u and the identity map on smooth, compactly supported functions extends to an embedding of $W^1(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ as a dense subspace.

first-sobolev-lemma

5.20 Lemma. *If D is a formally self-adjoint, constant coefficient, order one, elliptic operator on $\mathbb{R}^n$, then the operator $(D + iI)^{-1}$ maps $L^2(\mathbb{R}^n)$ continuously into $W^1(\mathbb{R}^n)$.*

Proof. It follows from the Plancherel theorem that $\|(D+iI)^{-1}u\|_1 \leq C \cdot \|u\|$. □

second-sobolev-lemma

5.21 Lemma. *Every compactly supported first order operator D on $\mathbb{R}^n$ extends to a bounded operator from $W^1(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Moreover the norm of D considered as an operator $W^1(R^n) \to L^2(\mathbb{R}^n)$ is bounded by a multiple of the supremum norm of the coefficients of D.*

Proof. The lemma follows from the fact (verified by the Plancherel theorem again) that each partial derivative ∂_j extends to a bounded operator from $W^1(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. □

Proof of Lemma 5.17. Let g be any smooth, compactly supported function on $\mathbb{R}^n$. Write[1]

$$(tD \pm iI)^{-1}g - g(tD_x \pm iI)^{-1} = t(tD \pm iI)^{-1}(gD_x - Dg)(tD_x \pm iI)^{-1}.$$

[1]Strictly speaking, the identity does not make sense, since D is not defined on the range of $(tD_x \pm iI)^{-1}$. This problem can be remedied by working with the operator closure of D.

Split the right hand side into two terms by putting
$$gD_x - Dg = gD_x - gD - [D, g].$$
The right hand side now becomes
$$(tD \pm iI)^{-1} \cdot (gD_x - gD) \cdot t(tD_x \pm iI)^{-1} - t(tD \pm iI)^{-1}[D, g](tD_x \pm iI)^{-1}.$$
The second term is of order t in L^2-operator norm and hence tends to zero in operator norm as $t \to 0$. Write the first term as
$$(tD \pm iI)^{-1} \cdot (gD_x - gD)(D_x + iI)^{-1} \cdot t(D + iI)(tD_x \pm iI)^{-1}.$$
The operators $t(D+iI)(tD_x \pm iI)^{-1}$ are uniformly bounded in t in the L^2-operator norm. The operator $gD_x - gD$ is a first order differential operator and we can make its coefficients small by choosing g supported near x, because the coefficients of D and D_x agree at x. According to Lemma 5.21, by suitable choice of g, we can make the norm of $gD_x - gD$, as an operator from $W^1(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, as small as we want. Since $(D_x + iI)^{-1}$ maps $L^2(\mathbb{R}^n)$ boundedly into $W^1(\mathbb{R}^n)$, it follows that the L^2-operator norm of the middle term may be made as small as we want by a suitable choice of g. Since the operators $(tD \pm iI)^{-1}$ are bounded uniformly in t we conclude, finally, that the first term in the previous display can be made as small as we wish in operator norm, uniformly in t, by suitable choice of g. This completes the proof. □

We can now state the main result of this chapter.

cpt-ell-theorem

5.22 Theorem. *A formally self-adjoint first-order elliptic operator on a closed manifold is essentially self-adjoint and has compact resolvent.*

Proof. For essential self-adjointness see Theorem 5.1. For compact resolvent, it suffices to show that every point of M has a neighborhood U such that, if g is a smooth function that is compactly supported within U, then the operator $g \cdot (D + iI)^{-1}$ is compact.

Take U to be a disk in a coordinate neighborhood on M. Since U is contractible, the bundle S on which D acts is isomorphic, over U, to the trivial bundle with fibre $\mathbb{C}^n$. Let $\Phi\colon U' \to U$ be the diffeomorphism from an open subset of $\mathbb{R}^n$ to U provided by the local coordinates, and let $g'\colon U' \to \mathbb{R}$ be the smooth, compactly supported function obtained by composing g with Φ. Denote by
$$A\colon L^2(U; S) \to L^2(U')^N$$
the invertible operator determined by the coordinates on U and the given trivialization of S. Let D' be a formally self-adjoint, compactly supported, order one, linear partial differential operator on $\mathbb{R}^n$, acting on N-vector valued functions, that agrees with D in a neighbourhood of $\operatorname{supp}(g')$ under the given identifications. Note, in particular, that D' is elliptic in a neighbourhood of $\operatorname{supp}(g')$. Since $D'A = AD$ on smooth sections of S compactly supported within U, it follows that
$$(D' + iI)^{-1}g'A - Ag(D + iI)^{-1} = (D' + iI)^{-1}(g'D' - D'g')A(D + iI)^{-1},$$
or that
$$Ag(D + iI)^{-1} = (D' + iI)^{-1}g'A - (D' + iI)^{-1}(g'D' - D'g')A(D + iI)^{-1}.$$
By Theorem 5.16, both terms on the right hand side are compact operators, so the theorem is proved. □

In particular, an operator of this kind is Fredholm. As we saw in Chapter 2, to obtain a non-trivial index from a self-adjoint Fredholm operator requires an additional datum—a *grading*. For differential operators the only gradings that we shall consider are those coming from a grading of the underlying vector bundle S.

5.23 Definition. A *grading* of the vector bundle S is an endomorphism ε of S satisfying $\varepsilon^2 = 1$, $\varepsilon = \varepsilon^*$. A differential operator D is *odd* with respect to the grading if $D\varepsilon + \varepsilon D = 0$ (as operators on smooth sections of S).

A grading of the vector bundle S automatically gives rise to a grading of the Hilbert space $L^2(M, S)$, and (if M is compact) the closure of D is an odd, self-adjoint operator with respect to this grading. It therefore has a well-defined Fredholm index

$$\operatorname{Ind}(D, \varepsilon) = \dim(\ker D \cap \{\varepsilon = 1\}) - \dim(\ker D \cap |\{\varepsilon = -1\}).$$

The project of the index theorem is to compute this quantity in terms of the symbol of D.

The index of the operator D in the sense of this chapter is computed by counting (having due regard to the grading) the number of linearly independent (strong) L^2 solutions of the equation $Du = 0$. In contrast, in Chapter **??** we dealt exclusively with ordinary, smooth solutions of the equation $Du = 0$ (for the special operator $D = d + d^*$). In this section we shall reconcile this difference by sketching the proof of the following *regularity principle*: if D is a first order, elliptic differential operator, if $Du = v$ in the strong sense, and if v is smooth, then u is smooth. Apart from reconciling the two notions of $\operatorname{Ind}(D, \varepsilon)$, the results of this section will not otherwise be used in the book.

We begin by defining higher Sobolev spaces.

5.24 Definition. Let u be a smooth, compactly supported function on $\mathbb{R}^n$. Let $s \in \mathbb{R}$. The *Sobolev* s*-norm* of u is the quantity $\|u\|_s$ defined by

$$\|u\|_s^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi.$$

The *Sobolev space* $W^s(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ in the Sobolev norm.

These are relevant to the problem of relating L^2-solutions of partial differential equations to smooth solutions by virtue of the following important fact:

sobolev-lemma

5.25 Lemma. *If* $s > \frac{n}{2} + k$*, then* $W^s(\mathbb{R}^n)$ *is included within* $C_0^k(\mathbb{R}^n)$*, the Banach space of* k*-times continuously differentiable functions on* $\mathbb{R}^n$ *whose derivatives up to order* k *vanish at infinity.*

Proof. We need to show that the C^k-norm of a smooth, compactly supported function on $\mathbb{R}^n$ is bounded by a multiple of the Sobolev s-norm on $\mathbb{R}^n$ whenever $s > \frac{n}{2} + k$. This will imply that the identity map on $C_c^\infty(\mathbb{R}^n)$ extends to a continuous map of $H^s(\mathbb{R}^n)$ into $C^k(\mathbb{R}^n)$, and this will imply the required result. The *Fourier inversion formula* asserts that

$$u(x) = \int e^{ix\cdot\xi} \hat{u}(\xi) \, d\xi,$$

for all smooth, compactly supported functions u, and hence that

$$\partial^\alpha u(x) = \int e^{ix\cdot\xi}(i\xi)^\alpha \hat{u}(\xi)\, d\xi.$$

Therefore, by the Cauchy-Schwarz inequality,

$$|\partial^\alpha u(x)|^2 \le \int (1+|\xi|^2)^{-s}|\xi|^{2\alpha}\, d\xi \cdot \int (1+|\xi|^2)^s |\hat{u}(\xi)|^2\, d\xi.$$

If $s > \frac{n}{2} + k$ and $k \ge |\alpha|$ then the first integral is finite. Taking square roots we get the required estimate $\sup_x |\partial^\alpha u(x)| \le \text{constant}\|u\|_s$. □

We are going to prove the following result:

local-ell-reg-thm **5.26 Theorem.** *Let D be a compactly supported, order one, linear partial differential operator on $\mathbb{R}^n$ which is elliptic over an open set $U \subseteq \mathbb{R}^n$. Let u and v be L^2-functions that are compactly supported within U. If $Du = v$ in the weak sense, and if $v \in W^s(\mathbb{R}^n)$ for some $s \ge 0$, then $u \in W^{s+1}(\mathbb{R}^n)$.*

regularity-theorem **5.27 Corollary.** *Let D be a formally self-adjoint, order one, linear elliptic partial differential operator on the sections of a smooth bundle S over a smooth, closed manifold M. If $Du = v$ in the weak sense, and if v is smooth, then u is smooth.*

Proof. Assume that $Du = v$ in the weak sense, and that v is smooth. To prove the result it suffices to show that if ϕ is a smooth function which is compactly support in a coordinate neighbourhood, then ϕu is smooth.

Let ϕ_1 be a smooth function which is compactly supported within a coordinate neighbourhood and which is equal to 1 on a neighbourhood $\operatorname{supp}(\phi)$. Let $u_1 = \phi_1 u$ and let $v_1 = [D, \phi_1]u + \phi_1 v$. Then $Du_1 = v_1$ in the strong sense. Since u_1 and v_1 are compactly supported in a coordinate neighbourhood, we can regard them as functions on $\mathbb{R}^n$. Replace D by a suitable compactly supported operator on $\mathbb{R}^n$ which is equal to the original D near the supports of u_1 and v_1. It is then easy to check that $Du_1 = v_1$ in the weak sense, and we are therefore in a position to apply Theorem 5.26. Since v_1 belongs to $W^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, it follows that $u_1 \in W^1(\mathbb{R}^n)$, and hence $\phi u = u_1 \in W^1(\mathbb{R}^n)$.

Now let ϕ_2 be a smooth function that is supported in the set where $\phi_1 = 1$, that is equal to 1 in a neighbourhood of $\operatorname{supp}(\phi)$. Let $u_2 = \phi_2 u$ and let

$$v_2 = [D, \phi_2]u + \phi_2 v = [D, \phi_2]u_1 + v.$$

Then $Du_2 = v_2$ in the weak sense, and in addition $v_2 \in W^1(\mathbb{R}^n)$ (since we just finished proving that $u_1 \in W^1(\mathbb{R}^n)$). We conclude from Theorem 5.26 that $u_2 \in W^2(\mathbb{R}^n)$, and hence $\phi u = \phi u_2 \in W^2(\mathbb{R}^n)$.

Continuing in this way, we see that $\phi u \in W^s(\mathbb{R}^n)$, for all s, and hence by Lemma 5.25, ϕu is a smooth function. □

The main step towards the proof of Theorem 5.26 is the following estimate, known as the *basic elliptic estimate*:

basic-est **5.28 Theorem.** *Let* D *be an order one linear partial differential operator on* $\mathbb{R}^n$ *which is elliptic over some open subset* $U \subseteq \mathbb{R}^n$. *Let* K *be a compact subset of* U *and let* $s \in \mathbb{N}$. *There exists* $\delta > 0$ *such that*

$$\|u\|_0 + \|Du\|_s \geq \delta\|u\|_{s+1}$$

for all smooth functions u *that are supported within* K.

The proof requires a few facts, which we shall present to the reader as exercises.

sob-fact1-ex **5.29 Exercise.** Show that for every $s \geq 0$ and every $\varepsilon > 0$ there exists $C > 0$ such that

$$\|u\|_s \leq C\|u\|_0 + \varepsilon\|u\|_{s+1}.$$

sob-fact2-ex **5.30 Exercise.** Let $s \in \mathbb{N}$. Show that multiplication by a smooth, compactly supported function determines a bounded operator on each $W^s(\mathbb{R}^n)$. Hint: Thanks to the Plancherel theorem, there is an equivalence of norms $\|u\|_s \approx \sum_{|\alpha| \leq s} \|\partial^\alpha u\|_0$.

sob-fact3-ex **5.31 Exercise.** Prove that D is a differential operator D of order q, and if the leading coefficients of D vanish at $x \in \mathbb{R}^n$, then for every $\varepsilon > 0$ there is a neighbourhood W of x and a constant $C > 0$ such that

$$\|Du\|_0 < \varepsilon\|u\|_0 + C\|u\|_{q-1}$$

for every u supported in W.

Proof of Theorem 5.28. The first step in the proof is to observe that if D is an order one, elliptic constant coefficient operator, and if $q \in \mathbb{N}$, then there exists $\delta > 0$ such that

$$\|u\|_0 + \|D^q u\|_0 \geq \delta\|u\|_q$$

for all smooth, compactly supported u. This is an exercise in Fourier theory, and it follows that for all q there is an equivalence of norms $\|u\|_0 + \|D^q u\|_0 \approx \|u\|_q$.

Before going on, if W is an open subset of U and s is a non-negative integer, then let us agree to say that s-*estimate holds over* W if the inequality of the lemma holds for the given value of s and all compact sets $K \subseteq W$ (with constant δ depending on K). Let us also note that for a general order one operator D, the assertion that for $q = s$ and $q = s+1$ and for every compact set $K \subseteq W$, there exists an equivalence of norms $\|u\|_0 + \|D^q u\|_0 \approx \|u\|_q$ for smooth functions u supported in K implies the s-estimate for D over W.

Let D be a variable coefficient elliptic operator, and for $x \in U$ let D_x be the constant coefficient operator obtained by freezing the coefficients of D at x. For every q, the leading coefficients of D^q and D_x^q agree at x. Therefore by Exercise 5.31, for every $\varepsilon > 0$ and every q there is a small neighbourhood W of x and a constant $C > 0$ for which

$$\|D^q u - D_x^q u\|_0 \leq \varepsilon\|u\|_q + C\|u\|_{q-1},$$

for every u supported in W. This follows from the fact that the leading coefficients of $D^q - D_x^q$ vanish at x. It follows from Exercise 5.29 that for every $x \in U$ and every q there is an equivalence of norms

$$\|u\|_0 + \|D^q u\|_0 \approx \|u\|_q$$

on the smooth functions u supported in a sufficiently small neighbourhood of x. This implies that for every $x \in U$ and every s there is a neighbourhood of x such that the s-estimate holds over that neighbourhood.

Now, given a compact set $K \subseteq U$ and $s \in \mathbb{N}$, cover K by finitely many open subsets of U over each of which the s-estimate holds, and let $\{\sigma_j\}$ be a smooth partition of unity on K which is subordinate to this cover. Let u be compactly supported in K and write

$$\begin{aligned}
\|u\|_{s+1} &= \|\sum_j \sigma_j u\|_{s+1} \\
&\le \sum_j \|\sigma_j u\|_{s+1} \\
&\lesssim \sum_j \|D\sigma_j u\|_s + \sum_j \|\sigma_j u\|_0 \\
&\le \sum_j \|\sigma_j D u\|_s + \sum_j \|[D, \sigma_j]u\|_s + \sum_j \|\sigma_j u\|_0 \\
&\lesssim \|Du\|_s + \|u\|_0 + \|u\|_s
\end{aligned}$$

(the notation $a \lesssim b$ means $a \le$ *constant* $\cdot\, b$). The basic estimate now follows from Exercise 5.29. □

Proof of Theorem 5.26. The theorem can be proved using a further application of the family $\{K_\varepsilon\}_{\varepsilon>0}$ introduced in the proof of Lemma 5.4. One can show that the operators introduced there have the following additional properties;

(i) If $v \in W^s(\mathbb{R}^n)$, then $K_\varepsilon v \to v$ in $W^s(\mathbb{R}^n)$ as $\varepsilon \to 0$.

(ii) For every $\varepsilon > 0$ the operator $[D, K_\varepsilon]$ extends to a bounded operator on $W^s(\mathbb{R}^n)$. If $v \in W^s(\mathbb{R}^n)$, then $[D, K_\varepsilon]v$ converges to zero in $W^s(\mathbb{R}^n)$.

Assume now that $Du = v$ in the weak sense, and that $v \in W^s(\mathbb{R}^n)$. If $u_n = K_{\frac{1}{n}} u$, then it follows from the properties above that $Du_n \to v$ in $W^s(\mathbb{R}^n)$ and $u_n \to u$ in $W^0(\mathbb{R}^n)$. It therefore follows from the basic elliptic estimate that $\{u_n\}$ is a Cauchy sequence in $W^{s+1}(\mathbb{R}^n)$. It follows that the L^2-limit, u, actually lies in $W^{s+1}(\mathbb{R}^n)$, as required. □

Let $\pi \colon X \to Y$ be a submersion between smooth manifolds. The manifolds may have boundaries, but if so, then we require that the boundary of X be the inverse image of the boundary of Y. The fibers $X_y = \pi^{-1}\{y\}$ are then smooth manifolds without boundary.

We shall assume that each fiber X_y is equipped with a smooth measure μ_y, and that if f is a smooth, compactly supported function on X, then the quantity $\int_{X_y} f(x)\, d\mu_y(x)$ is a smooth function of y.

Let S be a smooth Hermitian vector bundle over X and let S_y be its restriction to X_y. The Hilbert spaces $\mathcal{H}_y = L^2(X_y, S_y)$ form a continuous field of Hilbert spaces $\mathcal{H}$ over Y whose continuous sections are generated (in the sense of [**?**, Proposition 10.2.3]) by the smooth compactly supported sections of S.

Let D_y be a first-order linear partial differential operator acting on the sections of S_y and suppose that the family $D = \{D_y\}$ is *smooth* in the sense that if u is a smooth section of S on X, then the section Du defined by

$$(Du)\big|_{X_y} = D_y(u\big|_{X_y})$$

is also smooth.

We shall assume that each D_y is formally self-adjoint. To apply the index construction of the last section we shall need to obtain from D_y an operator that is self-adjoint in the sense of Hilbert space theory (see for example [**?**, Ch. 5]). For this the following concept is useful.

5.32 Definition (See [**?**, Definition 10.2.8])**.** The manifold X is *complete with respect to* D if there is a smooth, proper function $g\colon X \to [0, \infty)$ such that the commutator $[D, g]$ is a uniformly bounded endomorphism of S.

5.33 Proposition (See [**?**, Proposition 10.2.10])**.** *If* X *is complete with respect to* D*, then each formally self-adjoint operator* D_y *is essentially self-adjoint on the smooth, compactly supported sections of* S_y. □

Recall that an operator is essentially self-adjoint if its operator-theoretic closure is self-adjoint (see for example [**?**, Ch. 5] again). From now on we shall assume that X is complete for D, and in a slight abuse of notation we shall write D_y when in operator-theoretic contexts we actually mean the closure of D_y. Form the resolvent family

$$r(D) := (D + iI)^{-1} = \{(D_y + iI)^{-1}\}_{y \in Y}.$$

It is certainly a bounded operator on the continuous field $\mathcal{H}$. To say more, we shall suppose from here on that each operator D_y is *elliptic*. We can then draw the following conclusion.

rellich-prop2 **5.34 Proposition.** *If* f *is a smooth, compactly supported function on* X*, acting on the continuous field* $\mathcal{H}$ *as a family of multiplication operators, then* $f{\cdot}r(D)$ *is a compact endomorphism of* $\mathcal{H}$*.*

This is standard fare, but let us sketch a proof based on a C^*-algebra calculation.

cstar-tech-lemma **5.35 Lemma.** *Let* A *be a* C^**-algebra that includes* $C_0(X)$ *as a* C^**-subalgebra and let* a *be an element of* A *that commutes with* $C_0(X)$*. Suppose that for every* $x \in X$ *and every* $\varepsilon > 0$ *there is some* $f \in C_0(X)$ *such that* $f(x) = 1$ *and* $\|f{\cdot}a\| < \varepsilon$*. Then* $f{\cdot}a = 0$ *for every* $f \in C_0(X)$*.* □

Let B be the C^*-algebra of bounded continuous functions from $(0, 1]$ into the bounded operators on $\mathcal{H}$, and let J be the ideal of functions whose distance to the compact operators converges to zero at 0. Let $A = B/J$. The operator-valued function $a\colon t \mapsto (tD + iI)^{-1}$ determines an element of A, and so does every constant operator-valued function $f\colon t \mapsto f$, for every $f \in C_0(X)$. It suffices to show that the product $f \cdot a \in A$ is zero since

$$\lim_{t\to 0} f{\cdot}(tD + iI)^{-1}(D + iI)^{-1} = -if{\cdot}(D + iI)^{-1}.$$

The elements a and f commute in A, so it suffices to verify the estimates in Lemma 5.35 for each given x and $\varepsilon > 0$. The product $f \cdot a \in A$ depends only on the restriction of D to a neighborhood of the support of f, so we may as well assume that D is compactly supported and elliptic near the support of f. Furthermore, by choosing f to have sufficiently small support, we may assume that $p\colon X \to Y$ is actually a trivial vector bundle (since any submersion is locally isomorphic to a trivial vector bundle).

Using the basic elliptic estimate for constant coefficient operators we can choose f with sufficiently small support that $f \cdot a$ is ε-close to $f \cdot a'$, where a' is defined in the same way as a, but using an operator D' that restricts to the same constant coefficient elliptic operator in each vector space fiber of p. A Fourier transform calculation then shows that for every $t \in (0, 1]$ the operator $f \cdot (tD' + iI)^{-1}$ is compact, and the proof of Proposition 5.34 is complete.

If X is compact, then we may choose $f \equiv 1$ in Proposition 5.34, and conclude from Section **??** that D has a well-defined index in the K-theory group $K(Y)$. Thus a smooth family of elliptic operators on the fibers of a submersion with compact fibers and compact base has a well-defined *families index* in $K(Y)$ (compare [**?**]). But in the proof of the index theorem that we shall present here the manifold X will not be compact.

As a substitute for compactness we shall work with operators of the form $D + E$, where E is a suitable smooth self-adjoint endomorphism of S. The operators in the family $D + E$ are still essentially self-adjoint because X is complete with respect to $D + E$. The compactness of the resolvent $r(D + E)$ is guaranteed (in the cases of concern to us) by the following calculation.

compact-resolvent-prop

5.36 Proposition. *Assume that S is $\mathbb{Z}_2$-graded and that D is odd-graded. Let E be a smooth, odd-graded self-adjoint endomorphism of the Hermitian bundle S over X. Assume that*

(a) *The square of E is a proper scalar function from X to $[0, \infty)$.*

(b) *The anticommutator $DE + ED$ is a uniformly bounded smooth endomorphism of S.*

Then $r(D + E)$ is a compact operator on the continuous field $\mathcal{H}$.

Proof. We shall show that for every $\varepsilon > 0$ the family $r(D + E)$ lies within ε of a compact operator.

Choose a smooth, compactly supported real function f such that if $F = \gamma f$, where γ is the grading operator on S, then

$$\|(D + E + F)s\| \geq \varepsilon^{-1}\|s\|$$

for every compactly supported smooth section s. This is possible because first of all

$$(D + E + F)^2 = D^2 + (DE + ED) + [D, f]\gamma + E^2 + f^2.$$

It therefore suffices to choose f such that

$$E^2 + f^2 \geq \varepsilon^{-2} + \|DE + ED\| + \|[D, f]\|,$$

and this may be done because X is complete with respect to D.

The estimate implies that $\|r(D+E+F)\| < \varepsilon$. But then

$$r(D+E) - r(D+E+F) = r(D+E+F) \cdot \gamma f \cdot r(D+E),$$

and by Proposition 5.34 the right hand side is compact. □

5.37 Remark. Obviously the hypotheses can be relaxed in various ways. But they are adequate for our purposes as they stand.

5.5 THE HARMONIC OSCILLATOR

q-harmonic-lem

5.38 Lemma. *The kernel of* (*the closure of*) B_m *is spanned by the function*

$$v \mapsto \exp(-\tfrac{1}{2}\|v\|^2)I \in \mathrm{Cliff}(T_m M).$$

On the orthogonal complement of the kernel, B_m^2 *is bounded below by* 2*.*

Proof. We compute that

$$B_m^2 = \Delta + \|v\|^2 + (N - 2k)$$

where Δ is the Laplace operator and N is the *number operator* that acts as pI on all monomials $e_{i_1} \cdots e_{i_p}$. The lemma therefore follows from the well-known eigenvalue theory of the quantum harmonic oscillator $\Delta + \|v\|^2$. See for example [**?**, p. 12]. □

5.6 EXERCISES

5.7 NOTES

For more about elliptic theory, with a development similar to that given here, see [**?**]. We have emphasized the role of Hilbert space theory in showing the "approximate invertibility" of elliptic operators. An alternative approach, which is more classical and provides much more detailed information, is to construct the approximate inverse or "parametrix" directly by an iterative procedure. This idea goes back at least as far as the work of Hadamard, and in a modern formulation leads to the theory of *pseudodifferential operators*. For discussion see [**?**, **?**].

Chapter Six

Deformation Spaces

Let M be a smooth, closed manifold. Our proof the index theorem for operators on M will an auxilliary embedding of M into a finite-dimensional vector space V. In this chapter we shall introduce two geometric constructions related to the embedding that will play an important role in the proof. The first is the normal bundle, which will be familiar to many readers. The second is what we shall the *deformation space* of the embedding of M into V. This comes from algebraic geometry, and its counterpart there is called the *deformation to the normal cone.*

The deformation space is a substitute for a tubular neighborhood embedding of the normal bundle into V. While we could use tubular neighborhoods instead of the the deformation space, that latter is more functorial and for that reason simpler to handle.

The main result in this chapter is the formulation and proof a sort of proto-index theorem that uses the normal bundle and the deformation space to reduce the computation of the analytic index of an elliptic operator on M to the computation of the index of a family of operators on the tangent vector spaces of M. The normal bundle appears the parameter space for the family, and the deformation space is used to carry out the reduction.

Passing from the computation of a single analytic index to the computation of the index of an entire family may at first seem like no reduction at all, but progress will in fact have been made. This is because the operators in the family will be of a very simple type, consisting of the sum of a constant coefficient operator and a linear function. The archetype is the operator $d/dx + x$. As we shall see in Chapters **??** and **??**, computation of the families index in important cases reduces rather quickly to a matter of linear algebra and the characteristic class theory of Chapter **??**. The general case will be handled in Chapter **??**.

6.1 THE NORMAL BUNDLE

Let V be a smooth manifold of dimension n without boundary. The tangent bundle to V was introduced in Chapter **??**, but from now on we shall think of it in a more current mathematical way. Thus a *tangent vector* at a point $v \in V$ is a real-linear functional X on the space of smooth, real-valued functions on V such that the Leibniz rule

$$X(fg) = X(f)g(v) + f(v)X(g)$$

holds for every pair of smooth functions. The *tangent space* T_vV of all tangent vectors at v is an n-dimensional vector space. See the notes at the end of the chapter for further information about this and the other constructions in this section.

The tangent spaces assemble to form a smooth vector bundle TV over V, the *tangent bundle*. A section of TV, called a *vector field*, is smooth if and only if for every smooth function f, the quantity $X(f)$, evaluated pointwise on V, is a smooth function on V. This characterizes the smooth vector bundle structure on TV.

6.1 Example. If $V = \mathbb{R}^n$, then the tangent space at any point is spanned by the partial deriviatives $f \mapsto \partial f/\partial x_n$, evaluated at that point. To put it another way, if V is a vector space, then the tangent space at any point of V may be identified with the the vector space V itself: each vector $X \in V$ determines a tangent vector at $v \in V$ by the directional derivative formula

$$X(f) = \lim_{h\to 0} h^{-1}\big(f(v + tX) - f(v)\big).$$

The tangent bundle identifies in this way with the trivial bundle $M\times V$.

6.2 Example. ... vertical tangent vectors on a vector bundle ...

If $\Phi\colon M \to V$ is a smooth map from one manifold to another, and if $\Phi(m) = v$, then the *differential*

$$\Phi_*\colon T_m M \to T_v V$$

is defined by $\Phi_*(X)(f) = X(f \circ \Phi)$. It determines a map from TM to the pullback of TV along Φ.

Suppose now that M is an embedded submanifold of V of dimension p and let $q = n - p$. By definition, the topology that M carries as a smooth manifold agrees with the topology that it inherits as a subset of V. Moreover each point of M is included in an open neighborhood $U \subseteq V$ on which there exists a local coordinate system

$$x_1, \dots, x_p, y_1, \dots, y_q$$

for V such that

(a) the set $M \cap U$ is precisely the set of points for which $y_1 = \cdots = y_q = 0$, and

(b) the functions $x_1, \dots, x_p$ are coordinates on the open subset $M \cap U$ of M.

We shall say that such a local coordinate system on V is *adapted* to the submanifold M.

The smooth functions on M are precisely the restrictions to M of smooth functions on V. The inclusion of M into V identifies TM with a subbundle of the restriction of TV to M.

These details are provided mostly in order to fix our terminology, although we shall also use the above coordinates in the next section. For the moment our aim is only to introduce the following simple and probably familiar concept.

6.3 Definition. The *normal bundle* $N_V M$ for M, considered as a submanifold of V, is the quotient of the restriction of TV to M by the tangent bundle of M:

$$N_V M = TV|_M \;/\; TM.$$

The normal bundle is therefore a smooth vector bundle of rank $\dim(V) - \dim(M)$.

A normal vector (that is, a tangent vector on V, modulo the tangent vectors on M) determines a directional derivative functional on the space of all smooth, real-valued functions on V that vanish on M. This is because the tangent vectors on M, viewed as tangent vectors on V, vanish on all such functions.

In fact normal vectors at $m \in M$ may be identified with linear functionals X that are defined on the space of all smooth, real-valued functions on V that are locally constant on M, and that obey the Leibniz rule

$$X(fg) = X(f)g(m) + f(m)X(g)$$

on this space.

6.2 THE DEFORMATION SPACE ASSOCIATED TO AN EMBEDDING

We are going to construct a *deformation space* $\mathbb{N}_V M$ associated to an embedding of a smooth manifold M into a smooth manifold V (both without boundary, for simplicity). The essential features of this space are as follows:

(a) $\mathbb{N}_V M$ is a smooth manifold with boundary. Its dimension is the dimension of V, plus one.

(b) There is a submersion from $\mathbb{N}_V M$ onto the closed unit interval $[0, 1]$.

(c) The fiber of this submersion over $t = 0$ is the normal bundle $N_V M$, whereas the fiber over every $t \neq 0$ is a copy of V.

The deformation space is closely related to the concept of tubular neighborhood that is summarized in the following theorem.

6.4 Theorem. *Let M be a smooth embedded submanifold of a smooth manifold V (both without boundary, for simplicity). There is an open neighborhood U of M in V, and a diffeomorphism*

$$\Phi\colon U \xrightarrow[\cong]{} N_V M$$

such that Φ restricts to the natural identification of $M \subseteq U$ with the zero vectors in the normal bundle $N_V M$, and such that if X is a tangent vector for V at a point of M, then $\Phi_* X$ is the vertical tangent vector on $N_V M$ determined by normal vector associated to X,

We shall not need this result, except for motivation, and so we shall not prove it. The theorem can be strengthened by the addition of a uniqueness statement, up to isotopy. This however we shall not need at all.

6.5 Definition. We shall call the open set U above a *tubular neighborhood* of M in V, and we shall call the inverse of Φ a *tubular neigborhood embedding* of the normal bundle $N_V M$ into V.

Fix a tubular neighborhood U of M in V, as in the theorem, and let C be the set-theoretic complement of U in V. It is of course a closed subset of V. The deformation space $\mathbb{N}_V M$ is essentially the space

$$Z = V\times[0,1] \setminus C\times\{0\}.$$

This is an open subset of $V\times[0,1]$ and as such it is a smooth manifold with boundary. The projection mapping from Z to $[0,1]$ is a submersion, and fiber over $t = 0$ is (diffeomorphic to) the normal bundle $N_V M$, whereas the fiber over any $t \neq 0$ is a copy of V, as required.

The problem with Z is that it depends on a choice of tubular neighborhood, which is not by any means canonical. This is not an insurmountable difficulty, but the deformation space $\mathbb{N}_V M$ has the advantage of being canonically associated to the embedding of M into V. Once the somewhat abstract definition of $\mathbb{N}_V M$ has been internalized, subsequent calculations become much clearer and simpler.

We begin by defining $\mathbb{N}_V M$ as a set, and we do so in the most obvious way that fits our requirements.

6.6 Definition. Let M be a smooth, closed submanifold of a smooth manifold V without boundary. The *deformation space* $\mathbb{N}_V M$ associated to the inclusion of M into V is the disjoint union

$$\mathbb{N}_V M = N_V M \ \sqcup \ V\times(0,1],$$

where $N_V M$ is the normal bundle of M in V.

There is an obvious (set-theoretic) projection onto $[0,1]$ for which the fiber over $t = 1$ is $N_V M$. In fact there is also a projection

$$r\colon \mathbb{N}_V M \longrightarrow V\times[0,1]$$

that on $V\times(0,1]$ is the inclusion map, while on $N_V M$ is the projection to M, followed by inclusion into $V\times\{0\}$. Let us call this the *contraction map* (because it contracts normal vectors to zero).

6.7 Definition. We equip $\mathbb{N}_V M$ with the weakest topology (that is, the one with the fewest open sets) such that:

(a) The contraction map $r\colon \mathbb{N}_V M \to V \times [0,1]$ is continuous.

(b) If $f\colon V \to \mathbb{R}$ is a smooth function that vanishes on M, then the function

$$\delta f\colon \mathbb{N}_V M \longrightarrow \mathbb{R}$$

defined by the formulas

$$\delta f(X) = X(f) \quad \text{and} \quad \delta f(v,t) = \frac{f(v)}{t}$$

is continuous.

6.8 Lemma. *The deformation space $\mathbb{N}_V M$ is a Hausdorff topological space.*

Proof. □

6.9 Lemma. *Let U be an open subset of V. The natural set-theoretic inclusion of $\mathbb{N}_U(M \cap U)$ into $\mathbb{N}_V M$ is a homeomorphism onto an open s ubset.*

Proof. By item (a) in Definition **??**, $\mathbb{N}_U(M\cap U)$ is an open subset of $\mathbb{N}_V M$.... $\square$

local-coord-lemma

6.10 Lemma. *If $x_1, \ldots, x_p, y_1, \ldots, y_q$ are smooth local coordinates on V such that the functions x_i restrict to local coordinates on M, whereas the functions y_j vanish on M, then the functions*

$$x_1, \ldots, x_p, \delta y_1, \ldots, \delta y_q, t$$

constitute a topological local coordinate system on the deformation space: they determine a homeomorphism from the open subset where they are defined to an open subset of $\mathbb{R}^{p+q} \times [0, 1]$.

6.11 Remark. Let U be the open subset of V on which the smooth local coordinates $x_1, \ldots, x_p, y_1, \ldots, y_q$ are defined. The symbols x_j and t in the display refer to the functions on $\mathbb{N}_U(M \cap U)$ that are obtained by first applying the map $r\colon \mathbb{N}_U(M \cap U) \to U \times [0, 1]$ of item (a) of Definition **??**, then composing with the functions x_j and t defined on $U \times [0, 1]$. The functions δy_j on $\mathbb{N}_U(M \cap U)$ are of course defined as in item (b) of the definition.

Proof. The would-be coordinate functions on $\mathbb{N}_U(M\cap U)$ that are displayed in the statement of the lemma determine a map

$$\mathbb{N}_U(M\cap U) \to \mathbb{R}^p \times \mathbb{R}^q \times [0, 1]$$

that is continuous and one-to-one, and has open image. Viewing U as an open subset of $\mathbb{R}^p \times \mathbb{R}^q$ via the given coordinates, and identifying the normal bundle with $(M\cap U) \times \mathbb{R}^q$, the inverse map is given by the formula

$$(u, v, t) \mapsto \begin{cases} (u, u + tv, t) & t \neq 0 \\ (u, v, 0) & t = 0. \end{cases}$$

This map is continuous because its compositions with the maps in (a) and (b) above are continuous. $\square$

For form's sake, we note the following general topological consequence.

6.12 Corollary. *The topological space $\mathbb{N}_V M$ is paracompact.*

Next we want to equip $\mathbb{N}_V M$ with a smooth manifold structure. In fact this is straightforward, since the basic coordinate charts given by Lemma **??** constitute an atlas for a smooth manifold structure on $\mathbb{N}_V M$:

6.13 Proposition. *The transition function relating any two of the local coordinate systems described in Lemma **??** is a smooth map.*

Proof. $\square$

The reader is referred to the exercises for further general information about the deformation space. In the next section we shall turn to specific features of the deformation space associated to an embedding into a vector space, and the construction of families of differential operators parametrized by such deformation spaces.

6.3 FAMILIES OF OPERATORS OVER THE DEFORMATION SPACE

Now let M be a smooth manifold without boundary that is embedded as submanifold of a finite-dimensional real vector space V. Define a map

$$p\colon M\times V\times[0,1] \longrightarrow \mathbb{N}_V M$$

by the formulas

$$(m,v,t) \mapsto \begin{cases} p_m(v) \in N_V M & \text{when } t=0 \\ (m+tv,t) \in V\times(0,1] & \text{when } t\neq 0, \end{cases}$$

where p_m is the projection from V onto $V/T_m M$ (which is the fiber of the normal bundle $N_V M$ over the point $m \in M$).

6.14 Lemma. *The map* p *is a submersion.*

Proof. □

6.15 Remark. If we think of $M\times V$ as a trivial vector bundle over M, and the diagonal embedding of M into $M\times V$ as a section, then the diffeomorphism from $M\times V\times[0,1]$ to $\mathbb{N}_{M\times V}M$ given in Exercise **??**, identifies the submersion p with the map from $\mathbb{N}_{M\times V}M$ onto $\mathbb{N}_V M$ that is induced from the projection of $M\times V$ onto V, as in Exercises **??** and **??**.

Suppose now that S is a smooth complex vector bundle on M and that D is a first-order linear partial differential operator acting on the sections of S (in the next section D will be elliptic, but for the moment this need not be so).

Pull back the bundle S to $M\times V\times[0,1]$. Define a smooth family of operators on the fibers of p, acting on the sections of this pullback bundle, as follows.

(a) If $(v,t) \in V\times(0,1] \subseteq \mathbb{N}_V M$, then the fiber of p over (v,t) is the manifold

$$\left\{(m,t^{-1}(v-m),t) : m \in M\right\} \subseteq M\times V\times[0,1]$$

and so the coordinate projection onto M identifies the fiber with the manifold M. We define $D_{(v,t)} = tD$.

(b) If $(m,v) \in N_V M \subseteq \mathbb{N}_V M$, then the fiber of the map p over X is the manifold

$$\{m\} \times T_m M \times \{0\} \subseteq M\times V\times[0,1].$$

Let D_m be the *model operator* on $T_m M$, obtained from D by freezing the coefficients at m and dropping order zero terms. We define $D_{(m,v)} = -D_m$.

6.16 Lemma. *The operators above form a smooth family of elliptic operators. Moreover the manifold* $M\times V\times[0,1]$ *is complete with respect to this family.*

Proof. If we embed the bundle S over M as a summand of a trivial bundle, then we can reduce the lemma to the case where S is trivial, in which case the original operator D is a system of operators on scalar functions. This allows us to further reduce to the cases where D is either a vector field or multiplication by a function f on M. In the latter case the family is multiplication by the smooth function $(m, v, t) \mapsto tf(m)$ on $M \times V \times [0,1]$. In the former case, if X is a vector field on M, then the associated family of operators is given by the smooth vector field

$$X_{(m,v,t)} = (tX_m, -X_m, 0)$$

on $M \times V \times [0,1]$, where we identify the tangent space of the product manifold at (m, v, t) with $T_m M \times V \times \mathbb{R}$ and we consider $T_m M$ as a subspace of V via the given embedding of M.

As for completeness, if $g\colon V \to [0, \infty)$ is any smooth proper function, then its composition with the second coordinate projection on $M \times V \times [0,1]$ is a smooth proper function on the product manifold whose commutator with D is uniformly bounded. □

6.4 THE DEFORMATION SPACE AND THE INDEX THEOREM

Let M be a smooth, closed, manifold that is embedded into a smooth manifold V without boundary. Form the deformation space $\mathbb{N}_V M$ and regard the manifold V and the normal bundle $N_V M$ as embedded into $\mathbb{N}_V M$ as the two boundary parts, so to speak at $t = 1$ and $t = 0$, respectively. In particular, V and $N_V M$ are embedded as closed subsets of $\mathbb{N}_V M$, and so there are restriction maps in K-theory:

$$K(\mathbb{N}_V M) \longrightarrow K(V) \quad \text{and} \quad K(\mathbb{N}_V M) \longrightarrow K(N_V M).$$

Our first aim is to assemble from these a map from $K(N_V M)$ into $K(V)$.

tubular-lemma **6.17 Lemma.** *The restriction map*

$$K(\mathbb{N}_V M) \longrightarrow K(N_V M)$$

induced from the inclusion of $N_V M$ *into* $\mathbb{N}_V M$ *is an isomorphism.*

Proof. □

The lemma allows us to define the map we want, as follows.

6.18 Definition. We shall call the map $\iota_*\colon K(N_V M) \longrightarrow K(V)$ that fits into the commuting diagram

$$\begin{tikzcd} & K(\mathbb{N}_V M) \arrow[dl, "\cong"'] \arrow[dr] & \\ K(N_V M) \arrow[rr] & & K(V) \end{tikzcd}$$

the …

6.19 Remark. See Exercise **??** for another interpretation of this map in terms of tubular neighborhoods.

After these K-theoretic preliminaries we return to the families of operators that we began to analyze in the last section. Let us from here on assume that the manifold M is closed and that D is an elliptic operator on M.

Fix a spinor space S for V and let $X\colon V \to \mathrm{End}(\wedge V)$ be self-adjoint Clifford multiplication, as in Definition **??**. Consider X as a function

$$X\colon M\times V\times[0,1] \longrightarrow \mathrm{End}(\wedge V)$$

via the coordinate projection onto V.

Form the tensor product $SM\hat{\otimes}S$ with fibers $S_m M\hat{\otimes}S$. Form the operator $D\hat{\otimes}I$ on sections of $SM\hat{\otimes}S$ over $M\times V\times[0,1]$, and also the self-adjoint endomorphism $I\hat{\otimes}E$.

6.20 Lemma. *The anticommutator of* $D\hat{\otimes}I$ *and* $I\hat{\otimes}X$ *is a uniformly bounded endomorphism of* $SM\hat{\otimes}S$*, while* $(I\hat{\otimes}X)^2$ *is a proper function on* $M\times V\times[0,1]$.

Proof. The square of $I\hat{\otimes}X$ is the scalar function $\|v\|^2$, which is certainly a proper function on $M\times V\times[0,1]$. The anticommutator of $D\hat{\otimes}I$ and $I\hat{\otimes}X$ is the same as the commutator of $D\otimes I$ and $I\otimes X$ on the sections of $SM\otimes S$. □

According to Proposition 5.36 there is therefore an index class

$$\mathrm{Ind}(D\hat{\otimes}I + I\hat{\otimes}X) \in K(\mathbb{N}_V M),$$

as required.

6.21 Lemma. $\iota_*\big(\mathrm{Ind}(D\hat{\otimes}I + I\hat{\otimes}X)\big|_{N_V M}\big) = \mathrm{Ind}(D\hat{\otimes}I + I\hat{\otimes}X)\big|_V \in K(V).$ □

6.22 Lemma. $\mathrm{Ind}(D\hat{\otimes}I + I\hat{\otimes}X)\big|_V = \mathrm{Ind}(D)\cdot\beta(V) \in K(V)$

Proof. The map

$$q\colon M\times V\times[0,1] \longrightarrow V\times[0,1]$$
$$q\colon (m,v,t) \mapsto (tm+v,t)$$

is a submersion, and every fiber

$$q^{-1}\{(v,t)\} = \big\{(m, v-tm, t) : m\in M\big\} \subseteq M\times V\times[0,1]$$

is isomorphic to M via the projection to M. Construct the smooth family D that is the Dirac operator on each fiber, and then form the family

$$D\hat{\otimes}I + I\hat{\otimes}E,$$

acting on sections of $SM\hat{\otimes}S$ by using the same self-adjoint Clifford multiplication endomorphism as E as before. We are re-using notation, but this is not especially reckless because the restriction to $V \cong V\times\{1\}$ of the new family is identical to the same restriction of the old one. However the restriction to $V \cong V\times\{0\}$ of the new family, which has the same index as the restriction to $V\times\{1\}$ by homotopy invariance of K-theory, is the family of operators

$$D\hat{\otimes}I + I\hat{\otimes}E_v\colon L^2(M,SM)\hat{\otimes}S \longrightarrow L^2(M,SM)\hat{\otimes}S$$

Decompose $L^2(M,SM)$ into the kernel of D, direct sum its orthogonal complement, and decompose $L^2(M,SM)\hat{\otimes}S$ accordingly. On the second summand the above operators are uniformly bounded below by the first positive eigenvalue in the spectrum of D. On the first summand the operators are

$$I\hat{\otimes}E_v\colon \ker(D)\hat{\otimes}S \longrightarrow \ker(D)\hat{\otimes}S$$

Taking into account the grading on $\ker(D)$ we find that

$$\mathrm{Ind}(D\hat{\otimes}I + I\hat{\otimes}E)\big|_V = \mathrm{Ind}(D)\cdot\beta(S),$$

as required. □

6.23 Proposition. *Let D be a first-order, linear elliptic partial differential operator on a smooth, closed manifold M. If M is embedded into a finite-dimensional complex vector space V, then we have*

$$\mathrm{Ind}(D)\cdot\beta(V) = \iota_*\big(\mathrm{Ind}(D\hat{\otimes}I + I\hat{\otimes}X)\big|_{N_V M}\big)$$

in the group $K(V)$. □

6.5 EXERCISES

6.24 Exercise. Deformation space up to diffeomorphism via tubular neighborhood embeddings.

6.25 Exercise. Functoriality of the deformation space.

6.26 Exercise. Let U be an open subset of V. The natural set-theoretic inclusion of $\mathbb{N}_U(M\cap U)$ into $\mathbb{N}_V M$ is a diffeomorphism onto an open subset. □

6.27 Exercise. Suppose that V is a smooth vector bundle over M and that M is embedded into V via a vector bundle section $s\colon M \to V$. It need not be the zero section.

Each vector $v \in V$ determines a vertical tangent vector at each point in the fiber containing v, including the point that lies on the section s, and in this way the normal bundle $N_V M$ identifies with the vector bundle V itself. We shall use this identification in the following calculation. We shall also use the notation (m, v) for a point $v \in$ that lies in the fiber over $m \in M$.

The map from $V\times[0,1]$ to the deformation space $\mathbb{N}_V M$ that is given by the formulas

$$(m, v, t) \mapsto \begin{cases} (m, v) \in N_V M & \text{when } t = 0 \\ (m, s(m) + tv, t) \in V\times(0,1] & \text{when } t \neq 0 \end{cases}$$

is a diffeomorphism from $V\times[0,1]$ onto $\mathbb{N}_V M$.

6.28 Exercise. The tubular neighborhood theorem says that every embedding can be factored as a (zero) vector bundle section, followed by the inclusion of an open set, so Lemmas **??** and **??** in some sense describe all deformation spaces. See Exercise **??** for another way to characterize the deformation space.

6.29 Exercise. Characterization as a functor?

6.30 Exercise. Suppose given a commuting diagram of smooth maps

$$\begin{tikzcd} M \arrow[r] \arrow[d, equal] & W \arrow[d, "p"] \\ M \arrow[r] & V \end{tikzcd}$$

in which the horizontal maps are submanifold embeddings and the vertical map p is a submersion. The induced map from $\mathbb{N}_M W$ to $\mathbb{N}_V M$ is a submersion.

6.31 Exercise. products of embeddings

6.32 Exercise. The above map ι_* is equal to the map

$$j_*\colon K(N_V M) \longrightarrow K(V)$$

associated to a tubular neighborhood embedding $j\colon N_V M \to V$. Indeed, since the restriction map

$$K(\mathbb{N}_V M) \xrightarrow{\cong} K(N_V M)$$

is an isomorphism, it follows that the open inclusion of the tubular neighborhood $W = \iota(N_V M)$ into V induces an isomorphism

$$K(\mathbb{N}_W M) \xrightarrow{\cong} K(\mathbb{N}_V M).$$

The assertion therefore reduces to the case where $V = W$, and now the calculation in Example **??**, plus the homotopy invariance of K-theory, completes the proof.

6.33 Exercise. Index of an operator on an odd-dimensional manifold is zero.

6.34 Exercise. Deformations in the algebraic category

6.6 NOTES

Chapter Seven

The Index Theorem for the Dolbeault Operator

7.1 THE DOLBEAULT OPERATOR

dolbeault-example

Atiyah and Singer developed (or rediscovered) the Dirac operator to serve as a counterpart, in the realm of real manifolds, of the Dolbeault operator in complex manifold theory. Accordingly, we shall take a quick look at the Dobeault operator first.

In this section we shall assume that the reader has some very basic familiarity with complex manifold theory. Let M be a compact *complex hermitian manifold* of complex dimension k, and hence real dimension $n = 2k$. The space of ordinary 1-forms on M (with complex coefficients) decomposes as a direct sum

$$\Omega^1(M) = \Omega^{0,1}(M) \oplus \Omega^{1,0}(M),$$

with the first summand generated locally by the differentials of anti-holomorphic functions and the second by the differentials of holomorphic functions. The de Rham differential decomposes as a direct sum

$$d = \overline{\partial} + \partial \colon \Omega^0(M) \to \Omega^{0,1}(M) \oplus \Omega^{1,0}(M).$$

There is a corresponding decomposition of differential forms and the de Rham operator in higher degrees, so that for example

$$\Omega^r(M) = \oplus_{p+q=r} \Omega^{p,q}(M).$$

The space $\Omega^{0,q}(M)$ is naturally isomorphic to the space of smooth sections of the bundle $\wedge^q TM$, where here we regard TM as a *complex* vector bundle to define the exterior power.

We can consider the *Dolbeault complex*

$$\Omega^0(M) \xrightarrow{\overline{\partial}} \Omega^{0,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{0,k}(M)$$

and associated *Dolbeault operator* $D = \overline{\partial} + \overline{\partial}^*$. This is an elliptic operator, and its symbol is a familiar object. Namely, after we use the hermitian metric to identify T^*M and TM as smooth manifolds, the symbol of D can be identified with the Thom element for the complex vector bundle TM over M:

$$\sigma_D = b \colon \pi^* \wedge^* TM \to \pi^* \wedge^* TM.$$

Because of this we can use the results of Chapter **??** to compute the contribution of the Chern character of the symbol to the index formula. We get

$$\operatorname{Ind}(D) = \int_{T^*M} \tau(TM) u_{T^*M} \operatorname{Todd}(TM \otimes \mathbb{C})$$

where the class τ is defined as in Proposition **??** and the overall sign $(-1)^n$ in the index formula has been dropped since M is even-dimensional as a real manifold. Now M, being a complex manifold, is naturally oriented, and if we orient T^*M using local coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$, where $x_1, \ldots, x_n$ are oriented local coordinates on M, then we can compute the integral by first integrating along the fibers of T^*M. We get

$$\int_{T^*M} \tau(TM) u_{T^*M} \operatorname{Todd}(TM \otimes \mathbb{C}) = \int_M \tau(TM) \operatorname{Todd}(TM \otimes \mathbb{C}).$$

However this orientation on T^*M differs from the orientation introduced in the last chapter and used in the index formula by the sign $(-1)^{\frac{n(n-1)}{2}}$. Bearing this in mind, and since $(-1)^{\frac{n(n-1)}{2}} = (-1)^k$, we obtain the index formula

$$\operatorname{Ind}(D) = (-1)^k \int_M \tau(TM) \operatorname{Todd}(TM \otimes \mathbb{C}).$$

Now $TM \otimes \mathbb{C}$ is isomorphic, as a complex vector bundle, to $TM \oplus \overline{TM}$ (Exercise **??**). Hence

$$\operatorname{Todd}(TM \otimes \mathbb{C}) = \operatorname{Todd}(TM) \cdot \operatorname{Todd}(\overline{TM}).$$

¿From Exercise **??**, we therefore obtain the *Hirzebruch Riemann-Roch formula*

$$\operatorname{Ind}(D) = \int_M \operatorname{Todd}(TM).$$

7.1 Exercise. (For those who know some complex manifold theory.) Check Hirzebruch's formula for $\mathbb{C}P^1$. For extra credit, do the same for $\mathbb{C}P^n$ (in all cases you should get $1 = 1$). (See Exercise 4.25 for the computation of the right hand side.)

7.2 Lemma. *The Dolbeault operator is elliptic.*

7.3 Theorem. *The Hilbert space index of the Dolbeault operator with coefficients in* E *is equal to* $\chi(\mathcal{O}(E))$.

index-thm1

7.4 Theorem (Atiyah and Singer)**.** *Let* M *be a smooth, closed hermitian manifold. If* D *is the Dolbeault operator for* M *with coefficients in a hermitian vector bundle* E *over* M*, then*

$$\operatorname{Ind}(D) = \int_M \operatorname{Todd}(TM) \operatorname{ch}(E).$$

7.2 K-THEORY FORM OF THE INDEX THEOREM

Let M be a smooth, closed, almost-complex manifold. We shall assume, as we always may, that M is embedded as a smooth submanifold of a finite-dimensional complex vector space A. No relation is assumed between the embedding, the complex structure on A and the almost-complex structure on M.

We want to equip the normal bundle with a complex structure. This we may do, after some adjustments, as follows. First, place a hermitian inner product on

A and thereby, by taking the real part, place an inner product on the underlying real vector space. The restriction of the tangent bundle on the manifold A to M is the trivial bundle $M\times A$. So using the inner product we may identify the normal bundle, which is a certain quotient of $M\times A$, with the orthogonal complement of TM in $M\times A$. In this way we obtain an isomorphism of vector bundles

$$TM \oplus N_A M \cong M\times A.$$

Now let us make use of the complex structure on TM. Any complex vector bundle over M may be realized as a direct summand of a trivial complex vector bundle over M, and so there exists a finite-dimensional complex vector space B, a smooth complex vector bundle F over M and an isomorphism of complex vector bundles

$$F \oplus TM \cong M\times B.$$

We obtain vector bundle isomorphisms

$$F \oplus (M\times A) \cong F \oplus TM \oplus N_A M \cong (M\times B) \oplus N_A M.$$

The direct sum $(M\times B)\oplus N_A M$ is the normal bundle of the composite embedding

$$M \longrightarrow A \longrightarrow B\times A,$$

So if we set $V = B\times A$, then the above isomorphisms fix a complex structure on the normal bundle $N_V M$ for the embedding of M into the complex vector space V.

The complex structures on V and $N_V M$ just described are those that we shall use in the statement of the following theorem.

index-thm2

7.5 Theorem (Atiyah and Singer)**.** *Denote by* $\overline{N_V M}$ *the complex conjugate vector bundle. If* D *is the Dolbeault operator for* M *with coefficients in a hermitian vector bundle* E*, then*

$$\operatorname{Ind}(D)\cdot\beta(V) = (-1)^k \iota_*\big(\beta(\overline{N_V M})\cdot[E]\big) \in K(V),$$

where $\iota_*\colon K(N_V M) \to K(V)$ *is the map induced from the deformation of* $N_V M$ *into* V.

7.3 COHOMOLOGICAL FORM OF THE INDEX THEOREM

proof-sec

We shall now calculate $\operatorname{Ind}(D\hat{\otimes} I + I\hat{\otimes} E)|_{N_V M}$. Recall from Definition **??** that the opposite spinor bundle SN for the normal bundle $N_V M$ is defined so that there is an isomorphism of spinor bundles

$$M\times S \cong \overline{S}M \hat{\otimes} SN$$

for the trivial bundle $M\times V$. As a result there is an isomorphism

$$SM\hat{\otimes} S \cong \big(SM\hat{\otimes}\overline{S}M\big)\hat{\otimes} SN.$$

The continuous field of Hilbert spaces on which $(D\hat{\otimes} I + I\hat{\otimes} E)|_{N_V M}$ acts can therefore be written as the field with fiber

$$L^2(T_m M, S_m M)\hat{\otimes} S \cong L^2(T_m M, S_m M\hat{\otimes}\overline{S}_m M)\hat{\otimes} S_m N$$

over $(m, v) \in N_V M$. If we define an operator B_m on first tensor factor on the right hand side by

$$B_m = -D_m \hat{\otimes} I + I \hat{\otimes} E_m,$$

where the function E_m on $T_m M$ is self-adjoint Clifford multiplication, and if we denote by E_v self-adjoint Clifford multiplication by a normal vector v on $S_m N$, then

$$(D\hat{\otimes} I + I\hat{\otimes} E)_{(m,v)} \cong B_m \hat{\otimes} I + I \hat{\otimes} E_v$$

(one should be aware that the descriptions on the left and right use different tensor product decompositions).

We shall now compute that the index of the family on the right hand side. The first step is the following lemma, in which we shall use the canonical isomorphisms

$$S_m M \hat{\otimes} \overline{S}_m M \cong \mathrm{End}(S_m M) \cong \mathrm{Cliff}(T_m M),$$

so as to view B_m as an operator on $L^2(T_m M, \mathrm{Cliff}(T_m M))$. Note that according to our conventions, the first isomorphism in the display is grading-preserving if k is even and grading-reversing if k is odd.

Now form the one-dimensional continuous field of Hilbert spaces

$$\mathcal{K}_m = \ker(B_m) \subseteq L^2(T_m M, S_m M \hat{\otimes} \overline{S}_m M).$$

It is purely even-graded if k is even, and purely odd-graded if k is odd. The section given in lemma trivializes $\mathcal{K}$, and as a result

$$\begin{aligned} L^2(T_m M, S_m M \hat{\otimes} \overline{S}_m M) \hat{\otimes} S_m N &\cong \mathcal{K}_m \hat{\otimes} S_m N \oplus \mathcal{K}_m^{\perp} \hat{\otimes} S_m N \\ &\cong S_m N \oplus \mathcal{K}_m^{\perp} \hat{\otimes} S_m N, \end{aligned}$$

where the isomorphism between the first summands is grading-preserving or grading-reversing, according as k is even or odd. In the final direct sum decomposition the operator $B_m \hat{\otimes} I + I \hat{\otimes} E_v$ acts as the self-adjoint Clifford multiplication operator E_v on $S_m N$ and as an invertible operator on $\mathcal{K}_m^{\perp} \hat{\otimes} S_m N$, since

$$(B_m \hat{\otimes} I + I \hat{\otimes} E_v)^2 = B_m^2 \hat{\otimes} I + I \hat{\otimes} E_v^2,$$

while $B_m^2 \geq 2$ on $\mathcal{K}_m^{\perp}$. Using the additivity of the index, together with the triviality of the index of the second summand, we find that

$$\begin{aligned} \mathrm{Ind}(D\hat{\otimes} I + I \hat{\otimes} E)\big|_{N_V M} &= \mathrm{Ind}(B \hat{\otimes} I + I \hat{\otimes} E) \\ &= (-1)^k \beta(SN) \in K(N_V M), \end{aligned}$$

as required.

7.4 ALMOST-COMPLEX MANIFOLDS

7.6 Definition. A *complex structure* on a smooth, real vector bundle is a smooth endomorphism J of the bundle such that $J^2 = -I$. An *almost-complex structure* on a smooth manifold is a complex structure on the tangent bundle of the manifold.

Notice that the endomorphism J gives the bundle the structure of a complex vector bundle for which J acts as multiplication by $\sqrt{-1}$, and that the underlying real vector bundle of a complex vector bundle has a natural complex structure.

7.7 Definition. ... of a Dolbeault operator, using

$$[D, f]\, u = c\big(df\big)\, u.$$

for every smooth function f on M (viewed on the left-hand side of the display as a multiplication operator on sections of $\wedge^{0,*}M$).

Of course, *the* Dolbeault operator on a hermitian manifold, with coefficients in a hermitian holomorphic bundle E, is an example of a Dolbeault operator with coefficients in E, in the sense of the above definition.

Every Dolbeault operator D is elliptic, and so if M is closed, then (the self-adjoint extension of) D is Fredholm.

index-thm1

7.8 Theorem (Atiyah and Singer)**.** *Let M be a smooth, closed almost-complex manifold. If D is a Dolbeault operator for M with coefficients in a smooth complex vector bundle E over M, then*

$$\mathrm{Ind}(D) = \int_M \mathrm{Todd}(TM)\, \mathrm{ch}(E).$$

7.5 EXERCISES

7.6 NOTES

Chapter Eight

The Dirac Operator

DiracChapter

Let M be an oriented, Riemannian manifold. The signature operator D on M was described in Chapter **??**. If we square its symbol σ we find the key property that

$$\sigma(x, \xi)^2 = \|\xi\|^2 \cdot I,$$

which we used to infer that D is elliptic. Formulas of this type are common throughout K-theory and index theory. In this chapter we shall define and study the Dirac operator, which is in many respects the most important and most basic example of an elliptic operator.

8.1 CLIFFORD ALGEBRAS AND DIRAC OPERATORS

dirac-type-symbol-def

8.1 Definition. Let V be a euclidean vector bundle over some base X. A (complex) *Dirac-type symbol* associated to V consists of the following:

(i) A $\mathbb{Z}/2$-graded hermitian vector bundle S over X;

(ii) A real-linear vector bundle map

$$c\colon V \to \mathrm{End}(S)$$

whose values are all odd-graded, self-adjoint endomorphisms of S satisfying the relations

$$c(v)^2 = \|v\|^2 \cdot I,$$

for all $v \in V$.

8.2 Example. Suppose that V is the underlying real vector bundle of a complex hermitian bundle E, and let $S = \wedge^* E$. The formula

$$b(v)w = v \wedge w + v \lrcorner w$$

(which we used to define the Thom class in K-theory) is an example of a Dirac-type symbol $b\colon V \to \mathrm{End}(S)$. To put it another way, Dirac-type symbols generalize the Thom element construction that we introduced in the previous lecture.

complex-example

8.3 Remark. We can also define a *real* Dirac-type symbol in the same way, by replacing the complex vector bundle S with a real vector bundle. We will take a quick look at these at the end of the chapter.

We shall be most interested in the case where V is the cotangent bundle of a Riemannian manifold M, in which case we can view a Dirac-type symbol as the symbol of some elliptic operator on M. Notice that a Dirac-type symbol defines an elliptic endomorphism of the pullback $\pi^* S$ of S over V, and thus defines a K-theory class $[c] \in K(V)$ by the difference construction of **??**.

The action of vectors $v \in V$ on the bundle S via the map $c\colon V \to \mathrm{End}(S)$ is often called *Clifford multiplication*, thanks to its relation with the following algebraic construction.

8.4 Definition. Let V be a finite-dimensional euclidean vector space. The *complex Clifford algebra* $\mathbb{C}(V)$ is the complex, associative algebra with unit which is characterized up to canonical isomorphism by the following properties:

(i) There is a real linear map $c\colon V \to \mathbb{C}(V)$, such that $c(v^2) = \|v\|^2 I$, for all $v \in V$.

(ii) If A is any associative algebra with unit equipped with a real linear map $c_A\colon V \to A$ such that $c(v^2) = \|v\|^2 I$, for all $v \in V$, then there is a unique algebra homomorphism $\mathbb{C}(V) \to A$ such that the diagram

$$\begin{tikzcd} & V \arrow[dl, "c"'] \arrow[dr, "c_A"] & \\ \mathbb{C}(V) \arrow[rr] & & A \end{tikzcd}$$

commutes.

It is easy to check that if $v_1, \ldots, v_k$ is a basis for V, then the set of products $c(v_{i_1}) \cdots c(v_{i_p})$, where $i_1 < \cdots < i_p$, is a linear basis for $\mathbb{C}(V)$. Thus $\mathbb{C}(V)$ is finite-dimensional as a vector space, with

$$\dim(\mathbb{C}(V)) = 2^{\dim(V)}.$$

The algebra $\mathbb{C}(V)$ is $\mathbb{Z}/2$-graded by assigning the monomial $c(v_{i_1}) \cdots c(v_{i_p})$ even or odd degree, according as p is even or odd. A little less obvious is the following important fact:

cliff-iso-prop

8.5 Proposition. *If V has even dimension $2k$, then $\mathbb{C}(V)$ is isomorphic to the algebra of $2^k \times 2^k$ complex matrices.*

Proof (sketch/exercise). We shall construct an explicit representation from $\mathbb{C}(V)$ into the matrix algebra, and proving using a linear basis for $\mathbb{C}(V)$ that it is injective (and hence surjective too, by dimension counting). To do this, observe that if $v_1, \ldots, v_{2k}$ is an orthonormal basis for V, and if matrices $E_1, \ldots, E_{2k}$ are given such that

$$E_i^2 = I \quad \text{and} \quad E_i E_j + E_j E_i = 0 \quad \text{when} \quad i \neq j$$

then the formula

$$c(a_1 v_1 + \cdots + a_{2k} v_{2k}) = a_1 E_1 + \cdots + a_n E_n$$

defines a representation of the Clifford algebra. For example, if $k = 1$, then we can define

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

We leave it to the reader to work out suitable formulas for general k. (For $k = 2$ you will find them in Dirac's book on quantum mechanics.) □

8.6 Exercise. We can make this argument a little slicker if we are prepared to use the notion of *graded tensor product*, which we already discussed in Section 15.3 in connection with C^*-algebras. It is not hard to see from the universal property of Clifford algebras that $\mathbb{C}(V \oplus W) \cong \mathbb{C}(V) \widehat{\otimes} \mathbb{C}(W)$. On the other hand, we have explicitly computed above that if V is 2-dimensional, $\mathbb{C}(V)$ is isomorphic to the graded algebra $M_2(\mathbb{C})$ of endomorphisms of the graded vector space $\mathbb{C} \oplus \mathbb{C}$. Therefore we obtain $\mathbb{C}(V \oplus \mathbb{C}^2) \cong \mathbb{C}(V) \widehat{\otimes} M_2(\mathbb{C})$, and so by induction

$$\mathbb{C}(V) \cong M_{2^k}(\mathbb{C}),$$

where $\dim(V) = 2k$. This argument gives us the grading and $*$-algebra structure too (see the next remark).

8.7 Remark. It is easy to check that if V is any euclidean vector space, then there is a unique $*$-algebra structure on $\mathbb{C}(V)$ for which $c(v)^* = c(v)$ for all $v \in V$. If $\dim(V) = 2k$ then $\mathbb{C}(V)$ is $*$-isomorphic to the matrix algebra $M_{2^k}(\mathbb{C})$, with its usual $*$-algebra structure of conjugate transpose. In addition, we can find a grading preserving $*$-isomorphism, where $M_{2^k}(\mathbb{C})$ is graded as an algebra of block 2×2 matrices. We shall use these refinements of Proposition 8.5 at one or two points below.

product-symbol

8.8 Exercise. Following up on the previous exercise, suppose that c_1 and c_2 are Dirac-type symbols for bundles V_1 and V_2, acting on S_1 and S_2 respectively. Show that their *product*

$$c_1 \times c_2 = c_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} c_2$$

is a Dirac-type symbol for $V_1 \oplus V_2$ acting on $S_1 \widehat{\otimes} S_2$.

8.9 Remark. Proposition 8.5 is not true for odd-dimensional vector spaces, and this is the reason that we shall restrict to even-dimensional spaces for the rest of this lecture. There are odd-dimensional counterparts to the proposition and to most of what follows, but they are slightly more complicated and will not be discussed in these notes.

8.2 THE INDEX THEOREM FOR THE DIRAC OPERATOR

dirac-symbol-def

8.10 Definition. Let V be a euclidean vector bundle of *even* dimension $n = 2k$. A *Dirac symbol* associated to V is a Dirac-type symbol $c\colon V \to \mathrm{End}(S)$ such that the vector spaces S_x have (complex) dimension 2^k.

Dirac symbols are Dirac-type symbols of minimal dimension:

even-cliff-lemma **8.11 Lemma.** *If* $c\colon V \to \mathrm{End}(S)$ *is a Dirac-type symbol associated to a euclidean vector bundle, and if* $\dim(V) = 2k$*, then the fiber dimension of* S *is a multiple of* 2^k.

Proof. Starting with a euclidean vector bundle V, we can form the bundle $\mathbb{C}(V)$ of Clifford algebras, and it is clear that a Dirac-type symbol is the same thing as a homomorphism of bundles from $\mathbb{C}(V)$ into $\mathrm{End}(S)$, which is fiberwise a homomorphism of $\mathbb{Z}/2$-graded algebras. The fibers of S are therefore representation spaces of the Clifford algebra of $2k$-dimensional euclidean vector spaces. Since the Clifford algebras are all isomorphic to the matrix algebra $M_{2^k}(\mathbb{C})$, all such representations are multiples of the standard representation, of dimension 2^k. □

Dirac symbols play the same role in K-theory that orientations of vector bundles play in cohomology theory. It can be shown that if V is an even-dimensional euclidean vector bundle over X, and if $c\colon V \to \mathrm{End}(S)$ is a Dirac symbol, then the K-theory class $c \in K(V)$ freely generates $K(V)$ as a module over $K(M)$. Thus the existence of a Dirac symbol is a sufficient (and as it happens necessary) condition for the formulation of a Thom isomorphism theorem in K-theory.

orient-lemma **8.12 Lemma.** *Suppose that* V *is even-dimensional and that* $\sigma\colon V \to \mathrm{End}(S)$ *is a Dirac symbol. If* $v_1, \dots, v_{2k}$ *is any local orthonormal frame for* V*, then then locally-defined endomorphism*

$$\gamma = i^k c(v_1) \cdots c(v_{2k})$$

is equal to the grading operator of the bundle S*, up to a locally constant,* $\{\pm 1\}$*-valued function.*

Proof. The element γ defined by the above formula has the following properties: $\gamma^2 = 1$, and γ anticommutes with every $c(v)$. Using the explicit basis for $\mathbb{C}(V)$ given earlier, it is not hard to check that there are precisely two elements in any Clifford algebra with these properties, namely the grading operator and its negative. □

pos-oriented-def **8.13 Definition.** Let V be an oriented euclidean vector bundle and let $c\colon V \to \mathrm{End}(S)$ be a Dirac symbol. We shall say that the Dirac symbol is *positively oriented* if the operator γ of Lemma 8.12 associated to any oriented local orthonormal frame of V is equal to the grading operator of S. We shall say that the symbol is *negatively oriented* if γ is always minus the grading operator.

8.14 Remark. If any Dirac symbol $c\colon V \to \mathrm{End}(S)$ exists, then V must be orientable. If V is an oriented bundle over a connected base space, then every Dirac symbol for V must be either positively or negatively oriented.

We shall say a bit more later about conditions necessary to guarantee the existence of Dirac symbols. But it is easy to see that Dirac symbols are not necessarily unique. Indeed, if $c\colon V \to \mathrm{End}(S)$ is a Dirac symbol and if P is a complex line bundle, then the object

$$c \otimes \mathrm{id}_P\colon V \to \mathrm{End}(S \otimes P)$$

is also a Dirac symbol. Moreover we can also obtain a new Dirac symbol from $c\colon V \to \mathrm{End}(S)$ simply by reversing the grading of S.

8.15 Definition. Let $c_1\colon V \to \mathrm{End}(S_1)$ and $c_2\colon V \to \mathrm{End}(S_2)$ be two Dirac-type symbols associated to an even-dimensional, euclidean vector bundle V. Denote by $\mathrm{Hom}_V(S_1, S_2)$ the vector bundle whose sections are the bundle homomorphisms $S_1 \to S_2$ which commute with the actions of V on S_1 and S_2 by Clifford multiplication.

8.16 Remark. Despite the fact that S_1 and S_2 are $\mathbb{Z}/2$-graded, we do *not* require that elements of $\mathrm{Hom}_V(S_1, S_2)$ be grading-preserving.

E-lemma **8.17 Lemma.** *Let* $c_1\colon V \to \mathrm{End}(S_1)$ *and* $c_2\colon V \to \mathrm{End}(S_2)$ *be two Dirac symbols associated to an even-dimensional, oriented euclidean vector bundle* V. *The bundle* $L = \mathrm{Hom}_V(S_1, S_2)$ *is a line bundle, and* $c_2\colon V \to \mathrm{End}(S_2)$ *is isomorphic to the tensor product* $c_1 \otimes \mathrm{id}_L\colon V \to \mathrm{End}(S_1 \otimes L)$.

Proof. Since the actions of the fibers V_x on S_{1x} and S_{2x} correspond to equivalent irreducible representations of the Clifford algebra $\mathbb{C}(V_x)$, it follows from Schur's lemma in representation theory that the fibers of $L = \mathrm{Hom}_V(S_1, S_2)$ are one-dimensional vector spaces. Thus L is a line bundle, as claimed. The isomorphism in the statement of the lemma comes from the canonical evaluation map

$$S_1 \otimes \mathrm{Hom}_V(S_1, S_2) \to S_2,$$

so the proof of the lemma is complete. □

Let $c\colon V \to \mathrm{End}(S)$ be a Dirac-type symbol. Denote by $\overline{S}$ the complex conjugate bundle of S. The complex conjugate of the hermitian inner product on S is a hermitian inner product on $\overline{S}$ and of course the grading on S determines a grading on $\overline{S}$. The bundles $\mathrm{End}(S)$ and $\mathrm{End}(\overline{S})$ are identical to one another, since a complex-linear endomorphism of S is also a complex-linear endomorphism of $\overline{S}$. It follows that the map $c\colon V \to \mathrm{End}(S)$ determines a map $\overline{c}\colon V \to \mathrm{End}(\overline{S})$, which is a new Dirac-type symbol.

8.18 Definition. If $c\colon V \to \mathrm{End}(S)$ is a Dirac symbol (that is, a Dirac-type symbol of minimal dimension), then denote by L_S the line bundle

$$L_S = \mathrm{Hom}_V(\overline{S}, S).$$

signs-remark **8.19 Remark.** The canonical evaluation map given in the proof of Lemma 8.17 exhibits an isomorphism of Dirac symbols $L_S \otimes \overline{S} \cong S$. It is important to note that if the rank of V is $2k$ this isomorphism is orientation preserving or orientation reversing, according as k is even or odd. To see this, it suffices to note that if S is positively oriented for a given orientation on V, then $\overline{V}$ is positively or negatively oriented, according as k is even or odd.

8.20 Exercise. Show that if M is an auxiliary line bundle, and if we form the tensor product Dirac symbol $c \otimes \mathrm{id}_P\colon V \to \mathrm{End}(S \otimes P)$, then $L_{S\otimes P} \cong L_S \otimes P \otimes P$.

canonical-line-ex **8.21 Exercise.** Show that if V is the real bundle underlying a rank-k complex vector bundle E, if $S = \bigwedge^* E$, and if $b\colon V \to \mathrm{End}(S)$ is the Thom element, viewed as a Dirac symbol, then $L_S = \bigwedge^k E$. (Hint: if $w \in \bigwedge^k E$ show that the map from $\overline{S}$ to S defined by $s \mapsto w \lrcorner s$ commutes with Clifford action of V.)

8.22 Theorem. *Let V be an oriented, euclidean vector bundle of rank $2k$ over a compact manifold M. If $c\colon V \to \mathrm{End}(S)$ is a positively oriented Dirac symbol, then*

$$\mathrm{ch}(c) = \frac{\sqrt{\mathrm{ch}(L_S)}}{\widehat{A}(V)} u_V \in H^*(V)$$

chern-dirac *where $u_V \in H^*(V)$ is the cohomology Thom class of V.*

8.23 Remark. Note that since L_S is a line bundle, its Chern character is simply $e^{c_1(L_S)}$, where $c_1(L_S)$ is the first Chern class of L_S. The square root of the Chern character is therefore understood as $e^{\frac{1}{2}c_1(L_S)}$. Recall that the $\widehat{A}$ class is a similar square root (Exercise **??**); in fact, $\widehat{A}(V) = \sqrt{\mathrm{Todd}(V \otimes \mathbb{C})}$.

Proof of the Theorem (sketch). Consider the euclidean vector bundle $V \oplus V$. There are *two* natural ways to construct a Dirac symbol on this bundle:

(i) We may form the product symbol[1] $c \times c$ of two copies of the given Dirac symbol c;

(ii) Ignoring the given symbol c entirely, we may consider $V \oplus V$ as the underlying real vector bundle of the complex vector bundle $V \otimes \mathbb{C}$, and then form the Thom element $b\colon V \otimes \mathbb{C} \to \mathrm{End}(\bigwedge^* V \otimes \mathbb{C})$.

We are going to compute the Chern character of each, and then determine the relation between these Chern characters. Finally, we shall deduce the theorem from this relation.

Let us denote by $\mathcal{C}(c) \in H^*(M)$ the cohomology class such that $\mathrm{ch}(c) = \mathcal{C}(c)u_V$. Thus $\mathcal{C}(c)$ is precisely the class which we need to determine to prove the theorem. By following the same line of reasoning that we used in the last chapter, one can prove a multiplicative property for $\mathrm{ch}(c \times c)$, and conclude that

$$\mathrm{ch}(c \times c) = \mathcal{C}(c) \cdot \mathcal{C}(c) \cdot u_{V \oplus V}.$$

As for the Thom element associated to the complex bundle $V \otimes \mathbb{C}$, we computed in the last chapter that

$$\mathrm{ch}(b) = \frac{1}{\mathrm{Todd}(V \otimes \mathbb{C})} u_{V \otimes \mathbb{C}}.$$

We are now going to show that

$$\mathrm{ch}(c \times c) = (-1)^k \,\mathrm{ch}(b) \cdot \mathrm{ch}(L_S). \tag{8.1}$$ chern-rel

In view of the fact that the Thom classes $u_{V \oplus V}$ and $u_{V \otimes \mathbb{C}}$ differ by the sign $(-1)^k$ (since the orientations on $V \oplus V$ and $V \otimes \mathbb{C}$ differ by that sign), it will follow that

$$\mathcal{C}(c)^2 = \mathrm{ch}(L_S)\,\mathrm{Todd}(V \otimes \mathbb{C})^{-1}.$$

[1] See Exercise 8.8.

To verify the relation 8.1 between Chern characters, we shall prove that the tensor product of the Dirac symbol in (ii) with the line bundle L_S is the Dirac symbol in (i) (up to a possible reversal of the grading, which accounts for the sign $(-1)^k$).

Let us begin by considering $\mathbb{C}(V)$, the bundle of Clifford algebras over V. Like any algebra, the Clifford algebra is a *bimodule* over itself, using the actions of left and right multiplication. These actions commute, but we can make them anticommute instead by introducing a small twist from the grading operator $\gamma \in \mathbb{C}(V)$:

$$\ell(u) \cdot x = ux, \qquad r(v) \cdot x = i\gamma x \gamma v.$$

The pair (ℓ, r) defines a Dirac-type symbol for $V \oplus V$, which by dimension counting is a Dirac symbol.

In fact, this Dirac symbol is isomorphic to the Dirac symbol (ii): the canonical isomorphism of vector spaces from $\mathbb{C}(V)$ to the complexified exterior algebra $\bigwedge^* V \otimes \mathbb{C}$ which maps $v_1 \cdot \cdots \cdot v_p$ to $v_1 \wedge \cdots \wedge v_p$ is a (grading-preserving) isomorphism of Dirac symbols.

Clifford multiplication determines a grading-preserving isomorphism $\mathbb{C}(V) \to \operatorname{End}(S)$, and the vector bundle $\operatorname{End}(S)$ is, in turn, isomorphic to $S \otimes \overline{S}$ via the map which associates to $s_1 \otimes s_2 \in S \otimes \overline{S}$ the endomorphism $s \mapsto \langle s, s_2 \rangle s_1$. It follows that there are vector bundle isomorphisms

$$\textstyle\bigwedge^* V \otimes \mathbb{C} \otimes L_S \cong \operatorname{End}(S) \otimes L_S \cong S \otimes \overline{S} \otimes L_S \cong S \otimes S.$$

All of the isomorphisms are grading-preserving, except the last one, which is grading-reversing if k is odd (see Remark 8.19). The Dirac symbol structure $c \times c$ on $S \otimes S$ corresponds under these isomorphisms to the structure on $\operatorname{End}(S) \otimes L_S$ given by the anticommuting actions

$$\ell'(u) \cdot x = ux, \qquad r'(v) \cdot x = \gamma x v.$$

This is not the same as the original Dirac symbol structure on $\bigwedge^* V \otimes \mathbb{C}$ (tensored with the identity on L_S) but it is isomorphic to it via the map $x \mapsto x\sqrt{\gamma}$, where the operator $\sqrt{\gamma}$ is defined to be 1 on the $+1$-eigenspace of γ and $i = \sqrt{-1}$ on the -1-eigenspace. We conclude that

$$\operatorname{ch}(c \times c) = (-1)^k \operatorname{ch}(b) \cdot \operatorname{ch}(L_S),$$

as required.

Having now shown that $\mathcal{C}(c)^2 = \operatorname{ch}(L_S) \operatorname{Todd}(V \otimes \mathbb{C})^{-1}$, we can now take square roots to obtain the formula

$$\mathcal{C}(c) = \pm \frac{\sqrt{\operatorname{ch}(L_S)}}{\widehat{A}(V)}.$$

To determine the sign, we just need to work out the degree zero part of $\mathcal{C}(c)$ in $H^0(M)$, and to do this we can restrict the bundle V to a single point in M. Over any single point, we can give V a complex structure, and the restriction of the Dirac symbol to our point is isomorphic to the Bott element. The isomorphism is grading-preserving or grading-reversing according as k is even or odd, because the Bott element is positively or negatively oriented according as k is even or odd. The computations in the previous chapter now tell us that the correct sign is $+1$ □

8.24 Definition. We shall call a (symmetric, first-order) differential operator a *Dirac operator* if its symbol is a Dirac symbol associated to T^*M.

A Dirac operator is necessarily elliptic. Our calculation of the Chern character of Dirac symbols allows us to write out the index formula for Dirac operators in fairly explicit terms.

index-dirac-thm

8.25 Theorem. *Let D be a Dirac operator associated to a positively oriented Dirac symbol σ on a compact oriented manifold M of dimension $2k$. Then*

$$\operatorname{Ind}(D) = (-1)^k \int_M \sqrt{\operatorname{ch}(L_S)}\,\widehat{A}(TM).$$

(where we recall that the class $\widehat{A}(TM)$ is by definition the square root of the class $\operatorname{Todd}(TM \otimes \mathbb{C})$*).*

Proof. We will deduce this from the general form of the Index Theorem 9.1. The idea is the same as in Section 7.1.

Substituting the formula of Theorem 8.22, which gives the Chern character of the symbol, into the index theorem 9.1, we get

$$\operatorname{Ind}(D) = \int_{T^*M} \frac{\sqrt{\operatorname{ch}(L_S)}}{\widehat{A}(TM)} \operatorname{Todd}(TM \otimes \mathbb{C}) u_{TM}.$$

Integrating over the fiber, we get

$$\operatorname{Ind}(D) = (-1)^k \int_M \sqrt{\operatorname{ch}(L_S)}\,\widehat{A}(TM),$$

as required. □

8.26 Example. Let M be a complex manifold of complex dimension k, and hence real dimension $2k$. The symbol of the Dolbeault operator $D = \bar{\partial} + \bar{\partial}^*$ acting on $S = \wedge^{0,*} T^*_{\mathbb{C}} M \cong \wedge^* TM$ is a Dirac symbol. It is positively or negatively oriented, according as k is even or odd. As the reader was asked to show in Exercise 8.21, the line bundle L_S is $\wedge^k TM$ (the highest exterior power of the complex bundle TM). The following exercise now reconciles the formula in Theorem 8.25 with the previously derived index formula for the Dolbeault operator.

8.27 Exercise. Let E be a complex vector bundle of rank k. Show that

$$\operatorname{Todd}(E) = \operatorname{ch}(\wedge^k E)\operatorname{Todd}(\overline{E}).$$

Deduce that

$$\sqrt{\operatorname{ch}(\wedge^n E)}\,\widehat{A}(E_{\mathbb{R}}) = \operatorname{Todd}(E).$$

8.3 THE SPINOR DIRAC OPERATOR

This short section requires some familiarity with principal bundle theory.

8.28 Definition. A Spin^c-*structure* on an oriented, even-dimensional Riemannian manifold is an isomorphism class of positively oriented Dirac symbols associated to T^*M.

Let us discuss in more detail the problem of determining whether or not an oriented Riemannian manifold admits a Spin^c-structure.

Consider the complex Clifford algebra of $\mathbb{R}^{2k}$. It is isomorphic to matrix algebra $M_{2^k}(\mathbb{C})$—let us fix such an isomorphism. The group $SO(2k)$ acts on $\mathbb{R}^{2k}$ and therefore on $\mathbb{C}_{\mathbb{C}}(\mathbb{R}^{2k})$, and therefore on $M_{2^k}(\mathbb{C})$ via the given, fixed, isomorphism. Since every automorphism of $M_{2^k}(\mathbb{C})$ is induced from a unique automorphism of C^{2^k}, up to a scalar multiple of the identity, we obtain a group homomorphism

$$SO(2k) \to U(2^k)/Z,$$

where Z denotes the center of the unitary group $U(2^k)$ (Z is isomorphic to S^1 and consists of scalar multiples of the identity).

8.29 Definition. Denote by $\mathrm{Spin}^c(2k)$ the group which fits into the pullback diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "="'] & \mathrm{Spin}^c(2k) \arrow[r] \arrow[d] & SO(2k) \arrow[d] \\ Z \arrow[r] & U(2^k) \arrow[r] & U(2^k)/Z \end{tikzcd}$$

Observe that the group $\mathrm{Spin}^c(2k)$ comes with canonical representations on the spaces $\mathbb{R}^{2k}$ and $\mathbb{C}^{2^k}$. The map $\mathbb{R}^{2k} \otimes \mathbb{C}^{2^k} \to \mathbb{C}^{2^k}$ which is induced from our fixed isomorphism $\mathbb{C}(\mathbb{R}^{2k}) \cong M_{2^k}(\mathbb{C})$ is $\mathrm{Spin}^c(2k)$-equivariant.

One can prove the following result.

8.30 Theorem. *An oriented Riemannian* $2k$*-manifold admits a* Spin^c*-structure if and only if the principal* $S0(2k)$*-bundle* F *of oriented frames admits a reduction* $\widetilde{F}$ *to the group* $\mathrm{Spin}^c(2k)$*. In this case, the cotangent bundle* TM *is given by*

$$T^*M = \widetilde{F} \times_{\mathrm{Spin}^c(2k)} \mathbb{R}^{2k},$$

and the formula

$$S = \widetilde{F} \times_{\mathrm{Spin}^c(2k)} \mathbb{C}^{2^k}$$

defines a hermitian bundle equipped with an action $T^*M \otimes S \to S$ *which is a Dirac symbol. In this way,* Spin^c*-structures correspond bijectively to reductions of the oriented frame bundle to* $\mathrm{Spin}^c(2k)$. □

In the real case, we shall confine our attention to manifolds of dimension $8k$. This is to accommodate the following result:

8.31 Proposition. *The real Clifford algebra* $\mathbb{R}(\mathbb{R}^{8k})$ *is isomorphic to the matrix algebra* $M_{2^{4k}}(\mathbb{R})$. □

We define real Dirac symbols in the $8k$-dimensional case by putting a minimal dimensionality requirement on the bundle S. We define a Spin-structure on an oriented Riemannian manifold of dimension $8k$ to be an isomorphism class of positively oriented real Dirac symbols.

By following a similar reasoning to that used in the previous section one can prove:

8.32 Theorem. *An oriented Riemannian $8k$-manifold admits a Spin-structure if and only if the principal $SO(8k)$-bundle F of oriented frames admits a reduction $\widetilde{F}$ to the group $\mathrm{Spin}(8k)$. Here $\mathrm{Spin}(8k)$ is the double cover of $SO(8k)$ which fits into the diagram*

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r] \arrow[d, "="'] & \mathrm{Spin}(8k) \arrow[r] \arrow[d] & SO(8k) \arrow[d] \\ Z \arrow[r] & SO(2^{4k}) \arrow[r] & SO(2^{4k})/Z \end{tikzcd}$$

(one can show that Spin$(8k)$ *is the simply connected double cover of* $SO(8k)$*). In this case, the bundle S is given by*

$$S = \widetilde{F} \times_{\mathrm{Spin}(8k)} \mathbb{R}^{2^{4k}},$$

and the action of T^*M *on it is induced from the action of* $\mathbb{R}^{8k}$ *on* $\mathbb{R}^{2^{4k}}$.

If the Dirac symbol $c\colon T^*M \to \mathrm{End}(S)$ is the complexification of a *real* Dirac symbol, then the line bundle L_c is trivial. In this case the index formula reads quite simply

$$\mathrm{Ind}(D) = \int_M \widehat{A}(M).$$

One of the interesting features of this formula is that in the real case, thanks to the fact that the reduction $\widetilde{F}$ is a covering space of the frame bundle F, there is a canonical connection on $\widetilde{F}$ which gives rise to a canonical affine connection on S, and ultimately a canonical operator (defined in terms of the Riemannian geometry of M) whose symbol is the Dirac symbol, namely

$$D = \sum \sigma(\omega_i)\nabla_{X_i},$$

where the sum is over a local frame $\{X_i\}$ and dual frame $\{\omega_i\}$. This operator has the following important property, known as the *Lichnerowicz formula*:

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4},$$

where κ is the scalar curvature function on M.

weitzenbock-remark

8.33 Remark. Formulas of the above type, known as *Bochner-Weitzenbock formulas*, can be proved for the squares of many natural geometric operators. A special feature of the Dirac operator associated to a Spin-structure is that for it the order zero term that appears in all such formulas is particularly simple and significant.

The Lichnerowicz formula has the following immediate consequence:

lichnerowicz-theorem **8.34 Theorem.** *Let M be a Riemannian manifold which admits a Spin structure. If the scalar curvature of M is everywhere positive then* $\int_M \widehat{A}(M) = 0$.

Proof. If $\kappa > 0$, then by the Lichnerowicz formula the Dirac operator is bounded below, and is therefore invertible. Hence its index is zero. □

8.4 EXERCISES

8.5 NOTES

A book-length exposition of Clifford algebras, their representation theory, and the associated Dirac operators can be found in [**?**]. The Riemann-Roch theorem and the signature theorem were first proved by Hirzebruch [**?**]. In his work, Thom's cobordism theory acts as an organizing principle in roughly the same way that K-theory does in the proof of the index theorem.

For exotic spheres see [**?**, **?**].

Chapter Nine

The Atiyah-Singer Index Theorem

GeneralTheoremChapter

In Chapter 5 we saw that a linear elliptic partial differential operator D on a smooth closed manifold has a Fredholm index,

$$\mathrm{Ind}(D) \in \mathbb{Z}.$$

In Chapter **??** we saw that associated to D there is a symbol class

$$\sigma_D \in K(T^*M).$$

In Chapter 4 we discussed the Chern character and characteristic classes of vector bundles. The *index problem* is to compute $\mathrm{Ind}(D)$ in terms of $\mathrm{ch}(\sigma_D)$. The solution is the famous Atiyah-Singer Index Theorem, which we shall state precisely in this short chapter.

9.1 STATEMENT OF THE INDEX THEOREM

Recall that the Todd class $\mathrm{Todd}(E)$ of a complex vector bundle E is the multiplicative characteristic class associated to the formal power series $x/(1-e^{-x})$. The Todd class belongs to $H^*(M)$, and the Chern character of the symbol to $H^*(T^*M)$; the product appearing under the integration sign stems from the fact that $H^*(T^*M)$ is a module over $H^*(M)$. The "integration" functional on the top-dimensional cohomology of T^*M is defined because T^*M is always an oriented manifold. We shall explain our choice of orientation more precisely below.

the-index-theorem

9.1 Theorem (Atiyah and Singer)**.** *Let D be a linear elliptic partial differential operator on a smooth, closed manifold M, and denote by $[\sigma_D] \in K(T^*M)$ its symbol class. Then*

$$\mathrm{Ind}(D) = (-1)^{\dim(M)} \int_{T^*M} \mathrm{ch}(\sigma_D)\, \mathrm{Todd}(TM \otimes \mathbb{C}).$$

9.2 Remark. Let D be an elliptic differential operator on an *odd*-dimensional manifold M. Denote by $\iota\colon T^*M \to T^*M$ the vector bundle map which is multiplication by -1 in each fiber of the cotangent bundle. Then ι is orientation-reversing, but $\iota^* \mathrm{ch}(\sigma_D) = \mathrm{ch}(\sigma_D)$. It therefore follows from the index formula that $\mathrm{Ind}(D) = 0$, and so it would be safe to drop the sign $(-1)^{\dim(M)}$ from Theorem 9.1. However, in every dimension there are nontrivial instances of the index formula involving *pseudo*differential operators. The statement of the index formula in 9.1 does not need to be altered in any way to handle these more general cases.

The first and major step toward solving the index problem is to recast the problem in K-theoretic terms. The following result will be obtained in Chapter 10:

9.3 Theorem. *For each smooth manifold* M *(compact or not), there is a homomorphism*

$$\alpha_M \colon K(T^*M) \to K(\mathrm{pt})$$

that has the following property: if M is *compact, and if* $\sigma_D \in K(T^*M)$ *is the symbol class of an elliptic operator* D *on* M*, then* $\alpha(\sigma_D) = \mathrm{Ind}(D)$ *in* $K(\mathrm{pt}) \cong \mathbb{Z}$.

index-construction

9.2 THE ROTATION ARGUMENT

Atiyah's paper on index theory and Bott periodicity: [**?**].

9.3 THE INDEX THEOREM FOR ORIENTED MANIFOLDS

In many of the applications of the index theorem the manifold M is itself oriented and it is possible to evaluate the integral over T^*M in two stages: first integrate over the fibers of T^*M and then integrate the result over the base space M. For example, we shall see that this is how the expression $\int_M L(TM)$ arises in the Hirzebruch signature theorem.

9.4 THE SIGNATURE OPERATOR

We shall conclude this chapter by returning to the signature operator which was discussed in Chapter **??**, and computing the Atiyah-Singer formula in for it.

Let V be an oriented euclidean vector bundle of rank $2k$. Borrowing from Chapter **??**, let us define a $\star$-operator on the real exterior algebra bundle $\bigwedge^* V$ by the formula

$$(\beta, \alpha)\,\mathrm{vol} = \beta \wedge \star\alpha \in \textstyle\bigwedge^n V \otimes \mathbb{C}.$$

If we denote by S the complexification of $\bigwedge^* V$, then the operator

$$\varepsilon\alpha = i^{k+p(p-1)} \star \alpha \qquad (\alpha \in \textstyle\bigwedge^p V)$$

determines a grading operator on S. The by now standard formula

$$c(v) \cdot x = v \wedge x + v \lrcorner x$$

determines a Dirac-type symbol $c\colon V \to \mathrm{End}(S)$. If V is the cotangent bundle of an oriented, Riemannian manifold, then this Dirac-type symbol is the symbol of the signature operator.

Define a characteristic class $\mathcal{C}(V)$ by the formula $\mathrm{ch}(c) = \mathcal{C}(V) u_V$. This is a characteristic class of oriented, even-dimensional vector bundles, but in fact it extends to a characteristic class of all even-dimensional vector bundles (compare Remark **??**). One can show that it is multiplicative.

The index of the signature operator is given by the formula

$$\begin{aligned}\mathrm{Ind}(D) &= \int_{T^*M} \mathcal{C}(T^*M) u_{T^*M} \,\mathrm{Todd}(T^*M \otimes \mathbb{C}) \\ &= (-1)^k \int_M \mathcal{C}(TM) \,\mathrm{Todd}(TM \otimes \mathbb{C})\end{aligned}$$

(we have used the Riemannian structure to identify TM and T^*M). Let us therefore attempt to compute the characteristic class $(-1)^k \mathcal{C}(V)\,\mathrm{Todd}(V \otimes \mathbb{C})$. It too is multiplicative, and it is therefore determined by its value on the real bundle $E_{\mathbb{R}}$ underlying the universal complex line bundle E on projective space (see the discussion in Section **??**).

9.4 Exercise. Let E be a complex line bundle (for instance, the universal one). Prove that the even-graded part of $\bigwedge^* E_{\mathbb{R}} \otimes \mathbb{C}$ for the grading operator ε is isomorphic to the direct sum of $\overline{E}$ and a trivial line bundle, whereas the odd-graded part is isomorphic to the direct sum of E and a trivial line bundle.

By repeating the argument we gave in Theorem **??**, where we computed the class τ which appears in connection with the Thom homomorphism, we deduce from the exercise that

$$\mathcal{C}(E) = \frac{(1+e^{-x}) - (1-e^{x})}{x} = \frac{e^{-x} - e^{x}}{x},$$

where x is the Euler class of E. Since $\mathrm{Todd}(E \otimes \mathbb{C}) = \mathrm{Todd}(E)\,\mathrm{Todd}(\overline{E})$ it follows that

$$\begin{aligned}(-1)^k \mathcal{C}(E)\,\mathrm{Todd}(E \otimes \mathbb{C}) &= -\frac{e^{-x} - e^{x}}{x} \frac{x}{1-e^{-x}} \frac{-x}{1-e^{x}} \\ &= x\frac{e^{x}-1}{e^{x}-1} \\ &= \frac{x}{\tanh x/2}\end{aligned}$$

Let us write the answer as

$$\frac{x}{\tanh x/2} = 2\frac{x/2}{\tanh x/2}.$$

Our conclusion is then that if $\mathcal{L}$ is the multiplicative characteristic class of real vector bundles associated to the power series $(x/2)\big/(\tanh x/2)$, then

$$\mathrm{Ind}(D) = 2^k \int_M \mathcal{L}(TM).$$

This result is usually reformulated in terms of the multiplicative class L associated to the power series $x/\tanh x$. The formula so obtained is the Hirzebruch signature theorem:

9.5 Theorem. *Let L be the multiplicative class of real vector bundles associated to the power series $x/\tanh x$. If D is the signature operator on an even-dimensional, closed, oriented Riemannian manifold, then*

$$\mathrm{Ind}(D) = \int_M L(TM).$$

Proof. In degree $2k$ the cohomology classes $L(TM)$ and $2^k\mathcal{L}(TM)$ are equal. $\square$

We conclude by outlining an important application of the Hirzebruch signature theorem to the construction of an *exotic sphere*. This is due to Milnor (1957) and it highlighted the importance of playing off against one another two sources of "integrality" in the signature theorem: the fact the the signature (or more generally the index of an elliptic operator) is an integer, and the fact that the Pontrjagin classes are integral cohomology classes. It is interesting that in *non*commutative geometry only the first source of integrality (index theory) is available to us.

The geometric input that is needed is a construction of manifolds with prescribed intersection form (remember that the *intersection form* is the form defined by the cup-product on the middle-dimensional cohomology). We will be considering manifolds W with boundary, whose boundary is topologically a sphere; if you don't want to work out a general theory of intersection forms for manifolds with boundary, just *define* the intersection form of such a manifold to be the intersection form of the topological manifold obtained by capping off the boundary with a disk.

A quadratic form over the integers is said to be *even* if it can be represented by a matrix all of whose diagonal entries are even, and *unimodular* if its determinant is ± 1. Milnor gave an explicit construction, sometimes called "plumbing," which will produce a smooth W with prescribed even intersection form; unimodularity implies that the boundary is topologically a sphere. In particular

9.6 Theorem (Milnor Plumbing)**.** *There is a smooth 8-dimensional manifold W with boundary, such that*

- *$\Sigma = \partial W$ is homeomorphic to S^7;*
- *W is parallelizable (its tangent bundle is trivial);*
- *The intersection form of W is the E_8 matrix,*

$$E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

The E_8 form is even, unimodular, and *positive definite*: it is the "smallest" integral quadratic form with these properties.

We are going to show that Σ is *not diffeomorphic* to S^7: it is an "exotic sphere." For, suppose that it were. Then we could form a smooth, closed 8-manifold M by attaching an 8-disk to ∂W. Applying the Hirzebruch signature theorem we get

$$\mathrm{Sign}(M) = \int_M L(TM).$$

By Exercise **??**, this gives

$$8 = \frac{1}{45}\left(7\int_M p_2(TM) - \int_M p_1(TM)^2\right)$$

where the p_1 and p_2 are the Pontrjagin classes. Recall, however, that the tangent bundle of W is trivial. Thus TM is obtained by "clutching" two trivial bundles over the 7-sphere, and in such circumstances it is easy to see that all but the highest Pontrjagin classes must vanish. We conclude that $\int_M p_1(TM)^2 = 0$ so

$$\int_M p_2(TM) = \frac{56}{45}$$

which contradicts the integrality of the Pontrjagin classes.

9.5 EXERCISES

9.6 NOTES

Atiyah and Singer showed in [**?**] how the index theorem could be deduced from suitable axioms for an "analytic index map" (our α). Our *construction* of α, which will be given in Chapter 10, differs from theirs; but so far as the axiomatics of this chapter go, we have followed them closely.

There is a symbiotic relationship between Bott periodicity and elliptic operator theory. Following Atiyah [**?**] we proved the Bott periodicity theorem by computing the index of a certain special operator, while Theorem 12.1 shows how the computation of the index of an elliptic operator can be reduced ultimately to an invocation of Bott periodicity. From this point of view, the content of the index theorem is a reduction of the index theory of *all* operators to that of a small class of examples, one in each dimension.

Chapter Ten

Groupoids

GroupoidChapter

The original papers of Atiyah and Singer constructed the index homomorphism by making use of the theory of *pseudodifferential operators*. In this book we shall take a slightly different approach, more in the spirit of the noncommutative geometry program of Alain Connes [**?**]. Connes' program involves the analysis of generalized spaces, many of which can be modeled using the notion of *groupoids*. In the next chapter we shall produce the index map as a by-product of the construction of a certain groupoid, the so-called *tangent groupoid* of a manifold.

10.1 SMOOTH GROUPOIDS

The following definition is short but probably opaque to anyone who has not encountered it before. We shall therefore take some time to explain its meaning as fully as we can.

10.1 Definition. A *smooth groupoid* is a small category in which every morphism is invertible, and for which the set of all morphisms and the set of all objects are given the structure of smooth manifolds; the source and range maps are submersions; and the composition law and inclusion of identities are smooth maps.

In a more detail, a smooth groupoid consists of the following things:

(a) A smooth manifold G, whose points constitute the morphisms in some category;

(b) A smooth manifold B whose points are the objects in the category;

(c) Two maps $r, s\colon G \to B$ which associate to morphisms their range and source objects and which are required to be submersions.[1] Thanks to the assumption that r and s are submersions, the set

$$G^{(2)} = \{ (\gamma_1, \gamma_2) \in G \times G \,:\, s(\gamma_1) = r(\gamma_2) \}$$

of composable pairs of morphisms is a smooth submanifold of $G \times G$.

(d) A smooth composition map $G^{(2)} \to G$.

(e) A smooth map $B \to G$ which maps an object $x \in B$ to the identity morphism at x.

[1] Recall that a smooth map between manifolds is a submersion if in suitable local coordinates it has the form of a projection $(x_1, \ldots, x_{p+q}) \mapsto (x_1, \ldots, x_p)$.

(f) A smooth map $G \to G$ which associates to each morphism an inverse morphism.

We will usually specify a groupoid by describing the spaces G and B, and the source, range and composition maps (the other maps are determined by these). When we say, even more briefly "let G be a smooth groupoid" the "G" we are referring to is the space of all morphisms, as in (a). This is similar to saying "let G be a Lie group," in that the other structure implicit in a Lie group is assumed to be provided as well.

In noncommutative geometry, which we shall discuss later in this book, it is customary to paint what might be called the *quotient space picture* of groupoid theory. In this view, one thinks of the morphisms in G as defining an equivalence relation on the manifold B: two objects are equivalent if there is a morphism between them. Two objects might be equivalent for more than one reason, and the groupoid keeps track of this. It is customary in mathematics to form the quotient space from an equivalence relation, but even in rather simple examples the ordinary quotient space of general topology can be highly singular, and for example not at all a manifold. The groupoid serves as a smooth stand-in for the quotient space in these situations, and using it one can study the cohomology of the quotient space, and even its geometry. These ideas have developed extensively by Alain Connes.

A second view of groupoid theory, which is better suited to our immediate purposes, is what we shall call the *families picture*. We shall think of the groupoid as a family of smooth manifolds

$$G_x = \{\gamma \in G : s(\gamma) = x\}$$

parameterized by $x \in B$. If η is a morphism in G, from x to y, then there is an associated diffeomorphism

$$R_\eta \colon G_y \to G_x$$

defined by $R_\eta(\gamma) = \gamma \circ \eta$. We shall therefore think of a groupoid as a smooth family of smooth manifolds equipped with intertwining diffeomorphisms R_η. From this point of view, having been given a groupoid G, it will be very natural to consider families of say differential operators D_x, one on each G_x, which are equivariant with respect to the R_η in the obvious sense.

man-ex **10.2 Example.** A manifold M may be viewed as a smooth groupoid, by taking both the object and morphism sets to be M, and the source and range maps to be the identity map $M \to M$. In the families picture, we are thinking of M as a 'family of points' — parameterized (tautologically) by M itself.

fam-ex1 **10.3 Example.** A Lie group G may be viewed as a smooth groupoid. The object set is a single-element set, and the set of morphisms from this single element to itself is G. In the families picture, we have one manifold—the underlying smooth manifold of G—and a family of self-maps of this manifold, given by the usual right-translation operators on G. An equivariant differential operator in this example is a right-translation-invariant differential operator on the Lie group G. Thus if, for example, $G = \mathbb{R}^n$, then an equivariant differential operator is a constant coefficient operator on $\mathbb{R}^n$.

bundle-example **10.4 Example.** If V is the total space of a smooth vector bundle over a manifold M, then we can view V as a groupoid as follows: the source and range maps are both equal to the projection from V to the base space M, and composition of morphisms is addition in the fibers of V. We are therefore viewing V as a smooth family of additive Lie groups over M. We shall be particularly concerned later with the case $V = TM$, but we could in principle also consider more complicated smooth families of Lie groups, in which the groups were neither abelian nor mutually isomorphic to one another.

fam-ex2 **10.5 Example.** Let M be a smooth manifold. The *pair groupoid* of M has object space M and morphism space $G = M \times M$. Its structure maps are as follows:

- Source map: $s(m_2, m_1) = m_1$.
- Range map: $r(m_2, m_1) = m_2$.
- Composition: $(m_3, m_2) \circ (m_2, m_1) = (m_3, m_1)$.

The spaces G_m all identify with M, and the translation operators $G_{m_2} \to G_{m_1}$ all become the identity map under these identifications. An equivariant family of differential operators in this example is nothing more than a single differential operator on the manifold M.

fam-ex3 **10.6 Example.** Examples 10.3 and 10.5 can be combined, after a fashion, as follows. Let A be a discrete group or a Lie group acting (on the left) on a smooth manifold M. The *transformation groupoid* $A \ltimes M$ has object space M and the following morphism space:

$$\{ (m_2, a, m_1) \in M \times A \times M : m_2 = a m_1 \}.$$

Obviously the morphism space identifies with the product $A \times M$ by projection onto the last two factors, but the above description makes the structure maps more transparent:

- Source map: $s(m_2, a, m_1) = m_1$.
- Range map: $r(m_2, a, m_1) = m_2$.
- Composition: $(m_3, a_2, m_2) \circ (m_2, a_1, m_1) = (m_3, a_2 a_1, m_1)$.

However an equivariant family of differential operators is not, as one might guess, the same thing as an A-equivariant differential operator on M. Instead it is a family of differential operators D_m on A, parameterized by $m \in M$, for which the operator D_m is equivariant for the right translation action of the isotropy subgroup A_m on A.

balanced-prod-example **10.7 Example.** Let W be a smooth manifold. An action of a discrete group Γ by diffeomorphisms is *principal* if every point $w \in W$ has a neighborhood U such that $U\gamma \cap U = \emptyset$ for all $\gamma \in \Gamma$ apart from $\gamma = e$. Let G be the quotient of $W \times W$ by the diagonal action of Γ. This is a smooth groupoid with object space W/Γ and the following structure maps:

- Source map: $s([w_2, w_1]) = [w_1]$ (the square brackets denote orbits under Γ-action).
- Range map: $r([w_2, w_1]) = [w_2]$.
- Composition: $[w_3, w_2] \circ [w_2, w_1] = [w_3, w_1]$.

10.8 Exercise. Determine the identity morphisms and inverses in all of the above examples.

10.9 Exercise. Show that an equivariant family of differential operators on the groupoid of Example 10.7 is the same thing as an ordinary differential operator on the quotient manifold W/Γ.

10.2 GROUPOID C*-ALGEBRAS

We are going to associate to any smooth groupoid a convolution C^*-algebra.

10.10 Definition. A *right Haar system* on a smooth groupoid G is a system of smooth measures, one on each of the *leaves*

$$G_x = \{\gamma \in G : s(\gamma) = x\},$$

with the properties that:

(i) If f is a smooth, compactly supported function on G then $\int_{G_x} f(\gamma)\, d\mu_x(\gamma)$ is a smooth function of x.

(ii) If η is a morphism from x to y then

$$\int_{G_x} f(\gamma)\, d\mu_x(\gamma) = \int_{G_y} f(\gamma \circ \eta)\, d\mu_y(\gamma).$$

10.11 Proposition. *Every smooth groupoid admits a right Haar system.* □

The proposition can be proved by adapting the standard construction of Haar measures on Lie groups: pick a 1-density (basically a top degree differential form) on G_x at the point id_x and do so in a way which varies smoothly with x. Then right-translate the densities around G to define a 1-density at every point with the required properties.

10.12 Definition. Let G be a smooth groupoid with right Haar system. Define a convolution multiplication and adjoint on the space $C_c^\infty(G)$ of smooth, compactly supported complex functions on the morphism space of G using the formulas

$$f_1 \star f_2(\gamma) = \int_{G_{s(\gamma)}} f_1(\gamma \circ \eta^{-1}) f_2(\eta)\, d\mu_{s(\gamma)}(\eta).$$

and

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

10.13 Proposition. *Let* G *be a smooth groupoid with right Haar system. With the above operations,* $C_c^\infty(G)$ *is an associative* $*$*-algebra.* □

Proof. Exercise. □

10.14 Definition. Let G be a smooth groupoid with right Haar system. Define representations

$$\lambda_x\colon C_c^\infty(G) \to \mathcal{B}(L^2(G_x))$$

by the formulas

$$\lambda_x(f)h(\gamma) = f \star h(\gamma) = \int_{G_{s(\gamma)}} f(\gamma \circ \eta^{-1})h(\eta)\, d\mu_{s(\gamma)}(\eta).$$

The *reduced groupoid* C^**-algebra* of G, denoted $C^*_\lambda(G)$, is the completion of $C_c^\infty(G)$ in the norm

$$\|f\| = \sup_x \|\lambda_x(f)\|_{B(L^2(G_x))}.$$

This construction is sufficiently general to encompass a number of standard examples.

10.15 Example. Let M be a manifold, which we regard as a groupoid as in Example 10.2. Then each leaf is a single point, and the Dirac measures form a Haar system. The associated groupoid C^*- algebra is the commutative algebra $C_0(M)$ of continuous functions on M vanishing at infinity.

10.16 Example. Let G be a Lie group, regarded as a groupoid as in Example 10.3. There is only one leaf, which is G itself, and a Haar system on the groupoid is just the same thing as a Haar measure on the group. Our construction tells us to complete the algebra $C_c^\infty(G)$ (under convolution) in the norm coming from its regular representation on $L^2(G)$. This is the standard construction of the *reduced* C^**-algebra* of a group (see [**?**]).

groupalg-euclid **10.17 Example.** One particular case is extremely important. Suppose that $G = \mathbb{R}^n$. Recall that a function on $\mathbb{R}^n$ is said to belong to the *Schwarz class* $\mathcal{S}(\mathbb{R}^n)$ if it and all its derivatives decrease at ∞ faster than any polynomial[2]. It is easy to check that convolution is well-defined on $\mathcal{S}(\mathbb{R}^n)$ (essentially because the volume of a ball in $\mathbb{R}^n$ grows only as a polynomial in the radius). Plancherel's theorem from Fourier analysis, which we already encountered in Chapter 5, then has the following sharpening.

schwarz-ft **10.18 Proposition.** *The Fourier transform extends to give a linear homeomorphism of* $\mathcal{S}(\mathbb{R}^n)$ *onto* $\mathcal{S}((\mathbb{R}^n)^*)$*, and to give a unitary isomorphism of* $L^2(\mathbb{R}^n)$ *onto* $L^2((\mathbb{R}^n)^*)$*. Moreover, the Fourier transformation converts convolution into pointwise multiplication, in the sense that*

$$\widehat{f * u} = \widehat{f} \cdot \widehat{u},$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ *and* $u \in L^2(\mathbb{R}^n)$.

[2] The space $\mathcal{S}(\mathbb{R}^n)$ has a natural Fréchet topology, whose seminorms are

$$\|f\|_{k,l} = \sup\{|\partial^\alpha f/\partial x^\alpha (1+|x|)^l : x \in \mathbb{R}^n, |\alpha| \le k\}$$

for natural numbers k, l.

(A proof may be found in [**?**], for example.) It follows that the convolution algebra $C_c^\infty(\mathbb{R}^n)$, acting on the regular representation by convolution, is unitarily equivalent to a dense subalgebra of $\mathcal{S}((\mathbb{R}^n)^*)$, acting on $L^2((\mathbb{R}^n)^*)$ by pointwise multiplication. We conclude that $C^*(\mathbb{R}^n)$ is canonically identified (via Fourier transform) with $C_0((\mathbb{R}^n)^*)$.

haar1-ex **10.19 Example.** If $G = M \times M$ (the *pair groupoid* of Example 10.5), then in any Haar system all the measures μ_m on $G_m = M \times \{m\} \cong M$ are equal to one another and conversely any smooth measure μ on M determines a Haar system. The convolution multiplication and adjoint are

$$f_1 \star f_2(m_2, m_1) = \int_M f_1(m_2, m) f_2(m, m_1)\, d\mu(m).$$

and

$$f^*(m_2, m_1) = \overline{f(m_1, m_2)}.$$

Thus $C_c^\infty(G)$ is the algebra of 'smoothing operators' on M — compactly supported smooth functions on M, multiplied according to the first display above. (It is helpful to think of these functions as 'continuous matrices', and of the equations above as 'continuous' versions of the familiar rules for calculating the product and adjoint of matrices.)

It turns out that the groupoid C^*-algebra $C^*_\lambda(G)$ for the pair groupoid is simply the C^*-algebra $\mathcal{K}$ of *compact operators* on $L^2(M)$. To prove this we invoke Lemma **??** which tells us that every compactly supported smoothing operator is compact as an operator on $L^2(M)$. Moreover, if $\{e_m\}$ is an orthonormal basis of $L^2(M)$ made up of compactly supported smooth functions, then all the rank-one operators

$$u \mapsto \langle u, e_m \rangle e_n$$

are compactly supported smoothing operators. Since such operators span a dense subalgebra of $\mathcal{K}$, it follows that the norm closure of the compactly supported smoothing operators is exactly the algebra of compact operators.

haar2-ex **10.20 Example.** If $G = TM$ (the tangent bundle) then a Haar system is a smoothly varying system of translation-invariant measures on the vector spaces $T_m M$. Assuming for simplicity that M is oriented, a translation-invariant measure on $T_m M$ is the same thing as a point in $\wedge^n T_m^* M$, and so we see that a smooth Haar system on TM is determined by a smooth measure on M. The convolution multiplication and adjoint in the groupoid algebra are

$$f_1 \star f_2(X, m) = \int_{T_m M} f_1(X - Y, m) f_2(Y, m)\, d\mu(Y).$$

and

$$f^*(X, m) = \overline{f(-X, m)}.$$

The groupoid C^*-algebra is therefore, so to speak, made up of sections of a bundle of C^*-algebras over M, whose fiber at $m \in M$ is the group C^*-algebra of the additive group T_mM. However, each of these C^*-algebras is of the kind we discussed in Example 10.17: the C^*-algebra of the additive group T_mM is the algebra $C_0(T^*_mM)$. An element of the groupoid algebra is therefore identified with a section which associates, to each $m \in M$, a function on the cotangent space T^*_mM. Such a section is of course just a function on the total space of the cotangent *bundle* T^*M. We conclude that

$$C^*_\lambda(TM) \cong C_0(T^*M).$$

10.21 Exercise. Fill in the details of the argument above.

10.3 MORITA EQUIVALENCES

10.4 ELLIPTIC OPERATORS ON GROUPOIDS

index-tech-thm

10.22 Proposition. *Let D be an essentially self-adjoint, linear elliptic partial differential operator on a smooth, closed manifold M. Let σ_D be the symbol of D and let $f \in C_0(\mathbb{R})$. The index asymptotic morphism $\alpha_t \colon C_0(T^*M) \to \mathcal{K}(L^2(M))$ maps $f(\breve{\sigma}_D) \in C^*_\lambda(TM)$ to the family $f(tD) \in \mathcal{K}(L^2(M))$. up asymptotic equivalence.*

For simplicity of notation we have omitted explicit mention of the coefficient bundle S. The proposition easily implies Proposition 11.19, since α_t comes from an element of $C^*_\lambda(\mathbb{T}M)$ by definition and $C_0((0,1];\mathcal{K})$ is a subalgebra of $C^*_\lambda(\mathbb{T}M)$.

We model our proof of Proposition 10.22 on the proof of Theorem 5.22 from Chapter 5. This proof had two parts, first a local estimate, and second an invocation of the local-global principle expressed by Lemma 5.18.

In the present case the required local estimate is the following.

ind-tech-lemma2

10.23 Lemma. *Let D be an selfadjoint, first order elliptic differential operator on a smooth, closed manifold M, and let $f(x) = (x \pm i)^{-1}$. For every $m \in M$ and every $\varepsilon > 0$ there is a function $g_m \in C(M)$ such that $g_m(m) = 1$ and*

$$\limsup_{t\to 0} \|\alpha_t(g_m f(\breve{\sigma}_D) g_m) - g_m \cdot f(tD) \cdot g_m\| < \varepsilon.$$

Proof of Lemma 10.22, assuming Lemma 10.23. Let A be the asymptotic algebra of $\mathcal{K}$, that is $C_b((0,1];\mathcal{K})/C_0((0,1];\mathcal{K})$. The functions $\alpha_t(\breve{\sigma}_D)$ and $f(tD)$ both define elements of A, and we need to show that they are equal. Let a be their difference. The algebra $C(M)$ is represented in A by pointwise multiplication operators, and a commutes with the image of $C(M)$ (to see this, note that a commutes asymptotically with $\alpha_t(\breve{\sigma}_D)$ by Remark 10.27, and it commutes asymptotically with $f(tD)$ as was shown in the proof of Theorem 5.22). Lemma 10.23 shows that the hypotheses of Lemma 5.18 are satisfied. Hence $a = 0$ and the result is proved. $\square$

Now we carry out the local analysis. The reader should compare this analysis with that of lemma 5.17.

Proof of Propostion 10.23. Fix $m \in M$ and fix a set of local coordinates in a neighborhood U of $m \in M$. Let D_0 be the constant coefficient operator (in these coordinates) which agrees with D at the point m. Fix a smooth, compactly supported function g_1 on U which is identically 1 in a neighborhood of m, and let g be any smooth function such that $g_1 g = g$. Let us write

$$\begin{aligned}\alpha_t(gf(\sigma_D)g) - gf(tD)g = \alpha_t(gf(\sigma_D)g) &- \alpha_t(gf(\sigma_{D_0})g)\\ &+ \alpha_t(gf(\sigma_{D_0})g) - gf(tD_0)g\\ &+ gf(tD_0)g - gf(tD)g\end{aligned}$$

(To compute the operator $gf(tD_0)g$, we identify U with an open set in $\mathbb{R}^n$ using the given local coordinates, extend D_0 to a constant coefficient operator on $\mathbb{R}^n$, form the resolvent $f(tD_0) = (tD_0 + iI)^{-1}$, and then use the local coordinates again to identify $gf(tD_0)g$ with an operator on U.)

The first term can be written as $\alpha_t\big(gf(\sigma_D)g - gf(\sigma_{D_0})g\big)$. The function $f(\sigma_D) - f(\sigma_{D_0})$ on T^*U vanishes on the fiber over $m \in U$. So if $\varepsilon > 0$ and if g is chosen to be supported sufficiently close to m then $\|g(f(\sigma_D)g - gf(\sigma_{D_0})g\| < \varepsilon$. Since α_t is uniformly bounded, it follows that the first term in the display is uniformly bounded by ε, for suitable g.

The second term in the display is asymptotic to zero. We are working in a coordinate neighborhood, so we may calculate in the tangent groupoid $\mathbb{TR}^n$. Using the explicit choice of linear extension operator $L\colon C_c^\infty(T\mathbb{R}^n) \to C_c^\infty(\mathbb{TR}^n)$ defined by equation 10.2, one sees that $\alpha_t(f(\breve{\sigma}_{D_0}))$ is exactly equal to $f(tD_0)$ for a constant coefficient operator D_0 on $\mathbb{R}^n$.

As for the third term in the display, we find that

$$\begin{aligned}&gf(tD)g - gf(tD_0)g\\ &\qquad\qquad = g(tD + iI)^{-1}g_1 \cdot tg_1(gD_0 - Dg)g_1 \cdot g_1(tD_0 + iI)^{-1}g.\end{aligned}$$

The multiplication operator g_1 is inserted to make it clear that each of the three terms on the right hand side is an operator on U. Let us break the right hand side into the sum of the terms

$$g(tD + iI)^{-1}g_1 \cdot tg_1(gD - Dg)g_1 \cdot g_1(tD_0 + iI)^{-1}.$$

and

$$g(tD + iI)^{-1}g_1 \cdot g_1 g(D_0 - D)g_1 \cdot tg_1(tD_0 + iI)^{-1}$$

In the first term, the left and right factors are uniformly bounded in t (in the L^2-operator norm), while the middle term has norm converging to zero as t converges to zero. As for the second term, the rightmost operator is uniformly bounded it t in the norm of operators from $L^2(U)$ to the Sobolev space $H^1(M)$. The norm of the middle factor, considered as an operator from $H^1(M)$ into $L^2(M)$, is bounded by the size of the coefficients of the differential operator $g_1 g(D_0 - D)g_1$. By choosing g to be supported sufficiently close to m, where the coefficients of D and D_0 agree, we can make this norm as small as we please. The leftmost operator is uniformly bounded in the L^2-operator norm. Overall, for any $\varepsilon > 0$ we can choose g, with $g(m) = 1$, so that the bottom threefold product has norm less than ε, for all t. We have therefore shown that for any $\varepsilon > 0$ and suitable g, with $g(m) = 1$, the third term in the top display is uniformly bounded by ε. This completes the proof of the proposition. $\square$

10.5 SMOOTH FAMILIES OF GROUPOIDS

Consider elements of the smooth groupoid algebra $C_c^\infty(\mathbb{T}M)$ that vanish in a neighborhood of $0 \in \mathbb{R}$. These are simply families of smoothing operators on M, depending smoothly on a parameter t and vanishing near $t = 0$. Their closure forms an ideal in $C_\lambda^*(\mathbb{T}M)$, and this ideal is identified with $C_0((0,1];\mathcal{K}(L^2(M)))$. It is clear that this ideal is contained in the kernel of ε_0.

amen-lemma

10.24 Lemma. *The ideal described above is equal to the kernel of ε_0. In other words, the sequence of C^*-algebras*

$$0 \longrightarrow C_0((0,1];\mathcal{K}(L^2(M))) \longrightarrow C_\lambda^*(\mathbb{T}M) \longrightarrow C_\lambda^*(TM) \longrightarrow 0$$

is exact.

Before we prove this slightly technical lemma, let us see how it will allow us to construct the index homomorphism. From the lemma we obtain an isomorphism of C^*-algebras

$$C_0(T^*M) = C_\lambda^*(TM) \to \frac{C_\lambda^*(\mathbb{T}M)}{C_0((0,1];\mathcal{K}(L^2(M)))}.$$

But the evaluation maps ε_t, $t > 0$, fit together to give a $*$-homomorphism of $C_\lambda^*(\mathbb{T}M)$ into the C^*-algebra $C_b((0,1];\mathcal{K}(L^2(M)))$ of *bounded*, continuous functions from $(0,1]$ to the compact operators. Thus we get a $*$-homomorphism from $C_0(T^*M)$ to the quotient $C_b((0,1];\mathcal{K})/C_0((0,1];\mathcal{K})$ and this quotient is — up to a trivial reparameterization — the *asymptotic algebra* $\mathcal{Q}(\mathcal{K})$ introduced in the proof of Proposition 15.17. Finally, a morphism into the asymptotic algebra of $\mathcal{K}$ is the same thing as an equivalence class of asymptotic morphisms to $\mathcal{K}$ itself. Our conclusion is that the lemma allows us to construct an asymptotic morphism

$$\alpha_t : C_0(T^*M) \dashrightarrow \mathcal{K}\ .$$

index-map-def

10.25 Definition. The index homomorphism $\alpha\colon K(T^*M) \to \mathbb{Z}$ is the K-theory map induced from the above asymptotic morphism α_t .

10.26 Remark. Note that the definition of the asymptotic morphism α can be made completely explicit, at least for the 'smooth elements' $C_c^\infty(TM)$ of $C_\lambda^*(TM)$. Namely, given such an element (which is a smooth function defined on a closed submanifold of the tangent groupoid), extend it in any way to a smooth compactly supported function on the whole tangent groupoid, and then evaluate the resulting smoothing operator at a small (but non-zero) value of t. Asymptotically (as $t \to 0$) this process is independent of the choice of extension and defines the asymptotic morphism α.

expl-am-rmk

ascom-rmk

10.27 Remark. As $t \to 0$ the operators $\alpha_t(f)$ become more and more 'localized' in M. One way of expressing this is the following statement: the norm of the commutator $[\alpha_t(f), g]$ tends to zero, as $t \to 0$, for any smooth function g on M (acting by multiplications). Indeed, it is easy to see that if $f \in C_c^\infty(\mathbb{T}M)$ then $[\varepsilon_t(f), g] = \varepsilon_t(h)$, for some element $h \in C_c^\infty(\mathbb{T}M)$ vanishing at $t = 0$.

We now turn to the proof of Lemma 10.24. What underlies the proof is the *amenability* of the groupoid $\mathbb{T}M$ (see [**?**]), but we shall give a direct argument which does not require any general concepts from the theory of amenable groupoids. It is convenient to handle the case $M = \mathbb{R}^n$ first. We shall need a lemma.

10.28 Definition. Let A be a unital C^*-algebra. A unital subalgebra $\mathcal{A} \subseteq A$ is *inverse closed* if, whenever $a \in \mathcal{A}$ is invertible in A, its inverse a^{-1} in fact belongs to $\mathcal{A}$. Inverse closure for non-unital algebras is defined by passing to unitalizations.

It is a standard result from elementary C^*-algebra theory that any C^*-subalgebra of a C^*-algebra is inverse closed. However, we shall be concerned with dense subalgebras that are *not* C^*-algebras. In particular we shall need to note that the Schwarz algebra $\mathcal{S}((\mathbb{R}^n)^*)$ (under pointwise multiplication) is a dense, inverse closed subalgebra of $C_0((\mathbb{R}^n)^*)$. Equivalently, by Fourier transform (10.18), the Schwarz algebra $\mathcal{S}(\mathbb{R}^n)$ (under convolution) is a dense, inverse closed subalgebra of $C^*_\lambda(\mathbb{R}^n)$.

holomorphic-lemma

10.29 Lemma. *Let A be a C^*-algebra and $\mathcal{A}$ a dense, inverse-closed $*$-subalgebra. Then any $*$-homomorphism ϕ of $\mathcal{A}$ into a C^*-algebra B extends uniquely to a $*$-homomorphism of A into B.*

Proof. If $a \in \mathcal{A}$, then $\mathrm{Spectrum}_{\mathcal{A}}(a^*a) = \mathrm{Spectrum}_A(a^*a)$ (because of inverse closure). But since ϕ is a homomorphism,

$$\mathrm{Spectrum}_B(\phi(a^*a)) \subseteq \mathrm{Spectrum}_{\mathcal{A}}(a^*a).$$

In a C^*-algebra, the norm of a selfadjoint element is equal to its spectral radius. It therefore follows that $\|\phi(a^*a)\|_B \le \|a^*a\|_A$, and therefore that $\|\phi(a)\|_B \le \|a\|_A$. Then ϕ is norm continuous, and extends by density. $\square$

Proof of lemma 10.24 for $M = \mathbb{R}^n$. The evaluation map ε_0 gives a $*$-homomorphism

$$\varepsilon_0\colon \frac{C^*_\lambda(\mathbb{T}M)}{C_0((0,1];\mathcal{K}(L^2(M))} \to C^*_\lambda(TM).$$

We shall use Lemma 10.29 to construct an inverse to this $*$-homomorphism.

As a first step let us construct a $*$-homomorphism

$$\phi\colon C^\infty_c(TM) \to \frac{C^*_\lambda(\mathbb{T}M)}{C_0((0,1];\mathcal{K}(L^2(M))} \tag{10.1}$$ phieq

which will be a right inverse to ε_0. This is done by using the construction of Remark 10.26: given a smooth kernel function on TM, extend it smoothly to a function on the tangent groupoid, and observe that any two such extensions differ by a smooth family of smoothing operators, vanishing at $t = 0$. By virtue of the construction $\varepsilon_0 \circ \phi$ is equal to the identity on $C^\infty_c(TM)$. Moreover, ε_0 maps $C^\infty_c(\mathbb{T}M)$ to $C^\infty_c(TM)$, and when restricted to this dense subalgebra $\phi \circ \varepsilon_0$ is equal to the quotient map $C^\infty_c(\mathbb{T}M) \to \dfrac{C^*_\lambda(\mathbb{T}M)}{C_0((0,1];\mathcal{K}(L^2(M))}$.

We cannot apply Lemma 10.29 directly in this situation because the subalgebra $C^\infty_c(TM)$ of $C^*_\lambda(TM)$ is usually not inverse closed. However, we can embed it in an inverse closed subalgebra as follows. Let $\mathcal{S}_c(T\mathbb{R}^n)$ denote the collection of smooth functions on the space $T\mathbb{R}^n$ that are of Schwarz class in the 'fiber' direction and compactly supported in the 'base' direction. $\mathcal{S}_c(T\mathbb{R}^n)$ is an algebra under the operation of *fiberwise convolution* (the convolution operation in the groupoid $T\mathbb{R}^n$). A straightforward extension of the remarks above (using the 'fiberwise Fourier transform') shows that it is an inverse-closed subalgebra of $C^*_\lambda(T\mathbb{R}^n)$. It is naturally topologized as an inductive limit of the Fréchet algebras $\mathcal{S}_K(T\mathbb{R}^n)$ made up of functions supported over compact subsets K of the base $\mathbb{R}^n$.

It is now clear from Lemma 10.29 that in order to complete the proof it will be enough to extend ϕ to a $*$-homomorphism (which we shall also call ϕ)

$$\mathcal{S}_c(TM) \to \frac{C^*_\lambda(\mathbb{T}M)}{C_0((0,1];\mathcal{K}(L^2(M))}.$$

Such an extension can be described by an explicit formula. We can identify $\mathbb{T}M$ with $\mathbb{R}^n \times \mathbb{R}^n \times [0,1]$, as in Example 11.10. Using this identification we can make an explicit choice of linear extension operator $L\colon C_c^\infty(TM) \to C_c^\infty(\mathbb{T}M)$: namely, extend the function $h(v_2, v_1)$ defined on $T\mathbb{R}^n$ to the function

$$\tilde{h}(v_2, v_1, t) = h(t^{-1}(v_2 - v_1), v_1) \tag{10.2}$$

expl-ex

on $\mathbb{T}\mathbb{R}^n$. Because of Lemma **??**, for each compact $K \subseteq \mathbb{R}^n$ the operator norms of the compact operators $\varepsilon_t L(h)$ are bounded, uniformly in t, in terms of finitely many of the seminorms of h in the algebra $\mathcal{S}_K(T\mathbb{R}^n)$ — the actual constants in the bound depend on the particular smooth measure on $\mathbb{R}^n$ that we choose to generate our Haar system, but over any compact set $K \subseteq \mathbb{R}^n$ any two such choices are equivalent. By definition of the norm on $C^*_\lambda(\mathbb{T}\mathbb{R}^n)$, then, $L\colon C_c^\infty(T\mathbb{R}^n) \to C^*_\lambda(\mathbb{T}\mathbb{R}^n)$ is continuous with respect to the topology of $\mathcal{S}_c(T\mathbb{R}^n)$ and therefore extends uniquely to a continuous $*$-homomorphism defined on the latter algebra. The proof (for the case of $\mathbb{R}^n$) is thus complete. □

Proof of Lemma 10.24 for the general case. Once again, the key point is to show that the homomorphism ϕ of equation 10.1 extends continuously to $C^*_\lambda(TM)$. If h is compactly supported in the domain of a coordinate chart then we may use the special case of the lemma already proved to show that

$$\|\phi(h)\| \leq \|h\|,$$

the norms being C^*-norms in each case. Now consider a compact subset K of M. A partition of unity argument shows that on functions h supported on K,

$$\|\phi(h)\| \leq C\|h\|$$

where C is a constant which might *a priori* depend on K.. However, the collection of elements supported on K is a C^*-subalgebra of $C^*_\lambda(\mathbb{T}M)$. We have shown that ϕ is a continuous $*$-homomorphism defined on this C^*-subalgebra, and it must therefore have norm ≤ 1. (This is a standard property of C^*-homomorphisms, which is implicit in the proof of Lemma 10.29.) Thus $C = 1$ in the inequality above, independent of K, and as a result, ϕ extends by continuity to all of $C^*_\lambda(TM)$, as required. □

10.6 APPENDIX: ELLIPTIC REGULARITY AND SMOOTHING KERNELS

We close this chapter by briefly considering the problem of representing operators $f(D)$ by *kernel functions* $k(x, y)$. Such a representation, if it exists, should take the form

$$f(D)u(m_2) = \int_M k(m_2, m_1)u(m_1)\, dm_1,$$

valid for all compactly supported u. Since the operators we are interested in act on $L^2(M, S)$, we shall need to consider kernels which are not scalar-valued but S-*valued*, in the sense that $k(x, y) \in \mathrm{Hom}(S_y, S_x)$ (thus k is a continuous section of the obvious associated bundle over $M \times M$). Our aim is to show that if D is elliptic and if $f \in \mathcal{S}$ is *rapidly decreasing*, then $f(D)$ is represented by a smooth S-valued kernel.

10.30 Proposition. *Let D be a formally self-adjoint, elliptic order one operator on a closed manifold M and let f be a continuous, rapidly decreasing function on $\mathbb{R}$. Then there is a smooth, S-valued kernel k representing $f(D)$.*

elliptic-package

10.31 Remark. A function f is *rapidly decreasing* if, for every k, the function $f(x)(1 + x^2)^k$ is bounded.

Proof. For simplicity we shall argue as if D acted on functions, not sections of S; we leave it to the reader to make the appropriate modifications which take S into account. Fix $\ell \geq 0$ and write the rapidly decreasing function f as a product

$$f(x) = (x^2 + 1)^{-\ell} g(x)(x^2 + 1)^{-\ell},$$

where g is also rapidly decreasing, and in particular bounded. It follows from the spectral theorem that $g(D)$ is a bounded operator on $L^2(M)$. Using the functional calculus we see that

$$f(D) = (D^2 + I)^{-\ell} g(D)(D^2 + I)^{-\ell}.$$

We shall prove the proposition by analyzing the operator $(I + D^2)^{-\ell}$. Since $(D^2 + I)^{-1} = (D + iI)^{-1}(D - iI)^{-1}$, it follows from Theorem 5.27 that the range of the operator $(I + D^2)^{-\ell}$ is the Sobolev space $H^{2\ell}(M)$. So by the Sobolev Lemma if $k \geq 0$ and if $\ell > \frac{n}{2}$ then $(I + D^2)^{-\ell}$ maps $L^2(M)$ continuously into $C(M)$. Taking Banach space adjoints, and using the fact that

$$\langle (I + D^2)^{-\ell} u, v\rangle = \langle u, (I + D^2)^{-\ell} v\rangle,$$

for all $u, v \in L^2(M)$, it follows that $(I + D^2)^{-\ell}$ extends to a continuous map of the dual space $C(M)^*$ into $L^2(M)$. Returning to our product decomposition of $f(D)$, we see that $f(D)$ extends to a continuous map of $C(M)^*$ into $C(M)$. Now, each element $m \in M$ determines an element $\delta_m \in C(M)^*$ by the formula $\delta_m(\phi) = \phi(m)$. We can therefore define a kernel function on $M \times M$ by the formula

$$k(m_2, m_1) = (f(D)\delta_{m_2})(m_1).$$

It may be verified that this is a continuous kernel which represents $f(D)$ in the required fashion, and an elaboration of the argument shows that k is in fact smooth. $\square$

Now let G be a smooth groupoid with compact object space B. In the same way that G can be equipped with a smooth Haar system by right-translating a smooth density on M (thought of as the space of identity morphisms) over all of the G_x, the fibers G_x can be equipped with a smoothly varying family of Riemannian metrics which is right-translation invariant. Since we are assuming that the object space of G is a *compact* manifold, the Riemannian metrics that we construct on the G_x by this process are all *complete*.

Let $\mathbb{D} = \{D_x\}$ be an equivariant family of first order, formally self-adjoint elliptic operators on the fibers of G. By invariance, and by the compactness of the object space of G, the principal symbols of the D_x have the property that, when evaluated on cotangent vectors of length one, they return values whose norm is uniformly bounded (over all the G_x). Because of this we can appeal to a theorem of Chernoff to deduce that each operator D_x is essentially self-adjoint, and moreover the functional calculus associated to D_x has an important supplementary property:

chernoff-thm

10.32 Theorem (Chernoff). *Let D be a symmetric, first order, elliptic operator on a complete Riemannian manifold W, and assume that the symbol of D is uniformly bounded by a constant $C > 0$ on cotangent vectors of length one. Then D is essentially self-adjoint. If $f\colon \mathbb{R} \to \mathbb{C}$ is a rapidly decreasing function, then the operator $f(D)$ is represented by a smooth kernel $k_f(w_2, w_1)$, that is,*

$$(f(D)\phi)(w_2) = \int_W k_f(w_2, w_1)\phi(w_1)\, dw_1.$$

If the Fourier transform of f is supported in an interval $[-R, R]$, then

$$\text{Support}(k_f) \subseteq \{\, (w_2, w_1) \,:\, d(w_2, w_1) \le CR \,\}.$$

□

10.33 Remark. For brevity, we shall say that the operator D has *finite propagation speed* C.

10.34 Example. Let $W = \mathbb{R}$ and $D = -i d/dx$. By Fourier theory, $f(D)$ is represented by the kernel $k_D(y, x) = \check{f}(y - x)$, where $\check{f}$ is the inverse Fourier transform (which is equal to the Fourier transforms, up to signs).

A second theorem in partial differential equations guarantees that the function k_f varies smoothly with the coefficients of D:

smooth-family-kernels

10.35 Theorem. *Let $s\colon W \to X$ be a submersion of smooth manifolds and assume that the fibers of s have been equipped with a smoothly varying family of complete Riemannian metrics (but the space X may, for example, be a manifold with boundary). Let $\{D_x\}$ be a family of symmetric, first-order, elliptic operators on the fibers of s, and assume that the family has uniformly bounded finite propagation speed. Let f be a function with compactly supported Fourier transform. Then the kernel functions k_f associated to the different D_x vary smoothly with x.* □

10.36 Remark. "Smooth" means that the kernel functions k_f taken together constitute a smooth function on the manifold $\{\, (w_2, w_1) \,:\, s(w_2) = s(w_1) \,\}$.

Now let G be a smooth groupoid (with compact object space) and let $\mathbb{D} = \{D_x\}$ be an equivariant family of first order, selfadjoint elliptic operators along the leaves of G. If f has compactly supported Fourier transform then we can form the kernel functions $k_f(\gamma_2, \gamma_1)$ associated to $f(D_x)$, which are defined on $G_x \times G_x$. From the equivariance of the family $\{D_x\}$ it follows that

$$k_f(\gamma_2, \gamma_1) = k_f(\gamma_2 \circ \eta, \gamma_1 \circ \eta),$$

for every morphism $\eta\colon y \to x$. So if we define a function $h\colon G \to \mathbb{C}$ by the formula $h(\gamma) = k_f(\gamma, \mathrm{id}_x)$, where $x = s(\gamma)$, then

$$k_f(\gamma_2, \gamma_1) = h(\gamma_2 \circ \gamma_1^{-1}) \quad \forall \gamma_1, \gamma_2 \in G_x.$$

The function h is smooth by Theorem 10.35 and compactly supported by Theorem 10.32. Checking the definitions, we arrive at the following theorem:

groupoid-fun

10.37 Theorem. *Let* $D = \{D_x\}$ *be a smooth, right-translation invariant family of elliptic operators on the leaves* G_x *of a smooth groupoid* G *with compact object space. There is a $*$-homomorphism*

$$\phi_D\colon C_0(\mathbb{R}) \to C^*_\lambda(G)$$

with the property that if x *is any object, and* $\lambda_x\colon C^*_\lambda(G) \to \mathcal{B}(L^2(G_x))$ *is the regular representation, then*

$$\lambda_x(\phi_D(f)) = f(D_x)\colon L^2(G_x) \to L^2(G_x)$$

for every $f \in C_0(\mathbb{R})$. □

Proposition 11.19 is a special case of this result.

10.7 NOTES

Chernoff's theorem is in [**?**]. The application to groupoid algebras was made in [**?**].

Chapter Eleven

The Tangent Groupoid

TangentGroupoidChapter

The purpose of this chapter is to carry out Construction 9.3, of the homomorphism $\alpha_M \colon K(T^*M) \to \mathbb{Z}$ which maps the symbol class of an elliptic operator D to the index of D.

11.1 THE TANGENT GROUPOID OF A SMOOTH MANIFOLD

Let M be a smooth manifold. The *tangent groupoid* $\mathbb{T}M$ is a smooth groupoid whose object space is the product $M \times \mathbb{R}$. In the families picture the fibers of the tangent groupoid of M consist of repeated copies of M, parameterized by pairs $(m,t) \in M \times \mathbb{R}$, with $t \neq 0$, together with the tangent spaces $T_m M$ parameterized by pairs $(m,0) \in M \times \mathbb{R}$. These are made to form the fibers of a single submersion $s \colon \mathbb{T}M \to M \times \mathbb{R}$ for a suitable manifold structure on $\mathbb{T}M$.

The next two definitions equip $\mathbb{T}M$ with a groupoid structure and a topology. Following them, we shall put a smooth manifold structure on $\mathbb{T}M$ and verify that $\mathbb{T}M$ is a smooth groupoid.

tangent-grpd-def

11.1 Definition. Let M be a smooth manifold without boundary. Denote by $\mathbb{T}M$ the set

$$\mathbb{T}M = TM \times \{0\} \cup M \times M \times \mathbb{R}^\times$$

(a disjoint union) equipped with the following groupoid operations:

- Source map: $s(X,m,0) = (m,0)$ and $s(m_2,m_1,t) = (m_1,t)$.
- Range map: $r(X,m,0) = (m,0)$ and $r(m_2,m_1,t) = (m_2,t)$.
- Composition:

$$(X,m,0) \circ (Y,m,0) = (X+Y,m,0)$$
$$(m_3,m_2,t) \circ (m_2,m_1,t) = (m_3,m_1,t).$$

Thus $\mathbb{T}M$ is a disjoint union of copies of the pair groupoid $M \times M$, parameterized by $t \neq 0$, and a single copy of the groupoid TM at $t = 0$.

11.2 Remark. For later purposes, we are using the redundant notation (X,m), where $X \in T_m M$, to describe points of TM.

11.3 Definition. We equip $\mathbb{T}M$ with a topology, as follows:

(i) The inverse images of open sets in $M \times \mathbb{R}$ under the maps

$$r, s \colon \mathbb{T}M \to M \times \mathbb{R}$$

are deemed to be open.

(ii) Let X be a tangent vector on M, let $f \colon M \to \mathbb{C}$ be a smooth function and let $\varepsilon > 0$. The set $U_{f,\varepsilon} \subseteq \mathbb{T}M$ defined by

$$U_{f,\varepsilon} \cap TM \times \{0\} = \{ (Y, m, 0) \, : \, |X(f) - Y(f)| < \varepsilon \}$$

and

$$U_{f,\varepsilon} \cap M \times M \times \mathbb{R}^* = \left\{ (m_2, m_2, t) \, : \, \left| X(f) - \frac{f(m_2) - f(m_1)}{t} \right| < \varepsilon \right\}$$

is deemed to be an open neighborhood of X in $\mathbb{T}M$.

The topology of $\mathbb{T}M$ is generated by all those sets that are 'deemed to be open' in (i) and (ii) above.

11.4 Remark. It is useful to think of a triple (m_2, m_1, t) as being an "approximate tangent vector" which acts on functions according to the difference quotient formula

$$f \mapsto \frac{f(m_2) - f(m_1)}{t}.$$

Item (ii) of the definition says that the approximate vector (m_2, m_1, t) is close to an actual tangent vector X if the difference quotient is close to $X(f)$.

11.5 Exercise. Show that the composition law in the groupoid $\mathbb{T}M$ is continuous.

The topology on $\mathbb{T}M$ is easily seen to be Hausdorff. Moreover it is locally Euclidean:

tangent-groupoid-top-prop **11.6 Lemma.** *Let M be a smooth manifold without boundary. If U is an open subset of M, then the set*

$$\mathbb{T}U = TU \times \{0\} \cup U \times U \times \mathbb{R}^\times$$

is an open subset of $\mathbb{T}M$. Moreover if $\phi \colon U \to \mathbb{R}^n$ is a diffeomorphism onto an open subset then the map

$$\Phi \colon \mathbb{T}U \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

defined by

$$\begin{cases} \Phi(X, m, 0) = (D\phi_m(X), \phi(m), 0) \\ \Phi(m_2, m_1, t) = (t^{-1}(\phi(m_2) - \phi(m_1)), \phi(m_1), t) \end{cases}$$

is a homeomorphism onto an open subset. □

11.7 Remark. We denote by $D\phi_m \colon T_m U \to \mathbb{R}^n$ the derivative of ϕ at $m \in U$ (we have made the usual identification $T_{\phi(m)}\mathbb{R}^n \cong \mathbb{R}^n$).

11.8 Exercise. Prove the lemma.

The maps Φ defined in the lemma determine an atlas of charts for a smooth manifold structure on $\mathbb{T}M$:

11.9 Lemma. *Let* $\Phi\colon \mathbb{T}U \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ *and* $\Psi\colon \mathbb{T}V \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ *be the maps associated to diffeomorphisms* $\phi\colon U \to \mathbb{R}^n$ *and* $\psi\colon V \to \mathbb{R}^n$*, as in the previous lemma. The composition* $\Psi \circ \Phi^{-1}$ *is defined on an open subset of* $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$*, and is a smooth map.*

Proof. The inverse Φ^{-1} is given by the formula

$$\Phi^{-1}(v_2, v_1, t) = \begin{cases} (\phi^{-1}(tv_2 + v_1), \phi^{-1}(v_1), t) & \text{if } t \neq 0 \\ (D\phi_{v_1}^{-1}(v_2), \phi^{-1}(v_1), 0) & \text{if } t = 0. \end{cases}$$

Using the notation $\theta = \psi \circ \phi^{-1}$, the composition $\Theta = \Psi \circ \Phi^{-1}$ is given by the formula

$$\Theta(w_2, w_1, t) = \begin{cases} (t^{-1}\big(\theta(tw_2 + w_1) - \theta(w_1)\big), \theta(w_1), t) & \text{if } t \neq 0 \\ (D\theta_{w_1}(w_2), \theta(w_1), 0) & \text{if } t = 0. \end{cases}$$

By a version of the Taylor expansion, there is a smooth, matrix-valued function $\widetilde{\theta}(h, w)$ such that

$$\theta(h + w) = \theta(w) + \widetilde{\theta}(h, w)h \quad \text{and} \quad \widetilde{\theta}(0, w) = D\theta_w.$$

So we see that

$$\Theta(w_2, w_1, t) = \begin{cases} \big(\widetilde{\theta}(tw_2, w_1)w_2, \theta(w_1), t\big) & \text{if } t \neq 0 \\ \big(D\theta_{w_1}(w_2), \theta(w_1), 0\big) & \text{if } t = 0. \end{cases}$$

This is clearly a smooth function. □

We have therefore obtained a smooth manifold $\mathbb{T}M$. It is clear that the maps $r, s\colon \mathbb{T}M \to M \times \mathbb{R}$ are submersions. The fibers of s are T_mM at $(m, 0)$ and M at (m, t), when $t \neq 0$. To show that $\mathbb{T}M$ is in fact a smooth groupoid, we shall consider first the special case where $M = \mathbb{R}^n$:

trn-ex

11.10 Example. The map $\Phi\colon \mathbb{T}\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined by

$$\Phi(v_2, v_1, 0) = (v_2, v_1, 0)$$
$$\Phi(v_2, v_1, t) = (t^{-1}(v_2 - v_1), v_1, t) \quad (t \neq 0)$$

is a diffeomorphism. Now consider the transformation groupoid

$$G = \{\, (w_2, a, w_1) \,:\, w_1, w_2 \in \mathbb{R}^n \times \mathbb{R}, a \in \mathbb{R}^n, w_2 = a \triangle w_1 \,\},$$

where the operation $\triangle$, an action of the group $A = \mathbb{R}^n$ on the space $\mathbb{R}^n \times \mathbb{R}$, is defined by

$$a \triangle (v, t) = (v + ta, t).$$

The space G identifies with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ by dropping w_2 from (w_2, a, w_1). Using this, we obtain from the diffeomorphism Φ a diffeomorphism $\Psi\colon \mathbb{T}M \to G$ defined by the formulas

$$\Psi(v_2, v_1, 0) = ((v_1, 0), v_2, (v_1, 0))$$
$$\Psi(v_2, v_1, t) = \big((v_2, t), t^{-1}(v_2 - v_1), (v_1, t)\big) \quad (t \neq 0).$$

It is evident that Ψ is actually an isomorphism of groupoids, from which it follows that the groupoid structure on $\mathbb{T}\mathbb{R}^n$ is smooth.

To summarize:

11.11 Proposition. *Denote by* $G = \mathbb{R}^n \ltimes \mathbb{R}^{n,1}$ *the transformation groupoid associated to the action of* $\mathbb{R}^n$ *on the space* $\mathbb{R}^{n,1} = \mathbb{R}^n \times \mathbb{R}$ *given by the formula*

$$a \triangle (v,t) = (v + ta, t) \qquad (a \in \mathbb{R}^n \text{ and } (v,t) \in \mathbb{R}^{n,1}).$$

The map $\Psi\colon \mathbb{T}\mathbb{R}^n \to G$ *which is given by the formulas*

$$\begin{aligned} \Psi(v_2, v_1, 0) &= ((v_1, 0), v_2, (v_1, 0)) \\ \Psi(v_2, v_1, t) &= \big((v_2, t), t^{-1}(v_2 - v_1), (v_1, t)\big) \quad (t \neq 0). \end{aligned}$$

is an isomorphism of smooth groupoids. □

11.12 Remark. The groupoid $\mathbb{T}\mathbb{R}^n$ only depends on the smooth structure of $\mathbb{R}^n$, whereas, superficially at least, the groupoid $G = \mathbb{R}^n \ltimes \mathbb{R}^{n,1}$ depends very much on the vector space structure of $\mathbb{R}^n$. The proposition shows that this apparent dependence is an illusion.

11.13 Proposition. *The structure maps are all smooth, and the source and range maps are submersions. Thus* $\mathbb{T}M$ *is a smooth groupoid.*

Proof. Since smoothness is a local property, we can check this in a coordinate neighborhood U. Since the construction of $\mathbb{T}U$ is coordinate-independent we can assume that $U = \mathbb{R}^n$, and thereby reduce to the example just considered. □

11.14 Remark. The non-compactness of the tangent groupoid in the 't-direction' is sometimes a nuisance, and it is therefore convenient sometimes to replace the whole tangent groupoid by the subgroupoid $t^{-1}[0,1]$, where $t\colon \mathbb{T}M \to \mathbb{R}$ is the obvious map. This subgroupoid is not quite a smooth groupoid by our definition, because it has a boundary; but the extra generality does not cause any trouble. In what follows we shall make use of both versions of the tangent groupoid, often without comment. Both versions contain the same geometric information (namely, the gluing near $t = 0$); the rest of the groupoid is just 'flab'.

par-remark

11.2 THE C*-ALGEBRA OF THE TANGENT GROUPOID

cstartgsec

Let M be a smooth manifold without boundary. To define a smooth Haar system on the tangent groupoid $\mathbb{T}M$, first fix a smooth measure μ on M. As we noted above, μ determines a family of translation invariant measures μ_m on the vector spaces $T_m M$. We define smooth measures on the fibers $\mathbb{T}M_{(m,t)}$ of the source map by the formulas

$$\mu_{m,0} = \mu_m \quad \text{on } \mathbb{T}M_{(m,0)} \cong T_m M$$

and

$$\mu_{(m,t)} = t^{-n}\mu \quad \text{on } \mathbb{T}M_{(m,t)} \cong M.$$

11.15 Lemma. *The above formulas define a smooth right Haar system on* $\mathbb{T}M$.

Proof. The measures certainly constitute a translation-invariant system (compare Examples 10.19 and 10.20 above). To prove they are smooth we shall make use of the diffeomorphisms Φ introduced in the previous section, or rather their inverses $\Theta = \Phi^{-1}\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{T}U$. Let us choose coordinates on $U \subseteq M$ so that the diffeomorphism $\phi\colon U \to \mathbb{R}^n$ from which Φ is defined is the identity in these local coordinates. Then

$$\Theta(X, m, 0) = (X, m, 0) \text{ and } \Theta(m_2, m_1, t) = (m_1 + tm_2, m_1, t), \text{ if } t \neq 0.$$

If we restrict to one of the fibers of the source map then we obtain the maps

$$\Theta_{m_1,0}(X) = X \in \mathbb{T}_{v_1} U$$

and

$$\Theta_{m_1,t}(v_2) = v_1 + tv_2 \in U.$$

The derivatives of these maps (expressed as matrices, using our chosen coordinates) are I in the first case and tI in the second. Now, to transfer the measures from the fibers of $\mathbb{T}U$ to the fibers (under projection onto the last two factors) of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, we must multiply by the determinant of these derivative matrices. That is, if $\Theta\colon A \to B$ is a diffeomorphism between open sets in $\mathbb{R}^k$, and if $\mu(b) = m(b)db$ is a smooth measure on B, then

$$\int_B f(b)\, d\mu(b) = \int_B f(b) m(b)\, db = \int_A f(\Theta(a)) m(\Theta(a)) \det(D\Theta_a)\, da.$$

In our case we see that the factor t^{-n} in the definition of $\mu_{m,t}$ cancels with $\det(D\Theta) = t^n$, and we obtain smoothly varying measures, as required. □

The C^*-algebra of the tangent groupoid comes equipped with a family of restriction $*$-homomorphisms

$$\varepsilon_0\colon C^*_\lambda(\mathbb{T}M) \to C^*_\lambda(TM) = C_0(T^*M)$$

and, for $t \neq 0$,

$$\varepsilon_t\colon C^*_\lambda(\mathbb{T}M) \to \mathcal{K}(L^2(M)).$$

On the subalgebra $C_c^\infty(\mathbb{T}M)$ these are defined by restricting functions on $\mathbb{T}M$ to the "slice" of $\mathbb{T}M$ over t, which is either the tangent bundle TM (when $t = 0$) or the pair groupoid $M \times M$ (when $t \neq 0$). Strictly speaking, when $t \neq 0$ the restriction $*$-homomorphism lands in the compact operators on the Hilbert space $L^2(M, t^{-n}\mu)$ associated to the measure μ scaled by t^{-n}. But this Hilbert space is obviously unitarily equivalent to the Hilbert space associated to μ itself: just multiply by $t^{\frac{n}{2}}$. The restriction homomorphisms will be used in the next section to construct the index homomorphism in K-theory.

11.3 THE TANGENT GROUPOID, THE SYMBOL AND THE INDEX

stg-sec

In this section we shall explain how the geometry of the tangent groupoid encodes the relationship between the *symbol* of a differential operator D (Definition **??**) and the index of D. For simplicity we shall first consider only differential operators acting on scalar-valued functions; the small extra complications needed to handle operators on sections of vector bundles will be addressed at the end of the section. As in Chapter 5, we shall for simplicity also restrict attention to *first order* differential operators (though this restriction is not, in fact, essential).

Recall that the symbol of the (first order) differential operator D on M is by definition the function on the cotangent bundle T^*M defined by

$$\sigma_D \colon df \mapsto i[D, f].$$

Because $\sigma_D(\xi)$ is a homogeneous function of ξ (that is, $\sigma_D(r\xi) = r\sigma_D(\xi)$), the operator of multiplication by σ_D (acting on functions on T^*M) corresponds under Fourier transformation to a family of first order, constant coefficient differential operators[1] acting on the fibers of TM. That is, the symbol corresponds to an equivariant differential operator on the *groupoid* TM. We will call this equivariant differential operator the *cosymbol* $\check{\sigma}_D$ of D.

11.16 Proposition. *The cosymbol of D is obtained from D by 'freezing coefficients'. More precisely, suppose that in local coordinates $x_1, \ldots, x_n$, D has the representation*

$$D = \sum_{j=1}^{n} a_j(x)\, \partial/\partial x_j + b(x).$$

Then

$$\check{\sigma}_D(0) = \sum_{j=1}^{n} a_j(0) \partial/\partial y_j,$$

where the y_j are coordinates in T_0M corresponding to the local coordinates x_j.

Proof. The symbol σ_D maps the cotangent vector ξ at 0 to $i \sum a_j(0)\xi_j$. The result follows. (See the discussion in Example **??**.) □

Now consider the tangent groupoid $\mathbb{T}M$ and its natural 'scaling' map $t \colon \mathbb{T}M \to [0, 1]$. Over each $t \neq 0$, $\mathbb{T}M$ becomes a copy of the pair groupoid $M \times M$, and we have seen in Example 10.5 that a differential operator on M gives rise to an equivariant differential operator on the pair groupoid. Over $t = 0$, $\mathbb{T}M$ is just the tangent bundle TM and a differential operator on M gives rise to a differential operator on *this* groupoid by the cosymbol construction. We claim that these constructions can be scaled so as to fit together smoothly.

11.17 Proposition. *Any first order differential operator D on a manifold M gives rise to an equivariant differential operator $\mathbb{D}$ on $\mathbb{T}M$, which restricts to tD itself over each $t \neq 0$ and to the cosymbol of D over $t = 0$.*

eq-prop

Proof. The proposition describes the construction of $\mathbb{D}$ completely and all we must do is check that it is smooth. For this we may compute in local coordinates, so

[1] In Chapter 5 we referred to these as the *model operators* for D.

let us assume that $M = \mathbb{R}^n$. Then we use the coordinates for $\mathbb{T}\mathbb{R}^n$ described in Example 11.10:

$$(y, x, t) \mapsto \begin{cases} (y, x, 0) & (t = 0) \\ (t^{-1}(y - x), x, t) & (t \neq 0) \end{cases}$$

Suppose that $D = \sum a_j(x)\partial/\partial x_j + b_j(x)$. In the chosen coordinates the operator $\mathbb{D}$ takes the form

$$\mathbb{D}u(y) = \begin{cases} \sum a_j(x)\partial u/\partial y_j & (t = 0) \\ \sum a_j(x + ty)\partial u/\partial y_j + tb(x + ty) & (t \neq 0) \end{cases}$$

when acting on the fiber $G_{(x,t)}$. It is clear that the coefficients vary smoothly, even at $t = 0$. □

groupoid-coeff-remark

11.18 Remark. In all interesting examples D acts not on functions but on sections of some vector bundle S over M. In this case, the bundle S can be pulled back over the source map $s\colon \mathbb{T}M \to M \times \mathbb{R} \to M$ of the tangent groupoid to yield what we should naturally call an *equivariant bundle* over the tangent groupoid $G = \mathbb{T}M$: each morphism η in G from x to y gives rise to a diffeomorphism $R_\eta\colon G_y \to G_x$ which is covered by an isomorphism of vector bundles $R_e ta^* S_{|G_y} \to S_{|G_x}$. There is a natural definition of a G-equivariant differential operator on such an equivariant bundle, and with this definition all the results of this section still hold for operators on bundles.

When in the next section we come to consider groupoid *algebras* we shall need to make a similar adjustment to handle bundle coefficients. Namely, form the vector bundle $\mathrm{End}(S)$ over G whose fiber over a morphism $\gamma\colon x \to y$ is the vector space $\mathrm{Hom}(S_x, S_y)$. This is isomorphic to the pullback of S^* along the source map s, tensored with the pullback of S along the range map. We can now define the groupoid algebra $C_c^\infty(G, \mathrm{End}(S))$ (with respect to a given Haar system) to be the algebra of smooth, compactly supported sections of $\mathrm{End}(S)$, equipped with the convolution multiplication

$$f_1 \star f_2(\gamma) = \int_{G_{s(\gamma)}} f_1(\gamma \circ \eta^{-1}) f_2(\eta)\, d\mu_{s(\gamma)}(\eta).$$

In the formula, the product $f_1(\gamma \circ \eta^{-1})f_2(\eta)$ is a composition of operators

$$S_{s(\eta)} \to S_{r(\eta)} \to S_{r(\gamma)}.$$

This algebra has natural regular representations on $L^2(G_x, S)$, for each object x, and using these we can define the C^*-algebra completion, just as we did for $C^*_\lambda(G)$.

Let M be a compact manifold and let D be a self-adjoint, first order elliptic operator on M. Recall that we have associated two K-theory classes to D: its *symbol class*, which is an element of $K(T^*M)$, and its *index* which is an element of $K(\mathcal{K}) = \mathbb{Z}$. In Construction 9.3 we asserted that there is a natural homomorphism $\alpha\colon K(TM) \to \mathbb{Z}$ which maps the symbol class of an elliptic operator to its index. In this section we are going to prove that the asymptotic morphism that we have constructed from the tangent groupoid (Definition 10.25) does the job.

The key to the proof is the following

dfam1 **11.19 Proposition.** *Let* D *be a first order, selfadjoint, odd-graded elliptic operator on sections of a bundle* S *over a compact manifold* M*, and let* $\mathbb{D}$ *denote the equivariant operator on the tangent groupoid* $G = \mathbb{T}M$ *associated to* D *(Proposition 11.17). There is a graded* $*$*-homomorphism* $\mathcal{S} \to C^*_\lambda(G)$*, denoted* $f \mapsto f(\mathbb{D})$*, which has the property that*

$$\varepsilon_t(f(\mathbb{D})) = \begin{cases} f(\check{\sigma}_D) & (t = 0) \\ f(tD) & (t \neq 0) \end{cases}$$

where ε_t *are the restriction homomorphisms of Section 11.2.*

Granted this proposition, the proof of our main result is just a matter of reviewing definitions.

alpha-gives-index **11.20 Theorem.** *The index homomorphism of Definition 10.25 carries the symbol class of a (first order, self-adjoint) elliptic operator to its index.*

Proof. The symbol class of D is the element of $K(T^*M)$ defined by the $*$-homomorphism $f \mapsto f(\sigma_D)$ from $\mathcal{S}$ to $C_0(T^*M; \mathrm{End}(S))$. Using the Fourier isomorphism $C_0(T^*M) \cong C^*_\lambda(TM)$, we can equivalently view the symbol class of D as the element of $K(C^*_\lambda(TM))$ defined by the $*$-homomorphism $f \mapsto f(\check{\sigma}_D)$. By proposition 11.19, this is the composite

$$\mathcal{S} \xrightarrow{f \mapsto f(\mathbb{D})} C^*_\lambda(\mathbb{T}M; \mathrm{End}(S)) \xrightarrow{\varepsilon_0} C^*_\lambda(TM; \mathrm{End}(S)).$$

The index class of D is the element of $K(\mathcal{K})$ defined by the $*$-homomorphism $f \mapsto f(tD)$, for any fixed (nonzero) value of t — equivalently we may if we wish let $t \to 0$ and consider this as an asymptotic morphism. That is, the index class is defined by the composite

$$\mathcal{S} \xrightarrow{f \mapsto f(\mathbb{D})} C^*_\lambda(\mathbb{T}M; \mathrm{End}(S)) \xrightarrow{\varepsilon_t} \mathcal{K}$$

using proposition 11.19 again. But the diagram

$$\begin{tikzcd} C^*_\lambda(\mathbb{T}M) \arrow[r, "\varepsilon_0"] \arrow[dr, dashed, "\varepsilon_t"] & C^*_\lambda(TM) \arrow[d, dashed, "\alpha"] \\ & \mathcal{K} \end{tikzcd}$$

commutes (up to asymptotic equivalence) because of the definition of α. It follows that α takes the symbol class to the index, as required. □

11.4 NOTES

The idea of using smooth groupoids and their associated C^*-algebras in the proof of the index theorem is due to Connes, see [**?**]. We have supplied some details in this chapter which are difficult to find in the existing literature.

Chapter Twelve

Generalizations of the Index Theorem

GeneralizationsChapter

In the final chapters of these notes we shall sketch some of the ways in which the index theorem can be extended. Our goal is to indicate some of the power and flexibility of the techniques that we introduced in earlier chapters. We shall not always give complete proofs, or indeed any proofs at all.

12.1 AXIOMS FOR THE ANALYTIC INDEX HOMOMORPHISM

index-axioms-sec

We explained in the introduction that in Chapter 10 we shall construct a homomorphism $\alpha_M \colon K(T^*M) \to \mathbb{Z}$ which implements the analytic index, in the sense that $\alpha_M[\sigma_D] = \mathrm{Ind}(D)$ for every elliptic operator D on a closed manifold M. Once we have the homomorphism α_M in hand, the proof of the index theorem is a matter of computing α_M in cohomological terms. This we shall do by showing that α_M satisfies the hypotheses of Theorem 12.1 below, whose conclusion is in effect the Atiyah-Singer index theorem.

Before stating Theorem 12.1 we need to establish several conventions. First, we shall regard $T^*\mathbb{R}^k$, the cotangent bundle of $\mathbb{R}^k$, as a k-dimensional complex vector space using the formula $i\cdot(x,\xi) = (-\xi, x)$, where $x \in \mathbb{R}^k$ and ξ is in the cotangent vector space at x, which we identify with $\mathbb{R}^k$ in the usual way. Having done this, we have at our disposal the Bott element $b \in K^*(T^*\mathbb{R}^k)$, as described in Section **??**.

Next, we shall consider T^*M as oriented in the following way. Choose local coordinates $\{x_1, \dots, x_n\}$ on M and corresponding coordinates $\xi_1, \dots, \xi_n$ in the fibers of T^*M. Then we deem that the list $x_1, \xi_1, \dots, x_n, \xi_n$ is an oriented local coordinate system on T^*M.

Finally, suppose that V is a smooth, orthogonal vector bundle over a smooth manifold M and denote by $q\colon T^*M \to M$ the canonical projection map for the cotangent bundle of M. We shall need to identify the total space of the cotangent bundle T^*V with the total space of the Hermitian vector bundle $q^*V \otimes \mathbb{C}$ over the manifold T^*M. This is not altogether straightforward since there is not even a canonical projection from T^*V to T^*M, but it can be accomplished in several ways. We shall describe one in detail because we shall have to examine this identification carefully in Chapter **??**.

Denote by $\pi\colon V \to M$ the projection mapping. The tangent bundle TV, which is a real vector bundle over the total space of V, fits into the following short exact sequence of vector bundles over V:

$$0 \longrightarrow \pi^*V \longrightarrow TV \longrightarrow \pi^*TM \longrightarrow 0.$$

The inclusion of π^*V into TV is obtained by a standard construction: we regard a vector $v \in V_m$ as a tangent vector at any point w in the fiber of V over m using the formula

$$v(\phi) = \frac{df}{dv} = \lim_{h\to 0} \frac{f(w+hv) - f(w)}{h},$$

where f is any smooth function on V. The projection mapping is the differential of $\pi\colon V \to M$: it sends a tangent vector X at $v \in V$ to the tangent vector $\pi_*(X)$ at $\pi(v) \in M$ defined by $\pi_* X(g) = X(f \circ \pi)$, where g is any smooth function on M.

Choose a splitting $s\colon \pi^* TM \to TV$ of the above short exact sequence of vector bundles over TM and use it to define a submersion $p\colon T^*V \to T^*M$ by the formula

$$p(\alpha_v)(X_{\pi(v)}) = \alpha_v(s(X_{\pi(v)})),$$

where $\alpha_v \in T^*_v V$ and $X_{\pi(v)} \in T_{\pi(v)}M$. Having defined $p\colon T^*V \to T^*M$, it is easy to define a diffeomorphism $F\colon T^*V \to q^*V \otimes \mathbb{C}$, as required. Given an element $\alpha_v \in T^*_v V$, since we already noted that $V_{\pi(v)}$ is naturally included into $T_v V$, we can restrict α_v to $V_{\pi(v)}$. Remembering that $V_{\pi(v)}$ is equipped with an inner product, we can associate to α_v a unique vector $v_\alpha \in V_{\pi(v)}$ such that

$$\alpha_v(w) = \langle w, v_\alpha \rangle, \qquad \text{for all } w \in V_{\pi(v)}.$$

We can therefore define a diffeomorphism from T^*V to $q^*V \otimes \mathbb{C}$ by means of the formula

$$\begin{gathered} F\colon T^*V \longrightarrow q^*V \otimes \mathbb{C} \\ F(\alpha_v) = \big(v + i v_\alpha, p(v)\big). \end{gathered}$$

The diffeomorphism depends on the choice of splitting map $s\colon \pi^*TM \to TV$, but any two diffeomorphisms obtained from the construction are homotopic through diffeomorphisms, and in particular through proper continuous maps. It follows that the homotopy class of the Thom $*$-homomorphism

$$\phi\colon \mathcal{S} \otimes C_0(T^*M) \to C_0(T^*V, \operatorname{End}(\wedge^* q^*V \otimes \mathbb{C}))$$

obtained from Definition **??** and the diffeomorphism F is independent of the choice of s, and so we obtain a canonical Thom homomorphism

$$\phi_V\colon K(T^*M) \to K(T^*V)$$

at the level of K-theory.

12.1 Theorem. *Assume that to every smooth manifold M there is associated a homomorphism $\alpha_M\colon K(T^*M) \to \mathbb{Z}$ with the following properties:*

(i) *If M_1 is embedded as an open subset of M_2 then the diagram*

$$\begin{array}{ccc} K(T^*M_1) & \xrightarrow{\alpha_{M_1}} & \mathbb{Z} \\ \downarrow & & \downarrow \\ K(T^*M_2) & \xrightarrow[\alpha_{M_2}]{} & \mathbb{Z} \end{array}$$

commutes.

(ii) *If* V *is a smooth, real vector bundle over* M*, and if* $\phi_V \colon K(T^*M) \to K(T^*V)$ *denotes the Thom homomorphism described above, then the following diagram commutes:*

$$\begin{tikzcd} K(T^*M) \arrow[r, "\alpha_M"] \arrow[d, "\phi_V"'] & \mathbb{Z} \arrow[d, "="] \\ K(T^*V) \arrow[r, "\alpha_V"'] & \mathbb{Z}. \end{tikzcd}$$

(iii) *If* $b \in K(T^*\mathbb{R}^k)$ *is the Bott element, then* $\alpha_{\mathbb{R}^k}(b) = 1$.

Then

$$\alpha_M(x) = (-1)^{\dim(M)} \int_{T^*M} \mathrm{ch}(x) \cdot \mathrm{Todd}(TM \otimes \mathbb{C}), \qquad (12.1)$$ index-conclusion

index-axioms *for every* M *and every* $x \in K(T^*M)$.

Proof. Consider first the case of Euclidean space $\mathbb{R}^k$. The Todd class for $\mathbb{R}^k$ is equal to 1 because the tangent bundle to $\mathbb{R}^k$ is trivial. Formula (12.1) for $M = \mathbb{R}^k$ therefore amounts to the assertion that

$$\alpha_{\mathbb{R}^k}(x) = (-1)^{\dim(M)} \int_{T^*M} \mathrm{ch}(x),$$

for all $x \in K(\mathbb{R}^k)$. It follows from hypothesis (iii) above and Remark **??** that if $b \in K(\mathbb{R}^k)$ is the Bott class, then

$$\alpha_{\mathbb{R}^k}(b) = 1 = (-1)^{\dim(M)} \int_{T^*M} \mathrm{ch}(b),$$

and so (12.1) holds for b. But according to the Bott Periodicity Theorem **??**, the element b generates all of $K(T^*\mathbb{R}^k)$. Thus formula (12.1) is correct for every element of $K(T^*\mathbb{R}^k)$.

Next, it follows from axiom (i) that formula (12.1) is correct for any open subset U of $\mathbb{R}^k$. (Of course, for such a U the tangent bundle is still trivial, so that the Todd class is again 1 and the formula reads $\alpha_U(x) = (-1)^k \int_{T^*U} \mathrm{ch}(x)$.) The tubular neighborhood theorem of differential topology tells us that if M is any manifold, and if V is the normal bundle for some embedding of M into some Euclidean space $\mathbb{R}^k$, then the total space of V is diffeomorphic to an open subset of $\mathbb{R}^k$. It follows that formula (12.1) holds for the total space of V. We shall finish the proof by deducing formula (12.1) for M using hypothesis (ii).

It follows from that hypothesis that if $x \in K(T^*M)$, then

$$\alpha_M(x) = \alpha_V(\phi_{T^*V}(x)) = (-1)^k \int_{T^*V} \mathrm{ch}(\phi_{T^*V}(x)).$$

If to the right-hand side of this equation we apply Proposition **??**, then we obtain the formula

$$\alpha_M(x) = (-1)^k \int_{T^*M} \mathrm{ch}(x)\tau(V \otimes \mathbb{C}),$$

where τ is the multiplicative characteristic class corresponding to the power series $(1 - e^x)/x$. (We are using here the fact that T^*V, as a complex vector bundle over T^*M, is isomorphic to $\pi^*(V \otimes \mathbb{C})$, as was indicated in the remarks preceding the statement of the theorem.) At this point, let us insert the result of Exercise **??**, which relates the τ-class to the Todd class. The result is that

$$\alpha_M(x) = (-1)^{\dim(M)} \int_{T^*M} \operatorname{ch}(x) \frac{1}{\operatorname{Todd}(V \otimes \mathbb{C})}.$$

To finish the proof, note that the direct sum $V \oplus TM$ is isomorphic to a trivial bundle (of dimension k). Thus

$$\operatorname{Todd}(V \otimes \mathbb{C}) \cdot \operatorname{Todd}(TM \otimes \mathbb{C}) = 1,$$

since the Todd class is multiplicative and since the Todd class of a trivial bundle is 1. Substituting this into the previously displayed equation we obtain the result. □

12.2 K-THEORY REFORMULATION OF THE INDEX THEOREM

reformulation-section

Underlying the cohomological form of the Atiyah-Singer index theorem proved in the preceeding chapters is a K-theory form that is more basic and more readily amenable to generalization. It is based on the following construction, which is really nothing more than the Thom homomorphism studied in Chapter **??**.

wrong-way-inj-def

12.2 Definition. Let $i\colon M \to N$ be an embedding of smooth manifolds. Define an associated homomorphism

$$i_!\colon K(T^*M) \to K(T^*N)$$

as follows. Let E be the normal bundle of T^*M in T^*N. Observe that $E \cong \pi^*V_{\mathbb{C}}$, where V is the normal bundle of M in N and where $\pi\colon T^*M \to M$ is the standard projection (see Section 12.1). In particular, E has the structure of a complex vector bundle over T^*M. Let $f\colon U \to E$ be a diffeomorphism from a tubular neighborhood U of $T^*M \subseteq T^*N$ to E, as provided by the tubular neighborhood theorem. Define $i_!$ to be the map which fits into the commutative diagram

$$\begin{array}{ccc} K(T^*M) & \xrightarrow{i_!} & K(T^*N) \\ \phi \big\downarrow & & \big\uparrow j \\ K(E) & \xrightarrow[f^*]{} & K(U), \end{array}$$

where ϕ is the Thom homomorphism and $j\colon U \to T^*N$ is the inclusion map.

wrong-way-htpy-lemma

12.3 Lemma. *The map* $i_!\colon K(T^*M) \to K(T^*N)$ *associated to an embedding* $i\colon M \to N$ *depends only on the homotopy class of* i *through embeddings.*

Proof. This follows easily from the homotopy invariance of K-theory. □

A key feature of the construction in Definition 12.2 is its functoriality:

embedding-funct-lemma

12.4 Lemma. *If* $i\colon M \to N$ *and* $j\colon N \to P$ *are embeddings of smooth manifolds, then*

$$j_! \circ i_! = (j \circ i)_! \colon K(T^*M) \to K(T^*P).$$

Proof (sketch). If U is an open set in N containing the image of M, and if Z is an open set in P containing the image of $U \subseteq N$, then the diagram

$$\begin{tikzcd} K(T^*M) \arrow[r, "i_!"] & K(T^*U) \arrow[r, "j_!"] \arrow[d] & K(T^*Z) \arrow[d] \\ & K(T^*N) \arrow[r, "j_!"'] & K(T^*P) \end{tikzcd}$$

commutes, where the vertical maps are induced by inclusions of open sets. Thanks to this observation and the tubular neighborhood theorem, the proof of the lemma reduces to the case where N is the total space of a smooth real vector bundle V over M and P is the total space of the pullback to V of a second smooth real vector bundle W over M. Here the lemma amounts to the assertion that the composition of Thom homomorphisms

$$K(T^*M) \longrightarrow K(E) \longrightarrow K(E \oplus F),$$

where E and F are the complexifications of V and W (pulled back to T^*M) is the Thom homomorphism for $E \oplus F$. This property of the Thom homomorphism, Lemma **??**, was a key step in our earlier proof of the Index Theorem. $\square$

k-top-index-def

12.5 Definition. Let M be a smooth manifold without boundary. The *topological index map* $\operatorname{Ind}^t\colon K(T^*M) \to \mathbb{Z}$ is the homomorphism that fits into the commutative diagram

$$\begin{tikzcd} K(T^*M) \arrow[r, "i_!"] \arrow[d, "\operatorname{Ind}^t"'] & K(T^*\mathbb{R}^k) \\ \mathbb{Z} \arrow[r, "="'] & K(pt), \arrow[u, "\cong"', "j_!"] \end{tikzcd}$$

where $i\colon M \to \mathbb{R}^k$ is an embedding of M into a Euclidean space and $j\colon pt \to \mathbb{R}^k$ is the inclusion of a point into $\mathbb{R}^k$.

It follows from Lemmas 12.3 and 12.4 that the topological index map is independent of the embedding $i\colon M \to \mathbb{R}^k$ used to define it. This is because any two embeddings of M into Euclidean spaces become homotopic through embeddings after a further embedding into a sufficiently larger Euclidean space.

The most important feature of the maps $i_!\colon K(T^*M) \to K(T^*N)$ is their compatibility with the index homomorphisms that we defined in Chapter 10. We shall denote by $\operatorname{Ind}^a\colon K(T^*M) \to \mathbb{Z}$ the analytic index homomorphism which takes the symbol class $[\sigma_D]$ of an elliptic operator to its index $\operatorname{Ind}(D)$. (This homomorphism was introduced in Chapter **??** and defined in Chapter 10, and in both places it was denoted by α.) In view of our definition of $i_!$, the following result expressing the compatibility between $i_!$ and the analytic index is just a reformulation of Theorem **??**.

wrong-way-compatibility-thm

12.6 Theorem. *If* $i\colon M \to N$ *is any embedding of smooth manifolds, then the diagram*

$$\begin{array}{ccc} K(T^*M) & \xrightarrow{i_!} & K(T^*N) \\ {\scriptstyle \mathrm{Ind}^a}\downarrow & & \downarrow{\scriptstyle \mathrm{Ind}^a} \\ \mathbb{Z} & \xrightarrow[=]{} & \mathbb{Z} \end{array}$$

is commutative. □

The following is Atiyah and Singer's K-theoretic formulation of their index theorem.

as-original

12.7 Corollary. *If* M *is any smooth manifold, then* $\mathrm{Ind}^a = \mathrm{Ind}^t\colon K(T^*M) \to \mathbb{Z}$.

Proof. We need only apply Theorem 12.6 and use the fact that the analytic index $\mathrm{Ind}_a\colon K(T^*\mathbb{R}^k) \to \mathbb{Z}$ is inverse to $j_!\colon K(pt) \to K(T^*\mathbb{R}^k)$ (for which, see Chapter **??**). □

The cohomological version of the index theorem that we gave in Chapter **??** follows from Theorem 12.6 by a computation of the topological index using the approach followed in Section 12.1.

12.3 COEFFICIENTS AND THE INDEX MAPS

mi-fo-sec

We can generalize the constructions of the previous section by incorporating a *coefficient* C^**-algebra*. Throughout this section, let A be an arbitrary C^*-algebra. (As we shall see in later sections of this chapter, specific geometric situations will dictate appropriate choices for A.) We are going to incorporate A in a very simple way into the construction of the analytic and topological index maps given in Section 12.2.

Let E be a complex vector bundle over a locally compact space X. The Thom homomorphism $\psi\colon K(X) \to K(E)$ was defined in Section **??** using a graded $*$-homomorphism

$$\phi\colon \mathcal{S} \otimes C_0(X) \to C_0(E, \mathrm{End}(\wedge^* E)).$$

By tensoring this $*$-homomorphism with the identity map on A, we obtain a graded $*$-homomorphism

$$\phi\colon \mathcal{S} \otimes C_0(X) \otimes A \to C_0(E, \mathrm{End}(\wedge^* E)) \otimes A$$

and hence a K-theory map from $K(C_0(X) \otimes A)$ to $K(C_0(E) \otimes A)$. Using this generalization of the Thom homomorphism we can define maps

$$i_!\colon K(C_0(T^*M) \otimes A) \to K(C_0(T^*N) \otimes A)$$

associated to embeddings $i\colon M \to N$ by following the method outlined in Section 12.2.

mf-topological-index

12.8 Definition. Let M be a smooth manifold and let A be a C^*-algebra. The *topological A-index map*

$$\mathrm{Ind}^t_A\colon K(C_0(T^*M)\otimes A)\to K(A)$$

is defined to be the map which fits into the commutative diagram

$$\begin{tikzcd} K(C_0(T^*M)\otimes A) \arrow[r, "i_!"] \arrow[d, "\mathrm{Ind}^t_A"'] & K(C_0(T^*\mathbb{R}^n)\otimes A) \\ K(A) \arrow[r, "="'] & K(A) \arrow[u, "\cong"', "j_!"] \end{tikzcd}$$

where $i\colon M\to\mathbb{R}^n$ is an embedding of M into a Euclidean space and $j\colon \mathrm{pt}\to\mathbb{R}^n$ is the inclusion of a point into $\mathbb{R}^n$.

The analytic index map $\mathrm{Ind}^a\colon K(T^*M)\to\mathbb{Z}$ was defined in this book using an asymptotic morphism

$$\alpha\colon C_0(T^*M)\rightsquigarrow \mathcal{K}(L^2(M)).$$

Tensoring this with the identity map on A produces an asymptotic morphism

$$\alpha\colon C_0(T^*M)\otimes A\rightsquigarrow \mathcal{K}(L^2(M))\otimes A$$

(compare Section **??**).

mf-analytic-index

12.9 Definition. Let M be a smooth manifold. The *analytic A-index map*

$$\mathrm{Ind}^a_A\colon K(C_0(T^*M)\otimes A)\to K(A)$$

is the K-theory map associated to the above asymptotic morphism.

The following generalization of the index theorem is due to Mischenko and Fomenko.

mis-fom-theorem

12.10 Theorem. *If M is any smooth manifold and A is any C^*-algebra, then topological and analytic A-index maps*

$$\mathrm{Ind}^t_A, \mathrm{Ind}^a_A\colon K(C_0(T^*M)\otimes A)\to K(A)$$

are equal to one another. □

The original K-theory form of the index theorem (our Corollary 12.7) is the case $A=\mathbb{C}$. Because we have made use of C^*-algebraic techniques throughout this book, no additional difficulties arise in incorporating the auxiliary algebra A into all the arguments of Chapter **??**, and thus proving the stronger Mischenko-Fomenko theorem also.

12.4 TRACES AND GENERALIZED DIMENSIONS

tra-sec

In order to apply the Mischenko-Fomenko theorem to a specific geometric problem we must do three things:

(a) Choose a suitable coefficient algebra A.

(b) Describe a procedure for generating elements of $K(C_0(T^*M) \otimes A)$.

(c) Calculate the group $K(A)$, or at least describe a nontrivial homomorphism from $K(A)$ to some more familiar group.

Later in this chapter we shall give two related examples illustrating (a) and (b). In the current section we shall focus on a procedure which allows one to find a homomorphism, as in (c), using a *trace* on the C^*-algebra A.

ft-def

12.11 Definition. A *trace* on an algebra A is a linear functional $\sigma\colon A \to \mathbb{C}$ such that $\sigma(ab) = \sigma(ba)$, for all $a, b \in A$.

Traces give rise to "dimension homomorphisms" from $K(A)$ to $\mathbb{C}$, and thus allow us to obtain numerical indices from the $K(A)$-valued indices provided by the Mischenko-Fomenko construction.

t-prop

12.12 Proposition. *Let σ be a trace on a unital algebra A and let p be an idempotent in $M_n(A)$. The sum $\sigma(p) = \sum_{i=1}^n \sigma(p_{ii})$ depends only on the K-theory class represented by p and the correspondence which maps p to $\sigma(p)$ determines a group homomorphism from $K(A)$ to $\mathbb{C}$.*

Proof. The basic point is that if $p \in M_n(A)$ and $q \in M_k(A)$ are idempotents, and if $p = uv$ and $q = vu$ for some matrices $u \in M_{n,k}(A)$ and $v \in M_{k,n}(A)$, then

$$\sigma(p) = \sigma(uv) = \sum_{i,j} \sigma(u_{ij}v_{ji}) = \sum_{i,j} \sigma(v_{ji}u_{ij}) = \sigma(vu) = \sigma(q).$$

The remaining details of the proof are left to the reader. □

We shall call the homomorphism $K(A)$ from $\mathbb{C}$ that is defined by this proposition the *dimension homomorphism* associated to the trace σ. Usually we shall use the same notation for the trace and for the associated dimension homomorphism.

c-star-gamma

12.13 Example. Let Γ be a discrete group and let $\{ \delta_g : g \in \Gamma \}$ be the canonical orthonormal basis of the sequence space $\ell^2(\Gamma)$. The *reduced group C^*-algebra* $C^*_\lambda(\Gamma)$ is the C^*-algebra of operators on $\ell^2(\Gamma)$ generated by the unitaries

$$u_\gamma\colon \ell^2(\Gamma) \to \ell^2(\Gamma)$$
$$u_\gamma\colon \delta_\eta \mapsto \delta_{\gamma\eta}.$$

(Alternatively, if we think of Γ as a zero-dimensional smooth groupoid, then $C^*_\lambda(\Gamma)$ is its groupoid C^*-algebra.) The *canonical trace* on $C^*_\lambda(\Gamma)$ is the trace functional $\sigma\colon C^*_\lambda(\Gamma) \to \mathbb{C}$ defined by $\sigma(T) = \langle T\delta_e, \delta_e \rangle$. As we shall see, the canonical trace on $C^*_\lambda(\Gamma)$ gives rise to a very important and interesting dimension function from the perspective of index theory.

12.14 Remark. The canonical trace is *positive* in the sense that $\sigma(T^*T) \geq 0$, for every T. If σ is a positive trace on a C^*-algebra, then the dimension homomorphism actually maps $K(A)$ into $\mathbb{R} \subseteq \mathbb{C}$. This is because every K-theory class is representable by a projection p (that is, by a self-adjoint idempotent), and for such p we have $\sigma(p) = \sigma(p^2) = \sigma(p^*p) \geq 0$.

If A is an algebra *without* unit and if σ is a trace on A, then the formula

$$\sigma(a + \lambda 1) = \sigma(a)$$

extends the trace σ to the algebra $\widetilde{A}$ obtained by adjoining a unit to A. In view of the fact that K-theory for nonunital algebras may defined by first adjoining a unit (compare Exercise **??**), it follows that the approach outlined in Proposition 12.12 determines a trace homomorphism

$$\sigma\colon K(A) \to \mathbb{C}$$

whether or not A is unital.

Non-unital examples will be very important to us. However, an unavoidable complication of the theory of traces for non-unital C^*-algebras is that many interesting examples are *not* defined on the whole of a C^*-algebra A, but only on some dense subset. This phenomenon is apparent even in the commutative case. For example if $A = C_0(\mathbb{R})$, then the most natural example of a trace is the functional

$$f \mapsto \int f(x)\, d\mu(x),$$

where μ is Lebesgue measure; but this functional is defined only on the dense subspace $L^1(\mathbb{R}, \mu) \cap C_0(\mathbb{R})$. In fact, a systematic approach to the problem of unbounded traces involves developing a sort of noncommutative integration theory. We shall not do that here, but simply cite some basic results.

tr-weight-def

12.15 Definition. Let A be a C^*-algebra and denote by A^+ the set of positive elements in A. A *tracial weight* on A is a function $\tau\colon A^+ \to [0, \infty]$ such that

$$\tau(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 \tau(a_1) + \lambda_2 \tau(a_2)$$

for all $\lambda_1, \lambda_2 \geq 0$ and all $a_1, a_2 \in A^+$, and such that $\tau(aa^*) = \tau(a^*a)$ for all $a \in A$. A tracial weight τ is *(lower) semicontinuous* if

$$\tau(\lim_{n\to\infty} a_n) \leq \liminf_{n\to\infty} \tau(a_n)$$

for all norm-convergent sequences $\{a_n\}$ in A^+. It is *densely defined* if the set $\{a \in A^+ : \tau(a) < \infty\}$ is dense in A^+.

Pursuing our analogy with integration theory, lower semicontinuity for a tracial weight corresponds to Fatou's lemma.

Let τ be a densely defined, semicontinuous tracial weight on a C^*-algebra A. It may be shown that the linear span in A of the positive elements a for which $\tau(a) < \infty$ is a dense hereditary[1] ideal I_τ in A, that τ extends by linearity to a trace on this ideal, and that the inclusion of I_τ into A induces an isomorphism on K-theory groups.

Because τ defines a dimension homomorphism $\tau\colon K(I_\tau) \to \mathbb{R}$, and $K(I_\tau)$ is canonically isomorphic to $K(A)$, we conclude that:

dd-prop

12.16 Proposition. *A densely defined semicontinuous tracial weight on a* C^*-*algebra* A *defines a dimension homomorphism* $K(A) \to \mathbb{R}$. □

[1] An ideal I in a C^*-algebra is *hereditary* if $0 \leq a \leq a'$ and $a' \in I$ imply $a \in I$.

12.17 Example. Let $A = \mathcal{K}$ and let τ be the usual operator trace defined by $\tau(T) = \sum \langle Te_i, e_i\rangle$ for $T \geq 0$, where $\{e_i\}$ is an orthonormal basis. This is a densely defined, semicontinuous tracial weight and the associated ideal I_τ is the ideal of *trace class operators*. The dimension homomorphism associated to τ assigns the rank of p to any finite rank projection p. It implements the canonical isomorphism from $K(\mathcal{K})$ to $\mathbb{Z}$.

As we saw in Chapter **??**, important elements of the group $K(A)$ can be constructed from graded $*$-homomorphisms $\mathcal{S} \to M_2(A)$, where $M_2(A)$ is regarded as a *graded* C^*-algebra. In particular the ordinary index of an elliptic operator arises in this way (with $A = \mathcal{K}$) and we shall see that various generalized indices can also be constructed in the same way (with other choices of A). We are going to compute the τ-dimension of a K-theory class arising in this way.

min-ideal-lemma **12.18 Lemma.** *Let $\psi\colon C_0(\mathbb{R}) \to A$ be a $*$-homomorphism. Let I be a dense hereditary ideal in A. If h is a compactly supported continuous function on $\mathbb{R}$, then $\psi(h) \in I$.*

Proof. We may assume that $h \geq 0$. Let $a = \psi(h^{1/2})$. There is a positive element $b \in A$ such that $ab = a$. For example if $g \in C_0(\mathbb{R})$ is nonnegative and equal to 1 on the support of h, then $b = \psi(g)$ will do the job. By density, there is a positive element $c \in I$ such that $\|b - c\| < \frac{1}{2}$. Let us write

$$a^2 = aba = a(b - c)a + aca.$$

From the inequality $\|b - c\| < \frac{1}{2}$ it follows that

$$a(b - c)a \leq \tfrac{1}{2}a^2.$$

As a result, $a^2 \leq 2aca$. But $aca \in I$ since I is an ideal, so $a^2 = \psi(h) \in I$ since I is hereditary. $\square$

index-trace-prop **12.19 Proposition.** *Let A be a C^*-algebra, and let τ be a densely defined, semicontinuous tracial weight on A. Let $\psi\colon \mathcal{S} \to M_2(A)$ be a graded $*$-homomorphism defining a K-theory class $[\psi] \in K(A)$, and let $\varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ be the grading operator in $M_2(\tilde{A})$. Then the τ-dimension of the K-theory class determined by ψ is equal to*

$$\tau(\varepsilon\psi(h)),$$

where h is any compactly supported function on $\mathbb{R}$ which is even and has $h(0) = 1$.

Notice that $\psi(h) \in I_\tau$ by Lemma 12.18, so the formula in the proposition makes sense.

Proof (sketch). Let U be the Cayley transform associated to ψ (recall that U is the unitary element of $M_2(\tilde{A})$ obtained by applying the $*$-homomorphism ψ to the function $x \mapsto (x + i)/(x - i)$). By construction (**??**), the K-theory class $[\psi]$ is represented by the difference of projections

$$P_1 = \tfrac{1}{2}(U\varepsilon + I) \quad \text{and} \quad P_0 = \tfrac{1}{2}(\varepsilon + I).$$

A short calculation reveals that we may write P_1 as

$$P_1 = \frac{1}{2}\Big(\big(1 - 2\psi(f) + 2i\psi(f^{1/2})\psi(g)\big)\varepsilon + 1\Big),$$

where $f(x) = (x^2+1)^{-1}$ and where g is the real-valued, continuous, odd function on $\mathbb{R}$ such that

$$g^2 = 1 - f \quad \text{and} \quad \lim_{x\to+\infty} g(x) = 1$$

(thus $g(x) = x(x^2+1)^{-1/2}$). Now if f is *any* even function in $C_0(\mathbb{R})$ such that $f(0) = 1$ and $1 \geq f \geq 0$, and if g is defined in terms of f in the manner displayed above, then the formula for P_1 displayed above defines a projection. If we take f to be compactly supported, then we obtain a projection Q_1 in the algebra I_τ (with ε adjoined). Viewed as a projection in A it is equivalent to P_1; in fact P_1 and Q_1 are path-connected through projections via the straight-line path connecting the compactly supported function f to $(x^2+1)^{-1}$. It follows that $[\psi]$ is represented by the formal difference $[P_0] - [Q_1]$ of projections in I_τ (with ε adjoined). But $\tau(P_0) = 0$ and $\tau(Q_1) = \tau(\varepsilon\psi(f))$, so the result follows. $\square$

12.5 INDEX THEORY FOR COVERING SPACES

higherind-sec

In this section and the next we shall use the ideas that we have developed so far to investigate index theory for covering spaces. The following definition underlies Example 10.7.

12.20 Definition. Let Γ be a discrete group. A (left) action of Γ on a topological space if W is a *principal action* if each $w \in W$ has neighborhood $U \subseteq W$ such that $\gamma U \cap U = \emptyset$ for each element $\gamma \in \Gamma$ other than the element $\gamma = e$.

12.21 Definition. Let Z be a topological space. A *principal* Γ*-space over* Z is a topological space W equipped with a principal action of Γ, together with a homeomorphism from the quotient W/Γ to Z.

We have stated the definitions for arbitrary topological spaces because in Chapter 14, when we discuss classifying spaces, it will be appropriate to work in roughly this generality. But in the current chapter we shall be exclusively concerned with smooth manifolds, and in this context we shall always assume that Γ acts by diffeomorphisms. Notice that a smooth manifold structure on a principal Γ-space W therefore determines a compatible smooth manifold structure on the quotient W/Γ, and *vice versa*.

uc-example

12.22 Example. Suppose that M is a connected smooth manifold whose fundamental group is Γ. The universal cover of M, equipped with the action of Γ by deck transformations, is a principal Γ-space over M.

Let W be a principal Γ-space over a smooth closed manifold M and let D be a linear partial differential operator on M that acts on the sections of a smooth complex vector bundle S over M. The pull-back of the bundle S to W is in a natural way a Γ-equivariant complex vector bundle on W. The operator D lifts in a natural way to a Γ-equivariant partial differential operator on W, acting on sections of the pull-back of S. We shall use the notation D_W for this lifted operator.

Suppose now that S is given the structure of a graded hermitian vector bundle, that M (and hence W) is equipped with a smooth measure, and that D is an operator of the sort we have studied in earlier chapters—a formally self-adjoint, odd-graded, elliptic first-order partial differential operator on M. We noted in Theorem 10.32 that D_W is essentially self-adjoint, and we can therefore use the functional calculus to form the operators $f(D_W)$, for $f \in \mathcal{S} = C_0(\mathbb{R})$. Our first aim is to follow the approach we took in Chapter **??** to define an analytic index of D_W in the K-theory group of a C^*-algebra of operators that contains all the operators $f(D_W)$.

12.23 Definition. If W and Z are topological spaces equipped with Γ-actions, then we shall denote by $W \times_\Gamma Z$ the quotient of $W \times Z$ by the product action of Γ.

12.24 Definition. Let W be a principal Γ-manifold. An operator K on $L^2(W)$ is a *Γ-equivariant, Γ-compactly supported smoothing operator* if there exists a smooth Γ-invariant complex-valued function k on $W \times W$ that is compactly supported as a function on $W \times_\Gamma W$, such that

$$Kf(w) = \int_W k(w, v) f(v)\, dv$$

for all $f \in L^2(W)$. All such operators are necessarily bounded, and collectively they constitute a $*$-algebra of operators on $L^2(W)$. We shall denote by $\mathcal{K}(\Gamma)$ the C^*-algebra generated by the Γ-equivariant, Γ-compactly supported smoothing operators on $L^2(W)$.

12.25 Remark. In fact, we need a small modification of this definition which incorporates the complex vector bundle S on whose sections D_W acts. Denote by $S \otimes S^*$ the complex vector bundle over $W \times W$ whose fiber over (w_1, w_2) is $S_{w_1} \otimes S^*_{w_2}$. A Γ-equivariant, Γ-compactly supported smoothing operator on $L^2(W, S)$ is then an operator of the same form as in the display above, but where k is a smooth, equivariant, Γ-compactly supported section of $S \otimes S^*$. We shall suppress S from our notation and denote simply by $\mathcal{K}(\Gamma)$ the C^*-algebra generated by this $*$-algebra.

equiv-ell-reg-prop

12.26 Proposition. *If D is a formally self-adjoint elliptic first-order operator on M, and if $f \in \mathcal{S}$, then $f(D_W) \in \mathcal{K}(\Gamma)$.*

Proof. This follows from Theorem 10.37 and the fact that $\mathcal{K}(\Gamma)$ is the C^*-algebra of the groupoid $W \times_\Gamma W$ described in Example 10.7 (with coefficients in S, as detailed in Remark 11.18). □

Thanks to Proposition 12.26 the correspondence $f \mapsto f(D_W)$ defines a grading-preserving $*$-homomorphism from the graded C^*-algebra $\mathcal{S}$ into $\mathcal{K}(\Gamma)$. Thus, using the construction in Section **??**, we obtain an "index class" in the K-theory of $\mathcal{K}(\Gamma)$.

balanced-prod-morita-lemma

12.27 Lemma. *The C^*-algebra $\mathcal{K}(\Gamma)$ is isomorphic to the tensor product algebra $C^*_\lambda(\Gamma) \otimes \mathcal{K}(L^2(M, S))$. Hence $K(\mathcal{K}(\Gamma)) \cong K(C^*_\lambda(\Gamma))$.*

Proof. Let $F \subseteq W$ be a bounded and measurable fundamental domain for the action of Γ. Then the unitary isomorphism

$$U \colon L^2(W, S) \to \ell^2(\Gamma) \otimes L^2(F, S)$$

defined by $(Uf)(\gamma, w) = f(\gamma^{-1}w)$ conjugates $\mathcal{K}(\Gamma)$ to $C^*_\lambda(\Gamma) \otimes \mathcal{K}(L^2(F, S))$. □

12.28 Remark. The choice of fundamental domain in the proof of Lemma 12.27 does not affect the associated isomorphism $K(\mathcal{K}(\Gamma)) \cong K(C^*_\lambda(\Gamma))$.

eq-ind-op-def **12.29 Definition.** Let D be a formally self-adjoint, odd-graded, elliptic first-order partial differential operator on a closed manifold M and let W be a principal Γ-space over M. Denote by $\operatorname{Ind}_\Gamma(D) \in K(C^*_\lambda(\Gamma))$ the index class associated to D_W by Proposition 12.26 and Lemma 12.27.

Following the pattern of the classical index theorem, we are going to identify this index class with the image of a certain element of the group $K(C_0(T^*M) \otimes C^*_\lambda(\Gamma))$ under the Mischenko-Fomenko analytic index map

$$\operatorname{Ind}^a_{C^*_\lambda(\Gamma)} \colon K(C_0(T^*M) \otimes C^*_\lambda(\Gamma)) \to K(C^*_\lambda(\Gamma)).$$

The Mishchenko-Fomenko index theorem will then give us an index theorem for $\operatorname{Ind}_\Gamma(D)$.

For the purposes of the following definition, let us recall that $C^*_\lambda(\Gamma)$ is generated by the group of unitary operators $\{u_\gamma : \gamma \in \Gamma\}$ described in Definition 12.13.

mishch-mod-def **12.30 Definition.** Let M be a smooth manifold (or indeed any locally compact Hausdorff space) and let W be a principal Γ-manifold over M. Denote by $\mathcal{E}_W$ the space of all continuous functions from W into $C^*_\lambda(\Gamma)$ such that

(i) $f(\gamma w) = u_\gamma f(w)$, for all $w \in W$ and all $\gamma \in \Gamma$; and

(ii) the function $w \mapsto \|f(w)\|$ belongs to $C_0(M)$.

12.31 Remark. Thanks to item (i) in the definition above, $\|f(w)\| = \|f(\gamma w)\|$ for all $\gamma \in \Gamma$ and all $w \in W$. As a result, the correspondence $w \mapsto \|f(w)\|$ that appears in item (ii) of the definition can be regarded as a function on the quotient space $W/\Gamma = M$, as implied in the definition.

The space $\mathcal{E}_W$ may be given the structure of a Hilbert module over the tensor product C^*-algebra $C_0(M) \otimes C^*_\lambda(\Gamma)$, as follows. Regarding the tensor product as the C^*-algebra of continuous functions from M into $C^*_\lambda(\Gamma)$ that vanish at infinity, and regarding a function on M as the same thing as a Γ-periodic function on W, the right action of $C_0(M) \otimes C^*_\lambda(\Gamma)$ on $\mathcal{E}_W$ is given by

$$(f_1 \cdot f)(w) = f_1(w) f(w),$$

while the inner product is given by the formula

$$\langle f_1, f_2 \rangle(w) = f_1(w)^* f_2(w).$$

mish-fom-bundle-lemma **12.32 Lemma.** *The action of* $C_0(M)$ *on* $\mathcal{E}_W$ *as Hilbert-module endomorphisms given by formula* $(f \cdot f_1)(w) = f(w) f_1(w)$ *determines a* $*$*-homomorphism from* $C_0(M)$ *into the* C^**-algebra of compact operators on the Hilbert module* $\mathcal{E}_W$. □

Proof. It suffices to prove that if $f \in C_0(M)$ is compactly supported within the image of an open set $U \subseteq W$ such that $\gamma U \cap U = \emptyset$, for $\gamma \neq e$, then the operator $f_1 \mapsto f \cdot f_1$ is compact. Define $f' \in \mathcal{E}_W$ by

$$\begin{cases} f'(\gamma v) = u_\gamma f(v) & \text{if } v \in U, \gamma \in \Gamma \\ f'(w) = 0 & \text{if } w \notin U\gamma \text{ for any } \gamma. \end{cases}$$

Let $g \in C_0(M)$ be compactly supported within the image of U and such that $gf = f$ and define $g' \in \mathcal{E}_W$ same way we defined f'. Then $f \cdot f_1 = f'\langle g', f_1\rangle$. □

mishch-bundle-remar

12.33 Remark. The Hilbert module introduced in Definition 12.30 can be viewed more geometrically, as follows. Let Γ act on $C^*_\lambda(\Gamma)$ (which we shall view as a right module over itself) via left-multiplication, using the unitaries u_γ. The quotient space

$$\mathcal{L}_W = W \times_\Gamma C^*_\lambda(\Gamma)$$

is then a bundle over M whose fibers are Hilbert modules over $C^*_\lambda(\Gamma)$. It is called the *Mishchenko line bundle* over M, and $\mathcal{E}_W$ is its Hilbert module of continuous sections that vanish at infinity.

mf-bundle-map-def

12.34 Definition. Let W be a principal Γ-manifold over a manifold M. Denote by

$$\nu_W \colon K(C_0(M)) \longrightarrow K(C_0(M) \otimes C^*_\lambda(\Gamma))$$

the K-theory map determined as in Section **??** by the $*$-homomorphism from $C_0(M)$ into $\mathcal{K}(\mathcal{E}_W)$ given in Lemma 12.32.

12.35 Remark. The map ν_W can be viewed more geometrically as follows. If we assume for simplicity that M is compact, then $K(C(M))$ is generated by the classes of vector bundles over M. A vector bundle E over M may be tensored with the Mishchenko line bundle $\mathcal{L}_W$ so as to obtain a bundle whose fibers are finitely generated and projective (indeed free) modules over $C^*_\lambda(\Gamma)$. The map ν_W associates to E this tensor product bundle, or equivalently its finitely generated and projective module of continuous sections. Thus ν_W is the operation of "tensor product with the Mishchenko line bundle".

Actually we are most interested in the case where M is replaced by T^*M, and W by T^*W (which is of course a principal Γ-manifold over T^*M). We obtain in this case a map

$$\nu_{T^*W} \colon K(C_0(T^*M)) \longrightarrow K(C_0(T^*M) \otimes C^*_\lambda(\Gamma)).$$

This is the first part of the index construction. It may be interpreted as associating to the symbol of an elliptic operator on M the symbol of a new operator that acts not on sections of an ordinary vector bundle S over M, but on sections of the tensor product of S with the Mishchenko line bundle.

12.36 Definition. Let M be a smooth manifold and let W be a principal Γ-manifold over M. The *topological Γ-index map*

$$\mathrm{Ind}^t_\Gamma \colon K(T^*M) \to K(C^*_\lambda(\Gamma))$$

associated to the principal Γ-space W is the composition

$$K(C_0(T^*M)) \xrightarrow{\nu_{T^*W}} K(C_0(T^*M) \otimes C^*_\lambda(\Gamma)) \xrightarrow{\mathrm{Ind}^t} K(C^*_\lambda(\Gamma)),$$

where ν_{T^*W} is the map of Definition 12.34 and Ind_t is the topological index map of Definition 12.8 (we have omitted the subscript $A = C^*_\lambda(\Gamma)$ to streamline the notation). Similarly, the *analytic Γ-index map*

$$\mathrm{Ind}^a_\Gamma \colon K(T^*M) \to K(C^*_\lambda(\Gamma))$$

associated to the principal Γ-space W is the composition

$$K(C_0(T^*M)) \xrightarrow{\nu_{T^*W}} K(C_0(T^*M) \otimes C^*_\lambda(\Gamma)) \xrightarrow{\mathrm{Ind}^a} K(C^*_\lambda(\Gamma)),$$

where Ind^a is the analytic index map of Definition 12.9.

Of course, it follows immediately from the definitions and Theorem 12.10 that the topological and analytic Γ-indices agree:

g-index-theorem **12.37 Theorem.** *Let W be a principal Γ-manifold over a smooth manifold M. The topological and analytic Γ-index maps*

$$\mathrm{Ind}^t_\Gamma, \mathrm{Ind}^a_\Gamma \colon K(T^*M) \to K(C^*_\lambda(\Gamma))$$

are equal to one another. □

The link with the analytic index of Definition 12.29 is provided by the next proposition. It states that the analytic Γ-index map, like the ordinary analytic index map, sends the symbol of an elliptic operator D on a closed manifold to the index of D:

g-analytic-index-prop **12.38 Proposition.** *Let W be a principal Γ-space over a smooth closed manifold M and let D be a first-order, odd-graded, elliptic self-adjoint partial differential operator on M. The analytic Γ-index map* $\mathrm{Ind}^a_\Gamma \colon K(T^*M) \to K(C^*_\lambda(\Gamma))$ *takes the symbol class of* D *to the element* $\mathrm{Ind}_\Gamma(D)$ *described in Definition 12.29.* □

This may be proved in the same general way that we proved Proposition 10.23, using an approximation of D_W near any given orbit of Γ by a constant coefficient operator on $\mathbb{R}^n \times \Gamma$.

12.6 TRACES AND THE L^2-INDEX THEOREM

In Chapter 14 we shall investigate the significance of the K-theoretic index theorem for covering spaces that we have just proved, and this will lead us straight into some central unsolved problems in noncommutative geometry. But first we shall use traces and dimension functions to derive a numerical index theorem for covering spaces.

Let us retain the notation of the previous section. We are going to compute the *image* of $\mathrm{Ind}^a_\Gamma(D) \in K(C^*_\lambda(\Gamma))$ under the dimension homomorphism induced from the canonical trace on $C^*_\lambda(\Gamma)$. In order to express the answer in its simplest form, it is convenient to introduce the notion of *Γ-dimension.*

gamma-dim-def

12.39 Definition. Let W be a principal Γ-manifold and let $\mathcal{M}(\Gamma)$ be the algebra of all bounded, Γ-equivariant operators on $L^2(W, S)$. Let $P\colon L^2(W,S) \to L^2(W,S)$ be the orthogonal projection onto the L^2-space of a fundamental domain in W. If T is a positive element of $\mathcal{M}$, define

$$\tau(T) = \operatorname{Tr}(PTP),$$

where the functional Tr on the right hand side is the usual operator trace: the sum of the diagonal elements in any matrix representation of PTP (this sum may be ∞).

12.40 Lemma. *The quantity $\tau(T)$ does not depend on the choice of fundamental domain made in Definition 12.39. Moreover if S is any operator in $\mathcal{M}(\Gamma)$, then $\tau(S^*S) = \tau(SS^*)$.*

Proof. We shall prove the second assertion first. Let $P_\gamma = U_\gamma P U_\gamma^*$, where $\gamma \in \Gamma$ and $U_\gamma\colon L^2(W) \to L^2(W)$ is the unitary operator associated to the action of γ on W. Then $\sum_\gamma P_\gamma = I$ (with convergence in the strong operator topology) and so using the definition of τ and basic properties of the operator trace we find that

$$\tau(S^*S) = \operatorname{Tr}(PS^*SP) = \sum_\gamma \operatorname{Tr}(PS^*P_\gamma SP) = \sum_\gamma \operatorname{Tr}(P_\gamma SPS^*P_\gamma).$$

But since S is Γ-equivariant it commutes with the unitary operators U_γ, so that

$$U_\gamma^*(P_\gamma SPS^*P_\gamma)U_\gamma = PSP_{\gamma^{-1}}S^*P.$$

Hence

$$\tau(S^*S) = \sum_\gamma \operatorname{Tr}(P_\gamma SPS^*P_\gamma) = \sum_\gamma \operatorname{Tr}(PSP_{\gamma^{-1}}S^*P) = \tau(SS^*).$$

To prove the first assertion, note that if Q is the projection associated to a second fundamental domain, then there is a unitary operator $U \in \mathcal{M}(\Gamma)$ such that $Q = UPU^*$. Therefore

$$\operatorname{Tr}(QTQ) = \operatorname{Tr}(UPU^*TUPU^*) = \operatorname{Tr}(PU^*TUP) = \tau(U^*TU) = \tau(T),$$

where in the last step we invoked the trace property of τ. □

tau-is-trace-lemma

12.41 Lemma. *The functional τ is a semicontinuous tracial weight. The restriction of τ to $\mathcal{K}(\Gamma)$ is densely defined.*

Proof. The semicontinuity τ follows from the semicontinuity of the ordinary operator trace. If $K \in \mathcal{K}(\Gamma)$ is a Γ-invariant, Γ-compactly supported smoothing operator with kernel function k, then

$$\tau(K^*K) = \int_{W\times_\Gamma W} |k(w_1, w_2)|^2 \, dw_1 \, dw_2,$$

and in particular $\tau(K^*K) < \infty$. It follows that the restriction of τ to $\mathcal{K}(\Gamma)$ is densely defined. □

12.42 Definition. If H is a Γ-invariant closed subspace of $L^2(W,S)$, then its Γ-*dimension* is the quantity

$$\dim_\Gamma(H) = \tau(P_H),$$

where P_H is the orthogonal projection onto H.

Unlike the ordinary dimension, the Γ-dimension of a subspace need not be integral. For example if Γ is a finite group with N elements, then the Γ-dimension of a subspace if $1/N$ times the ordinary dimension. For infinite groups the Γ-dimension can assume arbitrary nonnegative real values.

12.43 Example. Let $\Gamma = \mathbb{Z}^n$ and let $W = \mathbb{R}^n$, on which Γ acts by translations. Let E be a measurable subset of $\mathbb{R}^n$ and let H be the subspace of $L^2(W)$ consisting of those L^2-functions whose Fourier transforms vanish almost everywhere outside of E. Then H is a closed, Γ-invariant subspace of $L^2(W)$ and its Γ-dimension is equal to the measure of the set E.

We are going to sketch a proof of the following result:

l2-index-prop **12.44 Proposition.** *Let D be a first-order, odd-graded elliptic self-adjoint partial differential operator on a closed manifold M and let W be a principal Γ-space over M. If $\ker_\pm(D_W)$ are the even and odd-graded components of the kernel of D_W, then $\dim_\Gamma(\ker_\pm(D_W)) < \infty$. Moreover if $\sigma\colon K(C^*_\lambda(\Gamma)) \to \mathbb{R}$ is the dimension function associated to the canonical trace on $C^*_\lambda(\Gamma)$, then*

$$\sigma(\mathrm{Ind}_\Gamma(D)) = \dim_\Gamma(\ker_+(D_W)) - \dim_\Gamma(\ker_-(D_W)).$$

12.45 Remark. The quantity on the right hand side of the above display is usually called the L^2-*index* of the operator D (associated to the principal Γ-manifold W).

Proof (sketch): The semicontinuous tracial weight τ defines a dimension homomorphism

$$\tau\colon K(\mathcal{K}(\Gamma)) \to \mathbb{R}$$

using the construction of Proposition 12.16. By Proposition 12.19, we have

$$\tau(\mathrm{Ind}_\Gamma(D)) = \tau(\varepsilon h(D_W)),$$

where h is any compactly supported even function with $h(0) = 1$. Choose a sequence $\{h_n\}$ of such functions such that $h_n(t)$ decreases monotonically to 0 (as $n \to \infty$), for each fixed $t \neq 0$. By the functional calculus, the operators $h_n(D_W)$ tend (in the strong operator topology) to

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}$$

as $n \to \infty$, where $P_\pm$ are the orthogonal projections onto $\ker_\pm(D_W)$. It can be shown that τ is continuous with respect to monotone strong convergence (this is a version of the Monotone Convergence Theorem from measure theory) and therefore that $\tau(\mathrm{Ind}_\Gamma(D))$ is equal to

$$\lim \tau(\varepsilon h_n(D_W)) = \tau(\varepsilon P) = \dim_\Gamma(\ker_+(D_W)) - \dim_\Gamma(\ker_-(D_W)).$$

On the other hand, one can show that under the isomorphism

$$\mathcal{K}(\Gamma) \cong C^*_\lambda(\Gamma) \otimes \mathcal{K}$$

provided by Lemma 12.27, the tracial weight τ on $\mathcal{K}(\Gamma)$ is equal to $\sigma \otimes \mathrm{Tr}$, where Tr is the usual operator trace on $\mathcal{K}$. Using this one can see that the diagram

$$\begin{tikzcd} K(\mathcal{K}(\Gamma)) \arrow[rr, "\cong"] \arrow[dr, "\tau"'] & & K(C^*_\lambda(\Gamma)) \arrow[dl, "\sigma"] \\ & \mathbb{R} & \end{tikzcd}$$

is commutative, and this completes the proof. $\square$

The following result about the topological Γ-index map will provide us with a formula for the trace of the Γ-index of an elliptic operator on M. The proof is not at all difficult, but since it is best presented using ideas we have not developed in this book (the approach to characteristic classes using connections and curvature) we shall omit it.

g-topological-index-prop

12.46 Proposition. *Let W be a principal Γ-manifold over a smooth manifold M. The composition*

$$K(T^*M) \xrightarrow{\mathrm{Ind}^t_\Gamma} K(C^*_\lambda(\Gamma)) \xrightarrow{\sigma} \mathbb{C}$$

of the topological Γ-index map with the canonical trace is given by the formula

$$\mathrm{Ind}^t_\Gamma(x) = (-1)^{\dim(M)} \int_{T^*M} \mathrm{ch}(x)\,\mathrm{Todd}(TM \otimes \mathbb{C}).$$

In other words, the trace of the topological Γ-index is equal to the ordinary topological index. $\square$

Putting together Theorem 12.37, Proposition 12.38 and Proposition 12.46, we obtain the following result, known as the L^2-*index theorem*:

l2-index-thm

12.47 Theorem. *Let W be a principal Γ-manifold over a closed manifold M and let D be an elliptic operator on M. Then*

$$\begin{aligned} &\dim_\Gamma(\ker_+(D_W)) - \dim_\Gamma(\ker_+(D_W)) \\ &\qquad\qquad = (-1)^{\dim(M)} \int_{T^*M} \mathrm{ch}(\sigma_D)\,\mathrm{Todd}(TM \otimes \mathbb{C}). \end{aligned}$$

In other words the L^2-index of the operator D associated to the principal Γ-manifold W over M is equal to the ordinary topological index of D on M. $\square$

heateq-rmk

12.48 Remark. Another way of formulating the L^2-index theorem is to say that the analytic index of D on M is equal to the L^2-index of D associated to any principal Γ-manifold W over M. This assertion is equivalent to the one in Theorem 12.47 by the ordinary index theorem. It bypasses topology and may be proved by purely analytic methods. This was in fact the original approach.

The L^2-index theorem should not be viewed as a negative result, despite the fact that it shows that the L^2-index of D, which is a part of the equivariant index of Definition 12.29, is not a new invariant but rather the ordinary analytic index of D. The equality of the L^2-index and the ordinary analytic index has a number of important consequences. They arise because in certain circumstances it is possible to compute the L^2-index separately by analytic, geometric or representation-theoretic means. We shall conclude this section by sketching an example related to the following conjecture of Hopf:

hopf-conj **12.49 Conjecture.** *If M is a closed, connected, even-dimensional manifold with contractible universal cover, then $(-1)^{\frac{1}{2}\dim(M)}\chi(M) \geq 0$.*

hopf-prop **12.50 Proposition.** *If M is a closed, connected, even-dimensional manifold that may be equipped with a Riemannian metric of constant negative curvature, then $(-1)^{\frac{1}{2}\dim(M)}\chi(M) > 0$.*

Proof (sketch). Let W be the universal cover of M and let $\dim(M) = 2k$. By our hypotheses on M, it is an even-dimensional hyperbolic space. Using the geometric structure of W, it may be shown that the spaces $\mathcal{H}^p(W)$ of harmonic, square-integrable p-forms on W are zero unless $p = k$, while in the middle dimension there are non-zero, square-integrable harmonic forms. Now let D be the de Rham operator on M. The ordinary analytic index of D is the Euler characteristic of M, while

$$\dim_\Gamma(\ker_+(D_W)) - \dim_\Gamma(\ker_+(D_W)) = \sum_{p=0}^{2k}(-1)^p \dim_\Gamma(\mathcal{H}^p(W)).$$

Hence $\chi(M) = (-1)^k \dim_\Gamma(\mathcal{H}^k(W))$, and the proposition follows. □

atiyah-conj-rem **12.51 Remark.** Note that although the L^2-index theorem implies the identity

$$\sum_{p=0}^{2k}(-1)^p \dim(\mathcal{H}^p(M)) = \sum_{p=0}^{2k}(-1)^p \dim_\Gamma(\mathcal{H}^p(W)),$$

it is *not* true that the individual terms in these two alternating sums may be identified with one another.

12.7 A FAMILIES INDEX THEOREM FOR COVERING SPACES

We are going to study a generalization of the index theorem for covering spaces in which, roughly speaking, equivariant, or *periodic* operators on a principal Γ-manifold W are replaced by families of *almost periodic* operators on W. A good picture to keep in mind is that of the Kronecker foliation depicted in Figure 13.2 below, in which a torus is decomposed into a family of lines, each wrapping around the torus infinitely often and with dense image. A function on the torus restricts to an almost-periodic function on each line.

We shall begin by describing a geometric construction which includes the Kronecker foliation. Let W be a principal Γ-manifold over a smooth manifold M, as in the previous section. But now let X be second smooth manifold equipped with an action of Γ by diffeomorphisms. We shall not assume that the action of Γ on X

is principal. Construct the space $W \times_\Gamma X$ by forming the quotient of $W \times X$ by the product action of Γ. The product action of Γ on $W \times X$ is principal, and therefore the quotient $W \times_\Gamma X$ is a smooth manifold. We are going to study the index theory of certain operators on $W \times_\Gamma X$; these operators will not however be elliptic in the conventional sense.

For each point x of X there is an immersion of W into $W \times_\Gamma X$ given by the formula

$$i_x \colon w \mapsto (w, x) \in W \times_\Gamma X.$$

If the action of Γ on X is free, then each of the maps i_x is injective and so embeds W as a submanifold of $W \times_\Gamma X$, although typically not as a closed submanifold. In general, the image of each i_x is a covering space of M intermediate between W and M. Anticipating the terminology of foliation theory to be discussed in the next chapter, let us call the submanifolds $i_x[W]$ the *leaves* of $W \times_\Gamma X$.

kronecker-example

12.52 Example. Let $\Gamma = \mathbb{Z}$ and let $W = \mathbb{R}$, on which Γ acts by translation. Let X be the unit circle on which Γ acts by the action $n \colon z \mapsto e^{2\pi i\theta n} z$ associated to some $\theta \in \mathbb{R}$. The manifold $W \times_\Gamma X$ is diffeomorphic to a torus. If θ is irrational, so that the action of Γ on X is free, then the leaves of $W \times_\Gamma X$ are lines that wind infinitely often around the torus, each one being dense in the torus. This is the Kronecker foliation.

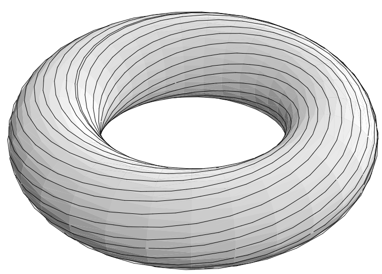

Figure 12.1 The Kronecker foliation. ir-slop-fig

The operators that we shall consider decompose into families of operators acting on the leaves of $W \times_\Gamma X$:

12.53 Definition. Let D be a linear partial differential operator acting on sections of some bundle S over $W \times_\Gamma X$. We shall say that D is a *leafwise operator* if its lift to the covering space $W \times X$ commutes with multiplication by the functions on $W \times X$ that factor through the projection from $W \times X$ to X.

A linear partial differential operator D is a leafwise operator if and only if its lift to $W \times X$ involves no derivatives in the X-directions, but only derivatives in the W-directions. If D is a leafwise operator, then its lift to $W \times X$ restricts to a partial differential operator D_x on each submanifold $W \times \{x\}$. This is because if s

is a section of the pull-back of S to $W \times X$, then the restriction of Ds to $W \times \{x\}$ depends only on the restriction of s to $W \times \{x\}$. The family of operators $\{D_x\}_{x\in X}$ is Γ-equivariant.

12.54 Definition. Let D be a first-order leafwise linear partial differential operator acting on sections of some bundle S over $W \times_\Gamma X$. We shall say that D is *leafwise formally self-adjoint* if each of the operators D_x is formally self-adjoint.[2] We shall say that D is a *leafwise elliptic* operator if each of the operators D_x is elliptic.

Assume that $W \times_\Gamma X$ is a closed manifold (that is, that both $M = W/\Gamma$ and X are closed) and let D be a leafwise formally self-adjoint, first order, leafwise linear partial differential operator on $W \times_\Gamma X$. Theorem 10.32 shows that each operator D_x is essentially self-adjoint. Assuming that D is leafwise elliptic we can therefore attempt to construct an analytic index for the family $\{D_x\}$ determined by D as we did in the last section (note that if X reduces to a single point, then we are in precisely the situation that we analyzed in the last section). The appropriate C^*-algebra for the index is as follows:

12.55 Definition. Let $K = \{K_x\}_{x\in X}$ be a family of bounded operators on $L^2(W)$. We shall call it an *Γ-equivariant, Γ-compactly supported family of smoothing operators* (or an *equivariant family of smoothing operators*, for short) if there is a smooth complex-valued function k on the three-fold product $W \times W \times X$ that is Γ-invariant for the product action and compactly supported as a function on $(W \times W) \times_\Gamma X$, such that

$$K_x f(w) = \int_W k(w, v, x) f(v)\, dv$$

for every $f \in L^2(W)$ and every $x \in X$. We shall denote by $\mathcal{K}(\Gamma, X)$ the closure of the $*$-algebra of equivariant families of smoothing operators on W in the norm

$$\|K\| = \sup_{x\in X} \|K_x\|.$$

12.56 Remark. As in the previous section we shall need a small modification of this definition which incorporates a complex vector bundle S on whose sections a leafwise operator acts. But we shall not dwell on this.

From now on we shall be dealing with a leafwise formally self-adjoint, leafwise elliptic, first-order leafwise linear partial differential operator on $W \times_\Gamma X$, acting on sections of a bundle S over $W \times_\Gamma X$. We shall further assume that S is graded and that D is odd-graded. Rather than repeat this long list of assumptions continually, we shall simply refer to D as a *leafwise elliptic* operator.

fam-ell-reg-prop

12.57 Proposition. *If D is a leafwise elliptic operator on $W \times_\Gamma X$, and if $f \in \mathcal{S}$, then $\{f(D_x)\} \in \mathcal{K}(\Gamma, X)$.*

Proof. The manifold $(W \times W) \times_\Gamma X$ is a smooth groupoid with object space $W \times_\Gamma X$, source and range maps the two natural projections, and composition law

$$[w_3, w_2, x] \circ [w_2, w_1, x] = [w_3, w_1, x].$$

[2]This is generally not the same as requiring that D itself be formally self-adjoint.

The C^*-algebra $\mathcal{K}(\Gamma, X)$ is the C^*-algebra of this groupoid, and the proposition therefore follows from Theorem 10.37 (together with the modification needed to take the bundle S into account detailed in Remark 11.18). $\square$

As in the previous section, it is convenient to identify $\mathcal{K}(\Gamma, X)$ with another C^*-algebra, up to Morita equivalence.

12.58 Definition. Denote by $C^*_\lambda(\Gamma, X)$ the C^*-algebra of the smooth groupoid $\Gamma \ltimes X$ associated to the action of Γ on X as in Example 10.6.

Lemma 12.27 from the previous section may be generalized as follows:

-balanced-prod-morita-lemma

12.59 Lemma. *The C^*-algebra $\mathcal{K}(\Gamma, X)$ is isomorphic to the tensor product algebra $C^*_\lambda(\Gamma, X) \otimes \mathcal{K}(L^2(M))$, where $M = W/\Gamma$.*

Proof. Let $F \subseteq W$ be a bounded and measurable fundamental domain for the action of Γ. Then the unitary isomorphism

$$U \colon L^2(W, S) \to \ell^2(\Gamma) \otimes L^2(F, S)$$

defined by $(Uf)(\gamma, w) = f(\gamma^{-1} w)$ conjugates the natural representation of $\mathcal{K}(\Gamma, X)$ on $L^2(W, S)$ associated to $x \in X$ to the natural representation of $C^*_\lambda(\Gamma) \otimes \mathcal{K}(L^2(M, S))$ on $\ell^2(\Gamma) \otimes L^2(M, S)$ associated to x. $\square$

fam-eq-ind-op-def

12.60 Definition. Let D be a leafwise elliptic operator on $W \times_\Gamma X$. Denote by $\mathrm{Ind}_{\Gamma, X}(D) \in K(C^*_\lambda(\Gamma, X))$ the equivariant index associated to D by Proposition 12.26 and Lemma 12.59.

As in the previous section, we can extract a numerical index from this K-theoretic index, although in the current context we shall need an additional piece of structure to do so.

12.61 Definition. Let μ be a positive, Γ-invariant Borel measure on the closed manifold X. Associate to μ the trace functional $\sigma_\mu \colon C^*_\lambda(\Gamma, X) \to \mathbb{C}$ given by the formula

$$\sigma_\mu(f) = \int_X f(e, x)\, d\mu(x).$$

(Recall that $C^*_\lambda(\Gamma, X)$ is a completion of the $*$-algebra of smooth, compactly supported functions on $\Gamma \ltimes X$. The trace σ_μ is defined initially on the smooth, compactly supported functions, then extended by continuity to $C^*_\lambda(\Gamma, X)$.)

To compute the value of the trace σ_μ on the K-theoretic index of a leafwise elliptic operator we shall need to extend the trace to a broader class of operators, as in the previous section:

12.62 Definition. Let $\mathcal{M}(\Gamma, X)$ be the algebra of all bounded, measurable,[3] Γ-equivariant families of operators on $L^2(W)$ parametrized by X. Let μ be a finite, Γ-invariant Borel measure on X and let $P \colon L^2(W) \to L^2(W)$ be the orthogonal

[3] A family $\{T_x\}$ is *measurable* if for every $v, w \in L^2(W)$ the function $\langle T_x v, w \rangle$ is a measurable function on X.

projection onto the L^2-space of a fundamental domain in W. If $T = \{T_z\}$ is a positive element of $\mathcal{M}(\Gamma, X)$, then define

$$\tau_\mu(K) = \int_X \mathrm{Tr}(PK_xP)\, d\mu(x).$$

If $P = \{P_x\} \in \mathcal{M}(\Gamma, X)$ is a measurable family of projections onto a family $H = \{H_x\}$ of subspaces of $L^2(W)$, define

$$\dim_{\Gamma,\mu}(H) = \tau_\mu(P).$$

The quantity $\tau(K)$ is independent of the choice of fundamental domain used in its definition and has the property that $\tau(T^*T) = \tau(TT^*)$, for every $T \in \mathcal{M}(\Gamma, X)$. If K is an equivariant family of smoothing operators with kernel function $k\colon W \times W \times X \to \mathbb{C}$, then

$$\tau_\mu(K^*K) = \int_{(W\times W)\times_\Gamma X} |k(w_1, w_2, x)|^2\, dw_1\, dw_2\, d\mu(x).$$

The proof of Proposition 12.44 sketched in the previous section may be adapted to provide a proof of the following result for families:

12.63 Theorem. *If D is a leafwise elliptic operator on $W \times_\Gamma X$, then*

$$\dim_{\Gamma,\mu}(\ker_\pm(D)) < \infty$$

and

$$\sigma_\mu(\mathrm{Ind}_{\Gamma,X}(D)) = \dim_{\Gamma,\mu}(\ker_+(D)) - \dim_{\Gamma,\mu}(\ker_-(D)).$$

□

There are also straightforward generalizations of the analytical and topological Γ-index maps from the previous section. These are maps

$$K(C_0(T^*W \times_\Gamma X)) \longrightarrow K(C_0(T^*M) \otimes C^*_\lambda(\Gamma, X)) \rightrightarrows K(C^*_\lambda(\Gamma, X)),$$

where the right-hand maps are the topological and analytic index maps associated to the C^*-algebra $A = C^*_\lambda(\Gamma, X)$ as in Definition 12.8 and Definition 12.9, while the left-hand map is obtained by the following modification of Definition 12.34.

12.64 Definition. Let $M = W/\Gamma$ and denote by $\mathcal{E}_{W,X}$ the Hilbert $C_0(M) \otimes C^*_\lambda(\Gamma, X)$-module of all continuous functions from W into $C^*_\lambda(\Gamma, X)$ such that

(i) $f(\gamma w) = u_\gamma f(w)$, for all $w \in W$ and all $\gamma \in \Gamma$; and

(ii) the function $w \mapsto \|f(w)\|$ belongs to $C_0(M)$,

Thus $\mathcal{E}_{W,X}$ is the Hilbert module of continuous sections, vanishing at infinity of the bundle $W \times_\Gamma C^*(\Gamma, X)$ over M.

The action of $C_0(W \times_\Gamma X)$ on $\mathcal{E}_{W,X}$ as Hilbert-module endomorphisms gives a $*$-homomorphism from $C_0(W \times_\Gamma X)$ into the C^*-algebra of compact operators on $\mathcal{E}_{W,X}$ and so determines a map

$$\nu_{W,X}\colon K(C_0(W \times_\Gamma X)) \to K(C_0(M) \otimes C^*_\lambda(\Gamma, X)).$$

Replacing W by T^*W and M by T^*M we obtain the map we need to complete our definition of the analytic and topological index maps.

12.65 Definition. The *topological* and *analytic* Γ*-index maps*

$$\mathrm{Ind}^t_{\Gamma,X}, \mathrm{Ind}^a_{\Gamma,X} \colon K(T^*W \times_\Gamma X) \to K(C^*_\lambda(\Gamma, X))$$

associated to W and X are the compositions

$$K(C_0(T^*W \times_\Gamma X)) \xrightarrow{\nu_{T^*M,X}} K(C_0(T^*W) \otimes C^*_\lambda(\Gamma, X)) \underset{\mathrm{Ind}^a}{\overset{\mathrm{Ind}^t}{\rightrightarrows}} K(C^*_\lambda(\Gamma, X)).$$

As is by now usual, the two indices agree, thanks to Theorem 12.10:

g-fam-index-theorem

12.66 Theorem. *Let W be a principal Γ-manifold over a smooth manifold M and let X be a smooth Γ-manifold. The topological and analytic Γ-index maps*

$$\mathrm{Ind}^t_{\Gamma,X}, \mathrm{Ind}^a_{\Gamma,X} \colon K(T^*W \times_\Gamma X) \to K(C^*_\lambda(\Gamma, X))$$

are equal to one another. □

Assume now that D is a leafwise elliptic operator on $W \times_\Gamma X$. Considered as an operator on $W \times_\Gamma X$ (rather than as a family of operators), D has a symbol which is an endomorphism of the pullback of the vector bundle S to $T^*(W \times_\Gamma X)$ (see Section 5.4). There is a natural projection map

$$T^*(W \times_\Gamma Z) \to T^*W \times_\Gamma X,$$

and the assertion that D is a leafwise operator implies that the symbol of D actually defines an endomorphism of the pullback of S to $T^*W \times_\Gamma X$. The assertion that D is leafwise elliptic implies that this latter endomorphism is elliptic in the sense of Definition **??**, and so defines a *leafwise symbol class*

$$\sigma_D \in K(C_0(T^*W \times_\Gamma X))$$

as explained in Section **??**. The analytic index map sends this symbol class to the analytic index of D:

12.67 Proposition. *If D is a leafwise elliptic operator on $W \times_\Gamma X$, then*

$$\mathrm{Ind}^a_{\Gamma,X}(\sigma_D) = \mathrm{Ind}_{\Gamma,X}(D).$$

□

12.68 Corollary. *If D is a leafwise elliptic operator on $W \times_\Gamma X$ and if μ is an invariant measure on X, then*

$$\sigma_\mu(\mathrm{Ind}^a_{\Gamma,X}(\sigma_D)) = \dim_{\Gamma,\mu}(\ker_+(D)) - \dim_{\Gamma,\mu}(\ker_-(D)).$$

□

In the remainder of this section we shall obtain from Theorem 12.66 a numerical index theorem that extends the L^2-index theorem presented in the last section. While the development of the numerical theorem is a fairly straightforward matter, it does involve one new geometric idea. To help introduce it, let us say that an open subset of $W \times_\Gamma X$ is *basic* if it is of the form $\pi[U \times X]$, where U is an open subset of W and $\pi \colon U \times X \to \pi[U \times X]$ is a diffeomorphism (π is the projection from $W \times X$ to $W \times_\Gamma X$).

12.69 Proposition. *Let W be an oriented smooth manifold equipped with an orientation-preserving, principal action of Γ. Let $n = \dim(W)$. Let X be a smooth, closed Γ-manifold and let μ be an invariant Borel measure on X. There is a unique linear functional*

$$C_\mu \colon \Omega^n_c(W \times_\Gamma X) \to \mathbb{R}$$

such that if $\alpha \in \Omega^n(W \times_\Gamma X)$, and if α is supported in a basic open set $\pi[U \times X]$, then

$$C_\mu(\alpha) = \int_X \left(\int_{U \times \{x\}} \pi^*(\alpha) \right) d\mu(x).$$

If α is exact as a compactly supported form, then $C_\mu(\alpha) = 0$.

Proof. Since every form $\alpha \in \Omega^n_c(T^*W \times_\Gamma X)$ is a finite sum of forms which are compactly supported within basic sets, there is clearly at most one linear functional with the required property. If α is supported in a basic set, then it follows from the invariance of the measure μ that the formula in the proposition for $C_\mu(\alpha)$ depends only on α and not on the choice of the open set $U \subseteq W$. This in turn implies that the formula determines a well-defined linear functional on $\Omega^n_c(T^*W \times_\Gamma X)$. If α is exact, so that $\alpha = d\beta$ for some compactly supported $(n-1)$-form β, then write $\alpha = \sum d\beta_j$ for forms $\beta_1, \ldots, \beta_N$ that are compactly supported in basic sets to conclude from Stokes' Theorem that $C_\mu(\alpha) = 0$. □

12.70 Definition. The functional $C_\mu \colon \Omega^n(W \times_\Gamma X) \to \mathbb{R}$ is called the *Ruelle-Sullivan* current on $W \times_\Gamma X$ associated to the invariant measure μ (a current is a continuous linear functional on the space of differential forms).

Because C_μ vanishes on exact forms it defines a linear functional on the (de Rham) cohomology group $H^n_c(W \times_\Gamma X)$. We shall write this functional as $\alpha \mapsto \int_{C_\mu} \alpha$.

12.71 Example. Consider the Kronecker foliation of Example 12.52, obtained from the action of $\mathbb{Z}$ on the circle X by an irrational rotation. If μ is Haar measure on X (normalized to have total volume 1), then μ is of course invariant for the given action of Γ. The associated Ruelle-Sullivan current maps the *integral* cohomology group $H^1(W \times_\Gamma X)$ to $\mathbb{Z} + \theta\mathbb{Z} \subseteq \mathbb{R}$. This shows that the Ruelle-Sullivan current need not correspond to an integral, or even a rational, cohomology class.

Replacing W by T^*W (which has a canonical orientation and orientation-preserving action of Γ) we obtain a Ruelle-Sullivan current

$$C_\mu \colon H^{2n}_c(T^*W \times_\Gamma X) \to \mathbb{R}.$$

Our interest in the Ruelle-Sullivan current stems from the following result. Like its simpler counterpart in Proposition 12.46, the formula below can be proved in a straightforward fashion using the approach to characteristic classes through connections and curvature.

12.72 Proposition. *Let W be a principal Γ-manifold over a smooth manifold M, let X be a smooth Γ-manifold, and let μ be a finite invariant measure on X. Then*

$$\sigma_\mu(\mathrm{Ind}^t_{\Gamma,X}(x)) = (-1)^{\dim(M)} \int_{C_\mu} \mathrm{ch}(x) \wedge \mathrm{Todd}(TM \otimes \mathbb{C})$$

for every $x \in K(T^*W \times_\Gamma X)$. □

Putting everything together, we obtain the following index theorem. It is a special case of Connes' index theorem for measured foliations, to be discussed in the next chapter, as well as a generalization of the L^2-index theorem from the previous section.

12.73 Theorem. *Let W be a principal Γ-manifold over a smooth manifold M, let X be a smooth Γ-manifold, and let μ be a finite invariant measure on X. If D is a leafwise elliptic operator on $W \times_\Gamma V$, and if μ is an invariant measure on X, then*

$$\begin{aligned}\dim_{\Gamma,\mu}(\ker_+(D)) - \dim_{\Gamma,\mu}(\ker_-(D)) \\ = (-1)^{\dim(M)} \int_{C_\mu} \mathrm{ch}(\sigma_D) \wedge \mathrm{Todd}(TM \otimes \mathbb{C}),\end{aligned}$$

where C_μ is the Ruelle-Sullivan current associated to the invariant measure μ. □

12.8 NOTES

The K-theory reformulation of the index theorem presented in Section 1 is the same as the approach of Atiyah and Singer to the index theorem in [**?**].

The index theory for covering spaces that we presented in Section 12.5 was invented by Novikov and his student Mishchenko [**?**, **?**]. A version of Theorem 12.10 appears in [].

The L^2-index theorem appears in [**?**]. A survey of its applications to the Hopf conjecture can be found in [] along with a great deal of additional information about Γ-dimensions. An important application of the L^2-index theorem not considered in our discussion is to representation theory. If G is a Lie group of isometries of a complete Riemannian manifold W, and if D_W is an equivariant elliptic operator on W, then the kernel of D_W is a representation space for G. If Γ is a discrete subgroup of G which acts principally and cocompactly on W, then D_W descends to an operator on $M = W/\Gamma$ to which we may apply the L^2-index theorem. If the index of the descended operator is non-zero, then the kernel of the operator D_W on W is nonzero, and so is a nonzero representation of G. For applications of this method to the construction of representations of G see [] or [].

The L^2-index of any elliptic operator D is an integer since it is equal to the ordinary index of D. But the L^2-index is the difference of the $\dim_\Gamma(\ker_+(D_W))$ and $\dim_\Gamma(\ker_-(D_W))$, and one might ask whether these quantities are themselves integers. Certainly they are not in general equal to their counterparts $\dim(\ker_+(D))$ and $\dim(\ker_-(D))$ (compare Proposition 12.50 and Remark 12.51). This integrality question was raised by Atiyah and remains a very challenging problem. See [] for a discussion of the case of the de Rham operator.

Invariant measures and their associated currents were studied in various contexts in the 1970s: see [**?**, **?**]. As the text suggests, their appearance in index theory is due to Connes. See [].

Chapter Thirteen

Index Theory for Foliations

FoliationsChapter

13.1 FOLIATIONS

foliations-section

13.1 Definition. Let M be a smooth manifold of dimension $n = p + q$. A p-dimensional *foliation atlas* for M is an atlas of smooth charts $\phi_\alpha \colon U_\alpha \to \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ (where U_α is open in M) such that the transition functions

$$\phi_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1},$$

which are diffeomorphisms from one open subset of $\mathbb{R}^p \times \mathbb{R}^q$ to another, have the form

$$\phi_{\alpha\beta}(x, y) = (f(x, y), g(y))$$

where $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$. In other words, the transition functions are required to respect the relation of "having the same $\mathbb{R}^q$ coordinate". We say that two foliation atlases are *equivalent* if their union is also a foliation atlas.

13.2 Definition. A p-dimensional *foliation* F on M is an equivalence class of p-dimensional foliation atlases. A manifold equipped with a foliation is called a *foliated manifold*. The charts appearing in a foliation atlas are called *flowboxes* for the foliation.

Let (M, F) be a foliated manifold. A *plaque* is a subset of M that is of the form $\mathbb{R}^p \times \{y\}$ in some foliation chart. We can define a new topology and manifold structure on M by taking the plaques as coordinate neighborhoods. In this topology M becomes a p-dimensional manifold with uncountably many connected components (except in trivial cases). These connected components are called the *leaves* of the foliation. Each leaf can be considered as a submanifold of M with its usual topology, and the partition of M into leaves looks locally like the partition of $\mathbb{R}^p \times \mathbb{R}^q$ into plaques. However, the global structure of the foliation may be extremely complicated: the leaves need not be compact, they may wind densely around in M (returning many times to the same foliation chart), and so on.

fam-fol-ex

13.3 Example. A submersion $\pi \colon E \to B$ (see Definition 13.9)gives rise to a foliation, whose leaves are the fibers of π. By definition, every foliation has this structure locally.

group-foliation-ex

13.4 Example. If a smooth manifold M is equipped with a locally free, smooth action[1] of a Lie group H then a foliated manifold is obtained by defining F to be the

[1]An action is locally free if its isotropy subgroups are discrete.

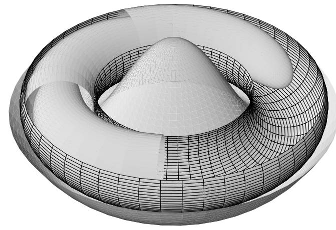

Figure 13.1 Parts of some leaves in the Reeb foliation. reeb-fig

bundle of tangent vectors on M which are tangent to the orbits of the action. Once again, we may obtain the irrational slope foliation on the torus as a particular case of this construction.

13.5 Example. A classical construction, due to Reeb, shows that there are 2-dimensional foliations of S^3. The construction depends on noting that S^3 can be obtained by identifying two solid tori along their boundaries. We regard each solid torus as the quotient of a solid cylinder by a discrete group of translations. Each such solid cylinder (such as $x^2 + y^2 \leq 1$, in Cartesian coordinates) can be foliated by translates of the surface obtained by rotating the curve $z = f(x)$ around the z-axis, where f is a smooth even function which increases to $+\infty$ as $x \to \pm 1$. The quotient of this foliation by the group of translations gives a foliation of the solid torus, having the torus itself as a boundary leaf; and by joining two copies of this foliation we get a foliation of S^3. Note that all the leaves except one are diffeomorphic to $\mathbb{R}^2$; the exceptional leaf is a 2-torus. See Figure 13.1.

inf-fol-rmk **13.6 Remark.** If (M, F) is a foliated manifold, the tangent vectors to the leaves form a subbundle TF of TM. Clearly, this subbundle has the *integrability property*: if X, Y are sections of TF, then their Lie bracket $[X, Y]$ is a section of TF also. Conversely, a classical theorem of Frobenius states that any subbundle of TM that has the integrability property is tangent to a foliation.

Let (M, F) be a foliated manifold.

13.7 Definition. A differential operator on M is said to be a *leafwise operator* if it restricts to a differential operator on each leaf. To put this another way, a differential operator is a leafwise operator if, when represented in a foliation chart $\mathbb{R}^p \times \mathbb{R}^q$, it only involves differentiation in the $\mathbb{R}^p$ directions.

Since TF is a subbundle of TM, T^*F is a quotient bundle of T^*M. The symbol of a leafwise operator vanishes on the kernel of $T^*M \to T^*F$, so it is a function on T^*F (with values in the endomorphisms of the bundle on which the operator acts). We'll call this function the *leafwise symbol*.

leaf-ell-def **13.8 Definition.** A leafwise operator is *leafwise elliptic* if its restriction to each leaf is elliptic. That is to say, a leafwise elliptic operator is one whose leafwise symbol is elliptic in the sense of Definition **??**.

In Example 13.3, the leafwise elliptic operators are simply the elliptic families that we discussed in the previous chapter. In Example 13.39, leafwise elliptic operators are closely related to higher index theory. We are going to study the index theory of leafwise elliptic operators in a way that will include both of these examples. As usual, the first stage is to construct a suitable groupoid that will reflect the geometry of the situation.

13.2 THE INDEX THEOREM FOR FAMILIES

We shall now generalize the index theorem by considering not a single elliptic operator but a *family* of elliptic operators.

mfam-def **13.9 Definition.** Let B be a smooth manifold. By a *manifold over* B we mean a smooth manifold M equipped with a submersion $\pi\colon M \to B$.

efam-def **13.10 Definition.** Let M be a manifold over B. A linear partial differential operator D acting on the sections of a bundle S over W, is *compatible* with the submersion from M to B if $D(\phi s) = \phi D(s)$, for every smooth section s and every smooth function $\phi\colon M \to \mathbb{C}$ which factors through the submersion $\pi\colon M \to B$.

If a linear partial differential operator D acting on the sections of a bundle S over M, is compatible with the given submersion from M to B, then ...

13.11 Definition. The B-*tangent bundle* of a manifold M over B is the vector bundle

$$T_B M = \ker(\pi_*\colon TM \to TB)$$

over M.

The symbols of a family of elliptic operators (as in Definition 13.10 above) provide an elliptic endomorphism of the pullback of S to $T_B M$, and thus the family has a *symbol class*

$$\sigma(D) \in K(T_B M)$$

which generalizes the class constructed in the single-operator case.

13.12 Definition. B-tangent groupoid.

This is the 'families' version of Construction 9.3. Here is a sketch of one way to carry it out. Recall that the we built the index map in the single-operator case from an asymptotic morphism associated to the *tangent groupoid* $\mathbb{T}M$. The tangent groupoid is a smooth family of groupoids depending on a real parameter t; for $t = 0$ it is the tangent bundle TM (considered as a bundle of abelian groups) and for $t \neq 0$ it is a rescaled copy of the pair groupoid $M \times M$. Now for a *family* of manifolds $\pi\colon W \to B$ we can carry out the tangent groupoid construction fiberwise on the family. In this way we obtain a groupoid $\mathbb{T}_B M$ which for $t = 0$ is $T_B M$ (considered as a bundle of abelian groups) and for $t \neq 0$ is a copy of the groupoid G of Definition 13.13. From this groupoid one obtains as in Chapter 10 an asymptotic morphism $C_0(T_B^* M) \rightsquigarrow C_r^*(G)$, and the associated K-theory homomorphism is the analytical index.

fam-gp-def

13.13 Definition. Let $\pi\colon M \to B$ be a family of manifolds over B. Then the *groupoid of the family* G is the smooth groupoid defined as follows:

- The object space is M;
- The morphism space is $\{(x, y) \in M \times M : \pi(x) = \pi(y) \in B\}$;
- The source and range maps are $s(x, y) = x$, $r(x, y) = y$;
- The composition law is $(x, y) \cdot (y, z) = (x, z)$;

In other words, G is a family of pair groupoids $M_x \times M_x$, parameterized by $x \in B$. A right Haar system for this groupoid is a smoothly varying family of Lebesgue measures on the fibers of π.

Recall from Example 10.19 that the C^*-algebra of the pair groupoid of a manifold M is simply the algebra of compact operators on $L^2(M)$. A similar calculation identifies the C^*-algebra of the groupoid G. To be precise, consider the space $C_c(E)$ of continuous, compactly supported functions on E. Given two such functions one can take their Hilbert space inner product on each fiber $\pi^{-1}(b)$, and this inner product then becomes a continuous function on B. Thus, the space $C_c(E)$ acquires a $C(B)$-valued inner product, and by completing relative to this inner product we obtain a Hilbert module (Section **??**) over $C(B)$. Now we have

famalg-morita

13.14 Lemma. *The C^*-algebra of the groupoid G is exactly the algebra of compact endomorphisms of the Hilbert $C(B)$-module described above.* □

(The proof follows the same lines as that of Example 10.19.) It follows from the discussion in Section **??** that there is a natural map from $K(C^*(G))$ to $K(B)$. In fact this map is an *isomorphism*, but we shall not need this.

Now suppose that M is a closed manifold and that $\{D_b\}$ is a family of elliptic operators over B. According to Theorem 10.37, the functional calculus determines a graded homomorphism

$$\phi\colon \mathcal{S} \to C_r^*(G), \qquad \phi\colon f \mapsto f(D).$$

Using the construction described in Chapter **??** and the discussion above, this graded homomorphism naturally determines and element of $K(B)$.

13.15 Definition. We call the element of $K(B)$ so defined the *analytical index* of the family of operators D_b.

aindfam-constr

13.16 Theorem. *Associated to the B-tangent groupoid there is a natural* analytical index map

$$\mathrm{Ind}_a\colon K(T_B M) \to K(B)$$

which sends the symbol of each elliptic family to the analytical index of the family. □

13.17 Remark. The construction needs to be elaborated to take into account the bundle S on which D is operating. We must replace the ordinary groupoid algebra by a version which has coefficients in S; the two versions have the same K-theory. The details are the same as in the single-operator case, which we discussed in Section 11.3. We will not repeat them here.

13.18 Example. If each operator D_b is invertible (as a linear operator on smooth sections), then it may be shown that the graded homomorphism ϕ is homotopic to zero. As a result, the analytical index of the family $\{D_b\}$ is zero. More generally, if there is a punctured neighborhood of $0 \in \mathbb{R}$ that does not meet the spectrum of any D_b, then the families $\{\ker D_b^+\}$ and $\{\ker D_b^-\}$ form vector bundles over B and their difference in $K(B)$ defines the analytical index the family $\{D_b\}$.

Now we shall define a topological index, and outline the proof of the *index theorem for families*.

fam-inj-def **13.19 Definition.** If $W \to B$ and $Z \to B$ are families of manifolds over the same base, let us define an *embedding* over B to be an embedding $W \to Z$ which makes the obvious diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ B \arrow[r, "="'] & B \end{tikzcd}$$

commute. An embedding of families can be factored (by the tubular neighborhood theorem again) into the inclusion of the zero-section of a vector bundle, followed by the inclusion of an open subset. The induced wrong way maps on K-theory are therefore topologically calculable. Let $i\colon Z \to W$ be an embedding of a smooth family W as a submanifold of a smooth family Z. Define an associated homomorphism

$$i_!\colon K(T_B Z) \to K(T_B Z)$$

as follows. . . .

As with the case considered in the previous section, it is not hard to show that the map $i_!\colon K(T_B M) \to K(T_B N)$ associated to the inclusion $i\colon M \to N$ of M as a submanifold of N does not depend on any of the choices made in its definition. Moreover, we obtain a functor:

13.20 Lemma. *If* $i\colon M \to N_1$ *and* $j\colon N_1 \to N_2$ *are embeddings of smooth manifolds over* B*, then* $j_! \circ i_! = (j \circ i)_!\colon K(T_B M) \to K(T_B N_2)$. □

13.21 Definition. . . . of topological index

$$\mathrm{Ind}_t\colon K(T_B M) \to K(B)$$

We now follow the pattern that we developed in the single-operator case.

13.22 Theorem. *If* $f\colon M \to N$ *is an embedding of manifolds over* B*, then the diagram*

$$\begin{tikzcd} K(T_B M) \arrow[r, "f_!"] \arrow[d, "\mathrm{Ind}_a"'] & K(T_B N) \arrow[d, "\mathrm{Ind}_a"] \\ K(B) \arrow[r, "="'] & K(B) \end{tikzcd}$$

commutes. □

13.23 Theorem. $\mathrm{Ind}_a = \mathrm{Ind}_t \colon K(T_B^* W) \to K(B)$.

13.24 Theorem.

$$\mathrm{ch}(\mathrm{Ind}\, D) = (-1)^n \oint \mathrm{ch}(\sigma_D)\, \mathrm{Todd}(T_B E \otimes \mathbb{C})$$

where $\oint$ *denotes integration along the fiber, an operation that passes from* $H^*(E)$ *to* $H^*(B)$.

There is an important nuance here. The cohomological formula of necessity computes the Chern character, $\mathrm{ch}(\mathrm{Ind}\, D) \in H^*(B)$. When B is a point, as in the case of the ordinary index theorem, $\mathrm{ch}\colon K(\mathrm{pt}) \to H^*(\mathrm{pt})$ is injective so that the Chern character captures all the information about the index. However, in general the Chern character is not injective (it loses all torsion information). For this reason, it is better to regard the K-theoretic statement as constituting the 'correct' form of the index theorem for families; the cohomological statement is just a homomorphic image of this.

13.3 THE FOLIATION GROUPOID

13.25 Definition. Tangent groupoid of a principal Γ-manifold

If Γ is infinite, no principal Γ-manifold can be compact, and therefore elliptic operators on such manifolds will not have indices in the usual Fredholm sense. However, we shall see that if we restrict attention to Γ-equivariant elliptic operators, we can define an index which takes values in a K-theory group associated to the Γ-action. In fact, the group in question is the K-theory of the C^*-algebra of the following groupoid.

13.26 Definition. Let X be a principal Γ-manifold over M. Construct a groupoid G as follows:

- The space of objects is M.
- The space of morphisms is the space of orbits of the diagonal action of Γ on $X \times X$. In other words, a morphism is an equivalence class of pairs $(x, x') \in X \times X$, two such pairs (x_1, x_1') and (x_2, x_2') being considered equivalent if there is $\gamma \in \Gamma$ such that $\gamma x_1 = x_2$ and $\gamma x_1' = x_2'$.

- The source and range maps are $s(x, x') = \pi(x)$, $r(x, x') = \pi(x')$, where $\pi\colon X \to M$ is the quotient map.

13.27 Exercise. Check that this is a smooth groupoid, and that when X is the universal cover of M (as in Example 12.22 above) it is in fact the *fundamental groupoid* of M as defined in Example **??**.

In terms of the "families picture" of groupoids, what we have here is a groupoid representing a family of copies of the universal cover of M relative to different base points in M, with non-trivial twisting provided by the deck transformations.

Once again (compare the discussion surrounding Lemma 13.14), we can identify the C^*-algebra of this groupoid as the algebra of compact operators on a suitable Hilbert module. Consider the vector space $C_c(X)$ of compactly supported continuous functions on X. We can equip this with a $C^*_r(\Gamma)$-valued inner product by defining

$$\langle f, g\rangle = \sum_{\gamma \in \Gamma} \left(\int f(x) g(\gamma^{-1} x)\, dx \right) \cdot [\gamma] \in C^*_r(\Gamma).$$

Completing $C_c(X)$ relative to this inner product, we obtain a $C^*_r(\Gamma)$-Hilbert module and one can prove

13.28 Lemma. *The C^*-algebra of the groupoid G is exactly the C^*-algebra of compact operators on the Hilbert $C^*_r(\Gamma)$-module described above.*

Thus we obtain a map $K(C^*_r(G)) \to K(C^*_r(\Gamma))$. Again, the map is in fact an isomorphism (because the Hilbert module in question is full), but we do not need to make use of this fact.

Let (M, F) be a foliated manifold. The partition of M into leaves defines an equivalence relation, which we call *leaf equivalence*. We wish to define a smooth groupoid which, in the quotient space picture of groupoids, represents the quotient of M by the leaf-equivalence relation, and which in the families picture represents the family which assigns to each point of M the leaf passing through that point. Actually, a small modification is necessary in order to define a manifold structure on this groupoid. We shall assemble the groupoid not just from pairs of leaf-equivalent points, but from *leafwise paths* connecting leaf-equivalent points.

13.29 Definition. A *leafwise path* in a foliated manifold (M, F) is a piecewise smooth path in M that lies in a single leaf (or, equivalently, whose tangent vector is everywhere tangent to the foliation).

Leafwise paths have a kind of 'foliated tubular neighborhood property' which is very important. To explain it, let us agree that if U is a flowbox (an open set in M foliated diffeomorphic to $\mathbb{R}^p \times \mathbb{R}^q$) then $\mathcal{P}(U)$ will denote the plaque set of U, that is the quotient of U by the relation of "having the same $\mathbb{R}^q$ coordinate". The plaque set $\mathcal{P}(U)$ is of course diffeomorphic to $\mathbb{R}^q$, and $\mathcal{P}$ is a functor on the category of flowboxes and inclusion maps: if $U \subseteq V$ are flowboxes, then there is a natural diffeomorphism of $\mathcal{P}(U)$ into $\mathcal{P}(V)$. In particular, suppose that V and V' are flowboxes that have nonempty intersection which is itself a flowbox. Then

$\mathcal{P}(V \cap V')$ is diffeomorphic both to a subset of V and to a subset of V', and thus we obtain a diffeomorphism from an open subset of $\mathcal{P}(V)$ to an open subset of $\mathcal{P}(V')$.

Now suppose that γ is a leafwise path from m_0 to m_1. We can cover γ by flowboxes $U_0, U_1, \ldots, U_n$ such that $U_i \cap U_{i+1}$ is a flowbox, and by iterating the above construction we get a diffeomorphism from an open subset of $\mathcal{P}(U_0)$ to an open subset of $\mathcal{P}(U_n)$. That is, we obtain the *germ* of a diffeomorphism from the space of plaques near m_0 to the space of plaques near m_1.

Standard arguments prove

hol-hom-lemma

13.30 Lemma. *The germ of the diffeomorphism obtained above is well-defined (independent of the choices involved in the construction). Moreover, if γ and γ' are leafwise paths which are leafwise homotopic (keeping endpoints fixed) then the germ associated to γ is the same as the germ associated to γ'.* □

13.31 Definition. This germ is called the *holonomy* of the path γ. Two leafwise paths in a foliated manifold with the same beginning and end points are *holonomy equivalent* if their holonomies are equal.

Holonomy is an equivalence relation; according to Lemma 13.30, it is a coarsening of the leafwise homotopy relation. Quite frequently it is much coarser. For example, it may be shown that for a dense G_δ set of beginning and final points, all paths are holonomy-equivalent to one another.

Holonomy is compatible with concatenation of paths: if γ is a leafwise path from m_0 to m_1, and if η is a leafwise path from m_1 to m_2, then the holonomy class of the concatenated path $\eta \vee \gamma$ from m_0 to m_2 depends only on the holonomy classes of γ and η. It follows that the set of holonomy classes of leafwise paths from a point m to itself is a group under concatenation.

13.32 Exercise. Let M be a smooth manifold equipped with a smooth, locally free action of a connected Lie group H, and let (M, F) be the associated foliated manifold (see Example 13.4). Assume that on a dense subset of M the action of H is in fact free. Show that the holonomy group of $p \in M$ is isomorphic to the isotropy group for the action.

13.33 Definition. Let (M, F) be a foliated manifold. Its *holonomy groupoid* $G(M, F)$ is given as follows:

- The space $G(M, F)$ of morphisms is the set of all holonomy classes of leafwise paths γ in M.
- The space of objects is the manifold M.
- The source and range maps assign to a path γ its initial and final points.
- Composition is given by concatenation of paths.

The identity morphisms are the constant paths; the inverse of a path $\gamma\colon [0,1] \to M$ is obtained by composing with an orientation-reversing diffeomorphism of $[0,1]$.

The space $G(M, F)$ is made into a manifold in the following way. Let γ be a leafwise path from m to m', and let U and U' be flowboxes containing m and m' and small enough that the holonomy of γ provides a diffeomorphism $\phi\colon \mathcal{P}(U) \cong \mathcal{P}(U')$. Let $U \times_\phi U'$ denote the set of pairs $(u, u') \in U \times U'$ such that

$$\phi(\pi(u)) = \pi'(u'),$$

where $\pi\colon U \to \mathcal{P}(U)$ and $\pi'\colon U' \to \mathcal{P}(U')$ are the obvious quotient maps. Clearly, $U \times_\phi U'$ is diffeomorphic to $\mathbb{R}^{2p+q}$. Moreover, each point of $U \times_\phi U'$ corresponds to a nonempty holonomy class of leafwise paths: those which start at u, finish at u', and remain close to γ. Thus $U \times_\phi U'$ is identified with a subset of G. We can define a topology and a manifold structure on G by using sets of the form $U \times_\phi U'$ as local coordinate charts. (It is of course necessary to check that the transition functions between any two such sets are smooth; this is a routine matter.)

13.34 Example. If our foliation is given by a submersion $\pi\colon E \to B$ (Example 13.3), then the groupoid of the foliation is just the groupoid G_π associated to the submersion in Definition 13.13.

13.35 Example. In the case of the foliated manifold (M, F) obtained from an effective, locally free action of a connected Lie group on a manifold M, the foliation groupoid $G(M, F)$ may be identified with the transformation groupoid $H \ltimes M$.

A Haar system for the groupoid $G(M, F)$ is given by a smoothly varying family of Lebesgue measures on the leaves; such Haar systems always exist. Using the techniques of Chapter 10, we can now define the *foliation* C^**-algebra* $C^*(M, F)$ to be the C^*-algebra of the groupoid $G(M, F)$. We think of this as the "function algebra" of the noncommutative space M/F — the *space of leaves* of the foliation.

A leafwise elliptic operator on (M, F) is the same thing as an equivariant elliptic operator on the groupoid G. From Theorem 10.37 we therefore obtain

13.36 Proposition. *The functional calculus for a leafwise elliptic operator* D *gives a* $*$*-homomorphism*

$$f \mapsto f(D)\colon \mathcal{S} \to C^*(M, F).$$

Consequently, a leafwise elliptic operator has a well-defined longitudinal index *in* $K(C^*(M, F))$. ☐

In the case of the foliation given by a submersion, this construction reduces to the families index of the previous chapter. Notice that we have again suppressed explicit mention of the bundle on which D acts; compare Remark **??**.

13.37 Remark. Although the topology that we have defined always makes $G(M, F)$ into a smooth manifold, it is not necessarily Hausdorff[2]. When G is not Hausdorff, the foliation groupoid C^*-algebra $C^*_\lambda(M, F)$ is still defined as a completion of a

[2]It *is* Hausdorff in many important cases, such as when all the leaves are simply connected, or when the foliation is not merely smooth but real analytic.

dense subalgebra $C_c^\infty(G(M,F))$, but the latter is now defined to be the linear span of all smooth, compactly supported functions on coordinate charts as defined above (these functions are extended by zero so as to obtain functions defined on all of $G(M,F)$, but these extended functions need not be continuous on $G(M,F)$). The algebraic operations, and the regular representations are defined as before.

13.38 Exercise. Show that the holonomy group of the toral leaf of the Reeb foliation is equal to its homotopy group $\mathbb{Z}^2$.

Let G denote the holonomy groupoid of the Reeb foliation. Let p be a point of the toral leaf, let γ be a constant path at p, and let γ' be a meridian of the torus that begins and ends at p. Show that the two paths γ and γ' define different points of G, but any two neighborhoods of them in G intersect. Thus G is not Hausdorff.

A leafwise elliptic operator on the foliated manifold (M,F) has a *leafwise symbol*, which gives rise to a K-theory class for the cotangent bundle T^*F to the foliation (see Definition 13.8). Using the same tangent groupoid techniques that have been employed in the previous chapter, we can construct an asymptotic morphism that gives rise to an *analytical (longitudinal) index map*

$$K(T^*F) \longrightarrow K(C^*(M,F))$$

which sends the symbol class of any leafwise elliptic operator to its longitudinal index. The *longitudinal index problem* for foliations is to find a means to compute this map.

We shall approach this problem by analogy with the other index theorems that were considered in the previous chapter. Thus, we plan to construct a topologically defined index group $\mathfrak{I}(M,F)$ and a topological index map $K(TF) \to \mathfrak{I}(M,F)$, in such a way that the analytical index factors through the topological index.

We can think of this as an abstract index theorem for foliations. As in other examples, the usefulness of this result will depend on the extent to which we can compute the index group.

13.4 THE INDEX THEOREM FOR FOLIATIONS

higher-fol-ex **13.39 Example.** The above construction can be generalized as follows: let Γ be a discrete group whose classifying space $B\Gamma$ is a compact manifold, and let $E\Gamma$ be the universal cover of $B\Gamma$. Suppose that Γ also acts on some other compact manifold N. Then the 'balanced product' $E\Gamma \times_\Gamma N$ — that is, the quotient of $E\Gamma \times N$ by the diagonal Γ-action — is a compact manifold and it is foliated by the images of $E\Gamma \times \{y\}$, $y \in N$. If the action of Γ on N is free, the leaves are all diffeomorphic to the universal cover $E\Gamma$.

13.40 Example. A very significant example in noncommutative geometry has been the *irrational slope foliation* of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, in which the leaves are the images of the lines $y = \alpha x + c$ for some fixed irrational α. Because α is irrational, each leaf is diffeomorphic to $\mathbb{R}$ and winds densely around the torus. See Figure 13.2, which shows a portion of one leaf of the irrational slope foliation. This foliation does not come from any submersion (because its leaves are not compact).

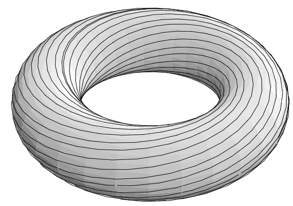

Figure 13.2 The irrational slope foliation. ir-slop-fig

Alain Connes' first version of the index theorem for foliated manifolds was analogous to the L^2-index theorem for coverings. The key concept here is that of an (invariant) *transverse measure* on a foliation.

13.41 Definition. A *transverse measure* on a foliated manifold (M, F) is a map which assigns to each flowbox U a Borel measure μ_U on the plaque set $\mathcal{P}(U)$, in such a way that if $U \subseteq V$ is an inclusion of flowboxes then μ_U is the restriction of μ_V to $\mathcal{P}(U)$.

Informally, a transverse measure is a measure on transversals that is invariant under holonomy.

13.42 Example. If F has a compact leaf L, then we can define a transverse measure on (M, F) by assigning a plaque with mass 1 if it belongs to L and with mass 0 otherwise. (This is called the *counting measure* associated to L.) In the case of the Reeb foliation, for example, it can be shown that the only transverse measures are scalar multiples of the counting measure associated to the unique compact leaf.

Let μ be a transverse measure on (M, F), and assume that M is compact and that F is oriented (that is, TF is oriented as a vector bundle over M). Let α be a p-form on M, where $p = \dim F$. Find a finite cover of M by flowboxes $\{U_i\}$, and choose a partition of unity $\{\phi_i\}$ subordinate to $\{U_i\}$. We can define a function on the plaque set $\mathcal{P}(U_i)$ by sending a plaque P to $\int_P \phi_i \alpha$ — notice that α is a p-form and P is an oriented p-manifold, so the integral is well-defined. Now define

$$C_\mu(\alpha) = \sum_i \int_{\mathcal{P}(U_i)} \left(\int_P \phi_i \alpha \right) d\mu_{U_i}(P).$$

13.43 Lemma. *The definition of* $C(\alpha)$ *is independent of the choices made in its construction.* C_μ *is a linear functional on the space of* p*-forms (a* p-current*). Moreover,* C_μ *is* closed*: that is,* $C_\mu(d\beta) = 0$ *for all* $(p-1)$*-forms* β. □

The proof of this lemma is elementary. The current C_μ is called the *Ruelle-Sullivan* current associated to the transverse measure μ. Because C_μ is closed it defines a linear functional on the de Rham cohomology group $H^p(M; \mathbb{R})$; that is, it defines a *homology class* $[C_\mu] \in H_p(M; \mathbb{R})$.

Using the Ruelle-Sullivan current we can define a trace on the groupoid algebra $C_c^\infty(G(M, F))$. Recall that the construction of the groupoid algebra makes implicit use of a *Haar system* on the groupoid. For the groupoid $G(M, F)$, a Haar system is simply a smooth family of Lebesgue measures on the leaves of F. Equivalently (since we are assuming that F is oriented) a Haar system is given by a p-form ω_F on M which restricts to a volume form on every leaf of F. Thus, from the transverse measure μ and the Haar system, we can construct an ordinary Borel measure ν on M by the formula

$$\int g d\nu = C_\mu(g\omega_F)$$

for a continuous function g on M.

Since M is the space of objects of the groupoid $G = G(M, F)$, there is a 'restriction to the diagonal' linear map

$$\delta\colon C_c^\infty(G) \to C^\infty(M).$$

Given a transverse measure μ, construct the linear functional ϕ_μ

$$k \mapsto \int \delta(k) d\nu = C_\mu(\delta(k)\omega_F)$$

on $C_c^\infty(G)$.

13.44 Lemma. *The functional ϕ_μ is tracial.*

Proof (sketch). Consider the case where the foliation is given by a product $M = F \times B$. In this case a transverse measure is simply a measure μ on B, and a Haar system is a family of Lebesgue measures λ_b on F, depending smoothly on $b \in B$. The groupoid algebra is the algebra of smooth functions on $F \times F \times B$, with composition law

$$k_1 \circ k_2(x, z, b) = \int k_1(x, y, b) k_2(y, z, b) d\lambda_b(y).$$

The functional ϕ_μ is given by the formula

$$\phi_\mu(k) = \iint k(x, x, b) d\lambda_b(x) d\mu(b).$$

Thus

$$\phi_\mu(k_1 \circ k_2) = \iiint k_1(x, y, b) k_2(y, x, b) d\lambda_b(y) d\lambda_b(x) d\mu(b),$$

and this is symmetrical in k_1 and k_2 by Fubini's theorem.

The general case is reduced to this one using foliation charts and partitions of unity. □

13.45 Lemma. *The functional ϕ_μ is normal.*

Proof. Exactly as in Lemma **??**. □

Extend ϕ_μ to an unbounded trace τ_μ on $C^*(M, F)$ according to the procedure of Section **??**. In turn, τ_μ defines a dimension function $\tau_{\mu*}\colon K(C^*(M, F)) \to \mathbb{R}$.

embed-tm **13.46 Exercise.** Let (M, F) be a foliation equipped with a transverse measure μ. According to Example **??**, a closed transversal W to F determines a homomorphism $K(W) \to K(C^*(M, F))$. Show that the diagram

$$\begin{tikzcd} K(W) \arrow[r] \arrow[dr] & K(C^*(M,F)) \arrow[d] \\ & \mathbb{R} \end{tikzcd}$$

commutes, where the right-hand vertical arrow is the dimension function $\tau_{\mu*}$ defined above and the left-hand vertical arrow is the dimension function defined by the restriction of the transverse measure to W (this induces a measure on W and therefore a trace on $C(W)$).

We can now state the *measured foliation index theorem*, which is the foliation analog of Atiyah's L^2 index theorem.

13.47 Theorem. *Let (M, F) be a compact foliated manifold equipped with a transverse measure μ, and let D be a leafwise elliptic operator on M. Denote by $\mathrm{Ind}(D)$ the longitudinal index of D in $K(C^*(M, F))$. Then*

$$\tau_{\mu*}(\mathrm{Ind}(D)) = (-1)^{\dim(F)} \langle [C_\mu], \mathrm{ch}(\sigma_D)\, \mathrm{Todd}(TF \otimes \mathbb{C}) \rangle$$

where C_μ denotes the Ruelle-Sullivan current associated to μ. □

Connes' first proof of this result used the heat equation method that we have discussed in Remark 12.48. The virtue of the heat equation method (when it works) is that it produces a *local formula* for an index. This is particularly useful in a situation (such as the foliation case) where local structure is simple although global structure may be extremely complicated. A later proof of the measured foliation index theorem, by Connes and Skandalis, derives it from the foliation index theorem that we have discussed in this chapter. The key idea is to use an embedding into Euclidean space to reduce the theorem to the result proved in Exercise 13.46.

13.48 Example. Consider the Reeb foliation, with μ being the counting measure associated to the compact (toral) leaf. The holonomy cover of this compact leaf is simply the universal cover. Restricting to the compact leaf thus gives a $*$-homomorphism from the C^*-algebra of the foliation to the C^*-algebra of the homotopy groupoid of the compact leaf, and μ_* factors through this homomorphism. We see that in this case the measured foliation index theorem reduces to Atiyah's L^2-index theorem for the compact leaf.

Using Proposition **??** we obtain a geometric result on foliations by surfaces.

ps-leaf **13.49 Proposition.** *Let F be an oriented 2-dimensional foliation of a compact manifold M, equipped with an invariant transverse measure μ. Suppose that F is given a Riemannian metric, and let Ω be the Gauss curvature 2-form of the leaves. If $\langle [C_\mu], \Omega \rangle > 0$ then the foliation has some compact leaves (indeed, the set of compact leaves has positive μ measure).*

Proof. We apply the measured index theorem to the leafwise de Rham operator: that is, the operator $D = d + d^*$ acting on the space of leafwise differential forms, graded by the degree of forms. The measured index is just a constant times $\langle [C_\mu], \Omega \rangle$ in this case. Since this is greater than 0, the positively graded part of the kernel of D must be nonzero, by Proposition **??**. Thus there must be some L^2 harmonic functions on the leaves (the spaces of harmonic functions and harmonic 2-forms are identified by Poincaré duality). But a complete Riemannian manifold (such as a leaf) admits L^2 harmonic functions if and only if it is compact. □

13.5 NOTES

The index theorem for families is due to Atiyah and Singer [**?**].

For the classical theory of foliations and many examples see the two-volume work by Candel and Conlon [**?**, **?**]. Foliation index theory was the first context in which Connes developed his noncommutative geometry, and it provides a rich source of ideas and examples. The basic reference for Connes' ideas about foliations and operator algebras is [**?**]. The general index theorem for foliations is proved in [**?**].

A proof of the index theorem for measured foliations which minimizes the use of C^*-algebraic ideas is in [**?**].

Chapter Fourteen

The Baum-Connes Conjecture

BCChapter

In the previous two chapters we showed that C^*-algebra K-theory plays a role in index theory by providing value groups for the analytic and topological indices of operators on covering spaces or foliations. In this final chapter we shall turn things around by using index theory to conjecturally describe, and in some cases actually calculate, C^*-algebra K-theory groups. We shall formulate the Baum-Connes conjecture, indicate several of its consequences, and describe some of the genuinely non-commutative geometric ideas which have led to the proof of the conjecture in certain instances.

14.1 THE ASSEMBLY MAP FOR COVERING SPACES

Let Γ be a discrete group and let W be a principal Γ-space over a smooth manifold M. In Section 12.5 we associated to W topological and analytic index maps

$$\mathrm{Ind}^t_\Gamma, \mathrm{Ind}^a_\Gamma \colon K(T^*M) \to K(C^*_\lambda(\Gamma)).$$

The main point in Section 12.5 was that these two maps are equal to one another. In this chapter we shall view either map as a means to construct elements in the K-theory group $K(C^*_\lambda(\Gamma))$. We should like to consider whether or not every class in $K(C^*_\lambda(\Gamma))$ is an index, and whether or not the only relations among these indices are consequences of relations among symbol classes in $K(T^*M)$. In other words, we should like to ask to what extent the index maps displayed above are isomorphisms.

A first issue that needs to be addressed is the obvious dependence of these questions on the choice of the manifold M and the principal Γ-space W over it. Our goal in this section is to deal with this point, at least for torsion-free groups Γ (the presence of torsion in Γ requires an additional idea, to be presented in the next section).

Principal Γ-manifolds form a category whose morphisms are the Γ-equivariant smooth maps. Since we shall want to keep track of quotient spaces, we shall think of morphisms of Γ-spaces as commuting diagrams

$$\begin{tikzcd} W_1 \arrow[r, "h"] \arrow[d, "\pi"'] & W_2 \arrow[d, "\pi"] \\ M_1 \arrow[r, "f"'] & M_2, \end{tikzcd}$$

in which f and h are continuous and h is Γ-equivariant (the vertical maps are the projections to the quotient spaces). We shall say that the map f of spaces is *covered* by the morphism h of principal spaces.

If f is an embedding of smooth manifolds, then there is an associated map

$$f_!\colon K(TM_1) \to K(TN_2),$$

as defined in Section 12.2. More or less as a result of its definition, the topological Γ-index is compatible with this map:

compat-higher

14.1 Proposition. *Associated to every morphism of principal Γ-spaces*

$$\begin{tikzcd} W_1 \arrow[r, "h"] \arrow[d] & W_2 \arrow[d] \\ M_1 \arrow[r, "f"'] & M_2 \end{tikzcd}$$

in which the map f is an embedding of manifolds there is a commuting diagram of index homomorphisms

$$\begin{tikzcd} K(TM_1) \arrow[r, "f_!"] \arrow[d, "\mathrm{Ind}^t_\Gamma"'] & K(TM_2) \arrow[d, "\mathrm{Ind}^t_\Gamma"] \\ K(C^*_\lambda(\Gamma)) \arrow[r, "="'] & K(C^*_\lambda(\Gamma)). \end{tikzcd}$$

Proof. This is a consequence of Theorem 12.10 (and, in the current context, the main step in the proof of that theorem). □

It will be convenient to slightly extend the construction of the "wrong-way" maps $f_!$ featured in the above lemma, as follows. Let $f\colon M_1 \to M_2$ be an arbitrary smooth map. Realize M_1 as an embedded submanifold of some euclidean space $\mathbb{R}^n$ and define an embedding $g\colon M_1 \to M_2 \times \mathbb{R}^n$ by means of the formula $g(x) = (f(x), x)$. Now define a K-theory map

$$f_!\colon K(TM_1) \to K(TM_2)$$

by means of the commutative diagram

$$\begin{tikzcd} K(TM_1) \arrow[r, "f_!"] \arrow[d, "="'] & K(TM_2) \arrow[d, "i_!"] \\ K(TM_1) \arrow[r, "g_!"'] & K(TM_2 \times T\mathbb{R}^n) \end{tikzcd}$$

in which the right-hand vertical map is the Bott periodicity isomorphism induced from the inclusion of a point into $T\mathbb{R}^n$. The definition agrees with our previous one if f is an embedding.

14.2 Proposition. *The map $f_!\colon K(TM_1) \to K(TM_2)$ defined above depends only on the homotopy class of f. The correspondence $f \mapsto f_!$ is a covariant functor.*

Proof. This is an easy consequence of the functoriality property for embeddings indicated in Lemma 12.4, together with the remarks made following our introduction of the K-theoretic topological index in Definition 12.5. □

Returning to principal spaces, if the smooth map $f\colon M_1 \to M_2$ is covered by a morphism h of principal Γ-spaces, then associated to the embedding

$$g\colon M_1 \to M_2 \times \mathbb{R}^n$$

defined above are embeddings of principal Γ-spaces

$$\begin{tikzcd} W_1 \arrow[r, "k"] \arrow[d] & W_2 \times \mathbb{R}^n \arrow[d] & W_2 \arrow[l, "i"'] \arrow[d] \\ M_1 \arrow[r, "g"'] & M_2 \times \mathbb{R}^n & M_2, \arrow[l, "i"] \end{tikzcd}$$

where $k(w) = (h(w), \pi(w))$ and where the maps i are obtained from the inclusion of 0 into $\mathbb{R}^n$. Applying Proposition 14.1 to this diagram we obtain the following strengthening of that result:

compat-higher2 **14.3 Proposition.** *Associated to* every *smooth morphism of principal* Γ*-spaces*

$$\begin{tikzcd} W_1 \arrow[r, "h"] \arrow[d] & W_2 \arrow[d] \\ M_1 \arrow[r, "f"'] & M_2 \end{tikzcd}$$

is a commuting diagram of topological index homomorphisms

$$\begin{tikzcd} K(TM_1) \arrow[r, "f_!"] \arrow[d, "\mathrm{Ind}^t_\Gamma"'] & K(TM_2) \arrow[d, "\mathrm{Ind}^t_\Gamma"] \\ K(C^*_\lambda(\Gamma)) \arrow[r, "="'] & K(C^*_\lambda(\Gamma)). \end{tikzcd}$$

□

Proposition 14.3 gives us the means to combine all the groups $K(TM)$ into one object, and so formulate an index homomorphism whose target is $K(C^*_\lambda(\Gamma))$ and whose source is independent of a choice of smooth manifold M or principal Γ-space over it.

ig1-def **14.4 Definition.** Let Γ be a torsion-free discrete group. A Γ-*symbol* is a triple (M, W, α) consisting of a principal Γ-space W over a smooth manifold M and a class α in the group $K(TM)$.

14.5 Remark. Our reason for limiting Definition 14.4 to torsion-free groups is that a modified definition will be required in order to fully account for all the classes in the group $K(C^*_\lambda(\Gamma))$ in geometric terms when Γ has elements of finite order. As we mentioned earlier, the modification will be presented in the next section.

14.6 Definition. Let Γ be a torsion-free discrete group. The *topological* K-*group* $K^{\mathrm{top}}(\Gamma)$ is the abelian group generated by all isomorphism classes of Γ-symbols (M, W, α), subject to the following relations:

(a) If W is a principal Γ-space over M, and if $\alpha_1, \alpha_2 \in K(TM)$, then

$$[M, W, \alpha_1] + [M, W, \alpha_2] = [M, W, \alpha_1 + \alpha_2]$$

in the group $K^{\text{top}}(\Gamma)$.

(b) If $f\colon M_1 \to M_2$ is a smooth map that is covered by a morphism of Γ-spaces $h\colon W_1 \to W_2$, and if $f_!(\alpha_1) = \alpha_2$, then

$$[M_1, W_1, \alpha_1] = [M_2, W_2, \alpha_2]$$

in the group $K^{\text{top}}(\Gamma)$.

14.7 Remarks. The zero element of $K^{\text{top}}(\Gamma)$ is represented by any cycle of the form $(M, W, 0)$. By taking disjoint unions of principal Γ-spaces (and noting that these disjoint unions are themselves principal Γ-spaces) we find that every class in $K^{\text{top}}(\Gamma)$ can be represented by a single Γ-symbol (M, W, α). Such a symbol represents the zero element of $K^{\text{top}}(\Gamma)$ if and only if there is some smooth map f, covered by a morphism of principal spaces, such that $f_!(\alpha) = 0$.

We can now formulate a version of the index map for covering spaces that gathers together all index problems associated to principal Γ-manifolds.

tf-assembly-def

14.8 Definition. Let Γ be a torsion-free discrete group. The *assembly map*

$$\mu\colon K^{\text{top}}(\Gamma) \to K(C^*_\lambda(\Gamma))$$

is the homomorphism that associates to a class $[M, W, \alpha]$ in $K^{\text{top}}(\Gamma)$ the image of $\alpha \in K(TM)$ under the index homomorphism

$$\text{Ind}^t_\Gamma\colon K(TM) \to K(C^*_\lambda(\Gamma))$$

associated to the principal Γ-space W.

14.9 Remark. The assembly map is well-defined by virtue of Proposition 14.3 and the definition of $K^{\text{top}}(\Gamma)$. Its range is the subgroup of $K(C^*_\lambda(\Gamma))$ generated by the index homomorphisms associated to all principal Γ-spaces over smooth manifolds without boundary.

14.10 Example. Suppose that Γ is the trivial one-element group.

14.2 THE ASSEMBLY MAP FOR PROPER ACTIONS

If Γ is a group that contains elements of finite order, then the construction outlined in the previous section certainly will not exhaust all the elements in $K(C^*_\lambda(\Gamma))$. Indeed, suppose H is a non-trivial finite subgroup of Γ. The element

$$p = \frac{1}{|H|} \sum_{h \in H} [h] \in C^*_\lambda(\Gamma).$$

is a projection and its trace is $|H|^{-1}$. But according to the L^2-index theorem, the trace of any K-theory class in the range of the assembly map is an integer.

To account geometrically for the K-theory classes determined by p and similar elements, we need to broaden the notion of principal Γ-space, as follows:

14.11 Definition. Let Γ be a discrete group. A topological space W equipped with an action of Γ is a *proper* Γ*-space* if for every point $w \in W$ there exists

(a) a finite subgroup $H \subseteq \Gamma$,

(b) a Γ-invariant open neigborhood U of $w \in W$, and

(c) a continuous equivariant map from U onto Γ/H.

A proper Γ-space is Γ*-compact* if the quotient space W/Γ is compact.

14.12 Remark. In the context of manifolds (our primary and nearly exclusive interest), we shall always assume that the action of Γ is via diffeomorphisms, as we did for principal actions.

If Γ is torsion-free, then there is no difference between the new notion of "proper Γ-space" and the notion of "principal Γ-space." However in the presence of torsion the new notion gives what we need to account for K-theory classes associated to finite subgroups of Γ. To see that this is so, we need to extend index theory from principal manifolds to proper manifolds.

As a first step, we shall associate to a Γ-equivariant elliptic operator on a Γ-compact proper Γ-manifold M an analytic index in the K-theory group $K(C^*_\lambda(\Gamma))$. One way to do this is to copy the approach that we took in Section 12.5. If D acts on sections of a Γ-equivariant, graded hermitian bundle S on M, and if D is an equivariant, formally self-adjoint, odd-graded, elliptic first-order partial differential operator acting sections of S, then D is essentially self-adjoint and if $f \in \mathcal{S}$, then the operator $f(D)$ lies in the C^*-algebra $\mathcal{K}(\Gamma)$ generated by the Γ-equivariant, Γ-compactly supported smoothing operators.

14.13 Definition. Let W be a proper Γ-manifold. We shall denote by $C^*_\lambda(\Gamma, W)$ the C^*-algebra of the crossed product groupoid $\Gamma \ltimes W$.

For the purposes of index theory, Γ-compact proper Γ-manifolds W are analogous to principal Γ-manifolds over closed manifolds M, and the K-theory group $K(C^*_\lambda(\Gamma, T^*W))$ is analogous to the K-theory group $K(T^*M)$.

Let S be a Γ-equivariant Hermitian vector bundle over a proper Γ-manifold W and let D be a Γ-equivariant first-order, formally self-adjoint, elliptic partial differential operator acting on sections of S.

$$\boxed{\text{symbol gives element of } K(C^*_\lambda(\Gamma, T^*W))}$$

14.14 Remark. If W admits a Γ-equivariant spin^c-structure, then the symbol construction exhausts the K-theory group $K(C^*(\Gamma, T^*W))$.

$$\boxed{\text{index gives element of } K(C^*_\lambda(\Gamma))}$$

14.15 Definition. Let Γ be a discrete group. A Γ-symbol is a pair (W, α) consisting of a smooth proper Γ-manifold W and a class $\alpha \in K(C^*_\lambda(\Gamma, T^*W))$.

$\boxed{\text{wrong way functoriality}}$

14.16 Definition. Let Γ be a discrete group. The *topological* K*-group* $\mathsf{K}^{\mathrm{top}}(\Gamma)$ is the abelian group generated by all isomorphism classes of Γ-symbols (W, α), subject to the following relations:

(a) If W is a proper Γ-space, and if $\alpha_1, \alpha_2 \in \mathsf{K}(C^*_\lambda(G, TW))$, then

$$[W, \alpha_1] + [W, \alpha_2] = [W, \alpha_1 + \alpha_2]$$

in the group $\mathsf{K}^{\mathrm{top}}(\Gamma)$.

(b) If $f\colon W_1 \to W_2$ is a smooth morphism proper of Γ-spaces, and if $f_!(\alpha_1) = \alpha_2$, then

$$[W_1, \alpha_1] = [W_2, \alpha_2]$$

in the group $\mathsf{K}^{\mathrm{top}}(\Gamma)$.

We can now repeat Definition 14.8 almost verbatim so as to obtain an assembly map for an arbitrary discrete group.

14.17 Definition. Let Γ be a discrete group. The *assembly map*

$$\mu\colon \mathsf{K}^{\mathrm{top}}(\Gamma) \to \mathsf{K}(C^*_\lambda(\Gamma))$$

is the homomorphism that associates to a class $[W, \alpha]$ in $\mathsf{K}^{\mathrm{top}}(\Gamma)$ the image of $\alpha \in \mathsf{K}(C^*_\lambda(\Gamma, T^*W))$ under the index homomorphism

$$\mathrm{Ind}_\Gamma\colon \mathsf{K}(C^*_\lambda(\Gamma, T^*W)) \to \mathsf{K}(C^*_\lambda(\Gamma)).$$

14.18 Example. Suppose that Γ is a finite group.

14.3 THE BAUM-CONNES CONJECTURE

14.19 Baum-Connes Conjecture. *If Γ is any discrete group, then the assembly map*

$$\mu\colon \mathsf{K}^{\mathrm{top}}(\Gamma) \to \mathsf{K}(C^*_\lambda(\Gamma))$$

is an isomorphism of abelian groups.

The kernel of the assembly map consists of all symbol classes $[W, \alpha]$ that have vanishing index. An element $[W, \alpha]$ of the kernel is non-zero if α is geometrically non-trivial in the sense that $f_!(\alpha)$ is never zero, for any f. So the Baum-Connes conjecture (for torsion-free discrete groups) amounts to the assertion that every element of $\mathsf{K}(C^*_\lambda(\Gamma))$ is an index, and that the only relations among these indices are those which are implied by the compatibility of the index with the K-theory maps associated to embeddings of proper Γ-manifolds $f\colon W_1 \to W_2$.

The conjecture was first formulated by Baum and Connes on the basis of rather modest evidence. However good progress on the conjecture has since been made for several classes of examples.

14.20 Definition. Let Γ be a group. A *matrix coefficient function* on Γ is any function $\phi\colon \Gamma \to \mathbb{C}$ of the form $\phi(g) = \langle v, \pi(g)v \rangle$, where π is a unitary representation of Γ on a Hilbert space H, and v is a unit vector in H.

14.21 Definition. A Γ group has the *Haagerup property* if the constant function 1 on Γ is a pointwise limit of c_0-matrix coefficient functions.

14.22 Theorem. *If Γ is a discrete group with the Haagerup property, then the Baum-Connes assembly map*

$$\mu\colon K^{\text{top}}(\Gamma) \longrightarrow K(C^*_\lambda(\Gamma))$$

is an isomorphism.

Suppose that M is compact and carries a Riemannian metric of strictly negative sectional curvature. A classical theorem of Hadamard in differential geometry then states that the universal cover $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^n$, and in particular it is contractible. Thus M is a compact $B\Gamma$. The validity of the Baum–Connes conjecture in this case was established quite recently by Lafforgue [**?**, **?**]; it is a difficult and subtle result.

14.23 Theorem. *Suppose that M is compact and carries a Riemannian metric of strictly negative sectional curvature. The assembly map*

$$\mu\colon K^{\text{top}}(\Gamma) \to K(C^*_\lambda(\Gamma))$$

is an isomorphism.

. . . significance . . . property T . . .

Another very significant aspect of Lafforgue's approach is that it applies to quite broad classes of non-discrete groups, with consequences for the representation theory of these groups. See Section 14.7.

14.4 UNIVERSAL SPACES

The following notion of universal space allows us to simplify $K^{\text{top}}(\Gamma)$ in important special cases. It also helps point the way toward generalizations of the Baum-Connes conjecture to classes of groups beyond discrete groups.

14.24 Definition. A proper Γ-manifold W is *universal* if and only if for every proper Γ-manifold X there exists a morphism of proper Γ-spaces from X to W, and this morphism is unique up to equivariant homotopy.

14.25 Remark. Clearly we can make a similar definition within the broader context of proper Γ-*spaces*. We have refrained from doing so immediately in order to avoid some niceties in general topology (having to do with paracompactness and related notions).

14.26 Example. If Γ is a finite group, then the one-point space is a proper Γ-manifold and it is clearly universal.

14.27 Proposition. *Let Γ be a discrete group. If W is a universal proper Γ manifold, then the map from $K(C^*(G, TW))$ into $K^{top}(\Gamma)$ sends $\alpha \in K(C^*(G, TW))$ to $[W, \alpha] \in K^{top}(\Gamma)$, is an isomorphism.*

Proof. The inverse map is defined by sending the class $[M_1, W_1, \alpha]$ in $K^{top}(\Gamma)$ to $f_!(\alpha) \in K(TM)$, where $f\colon M_1 \to M$ is covered by a map of principal Γ-spaces from W_1 to W. Note that f is unique up to homotopy, and because of this our prescription for the inverse is compatible with the defining relations for $K^{top}(\Gamma)$. □

In order to apply the proposition we need some means of identifying universal proper Γ-spaces. In fact there is a simple criterion.

universal-thm

14.28 Theorem. *A proper Γ-space W is universal if and only if the two projections from $W \times W$ to W are Γ-equivariantly homotopic to one another, and in addition each finite subgroup of Γ has a fixed point in W.*

Proof. □

14.29 Example. The manifold $W = \mathbb{R}^n$, equipped with the natural translation action of the group $\Gamma = \mathbb{Z}^n$, is a universal principal space. The two projection maps from $W \times W$ to W are homotopic via the straight line homotopy

$$(w_0, w_1) \mapsto (1-s)w_0 + sw_1 \qquad (s \in [0,1]).$$

It follows that $K^{top}(\Gamma) = K(T\mathbb{T}^n)$, where $\mathbb{T}^n$ is the torus $\mathbb{R}^n/\mathbb{Z}^n$.

symm-space-ex

14.30 Example. Let G be a connected, linear semisimple (or reductive) Lie group. It may be realized as a closed subgroup of some $GL(n, \mathbb{R})$ that is closed under the operation of matrix transpose, and having done so, the Lie algebra of G admits a *Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ in which the first summand consists of skew-symmetric matrices and is the Lie algebra of a maximal compact subgroup K, while the second summand consists of symmetric matrices. The homogeneous space $W = G/K$ identifies with $\exp(\mathfrak{p})$ and is therefore contractible. Moreover it admits a complete, G-invariant Riemannian metric, given on tangent vectors at the identity coset $e \in W$ by the formula

$$\|X\|^2 = \mathrm{Tr}(X^2) \qquad (X \in \mathfrak{p} = T_e W).$$

This metric has nonpositive curvature and as a result, a well-known theorem in Riemannian geometry implies that between any two points $w_0, w_1 \in W$ there is a unique geodesic $\phi_{w_1 w_2}$ such that $\phi_{w_0 w_1}(0) = w_0$ and $\phi_{w_0 w_1} = w_1$. The two projection maps from $W \times W$ to W are therefore homotopic via the geodesic homotopy

$$(w_0, w_1) \mapsto \phi_{w_0 w_1}(s) \qquad (s \in [0,1]).$$

Moreover by another well-known geometric result about nonpositively curved manifolds, every compact group of isometries of W has a fixed point. It follows from Theorem 14.28 that if Γ is a discrete subgroup of G, then W, equipped with the left-translation action of Γ, is a universal proper space.

14.31 Remark. In fact it is known that if G is *any* connected Lie group, then the homogeneous space G/K is universal for any discrete subgroup of G.

14.32 Conjecture. *If W is a universal proper Γ-manifold, then the index map*

$$\mathrm{Ind}_\Gamma \colon K(C^*_\lambda(\Gamma, T^*W)) \longrightarrow K(C^*_\lambda(\Gamma))$$

is an isomorphism. In particular, if Γ is torsion-free and W is a universal principal Γ-manifold with quotient M, then the index map

$$\mathrm{Ind}_\Gamma \colon K(T^*M) \longrightarrow K(C^*_\lambda(\Gamma))$$

is an isomorphism.

14.5 DUALITY

duality-sec

14.33 Definition. Let V be a finite-dimensional real vector space. A *lattice* in V is the abelian group generated by a basis for V.

If Γ is a lattice in V, then as we observed in Example **??**, the vector space V, equipped with the translation action of Γ, is a universal principal Γ-space. As a result,

$$K^{\mathrm{top}}(\Gamma) \cong K(TM)$$

where M is the torus V/Γ.

14.34 Definition. Let Γ be a lattice in a finite-dimensional vector space V. Let V^* be the dual vector space to V. Define

$$\Gamma^* = \left\{ \phi \in V^* \mid \phi[\Gamma] \subseteq \mathbb{Z} \right\}.$$

This is a lattice in V^*, called the *dual* of Γ. Let $M^* = V^*/\Gamma^*$. This is the *dual* of the torus $M = V/\Gamma$.

14.35 Lemma. *The C^*-algebra $C^*_\lambda(\Gamma)$ is isomorphic to the C^*-algebra of continuous functions on $M^* = V^*/\Gamma^*$ via the correspondence $a \mapsto \hat{a}$, where*

$$a = \sum a_\gamma[\gamma] \quad \Rightarrow \quad \hat{a}(\phi) = \sum a_\gamma e^{2\pi i \phi(\gamma)}$$

It follows, therefore, that the assembly map can be viewed as a homomorphism

$$\mu \colon K(TM) \longrightarrow K(M^*).$$

14.36 Definition. . . . of Poincaré line bundle

Let us investigate higher index theory for the free abelian group $\Gamma = \mathbb{Z}^n$. In this case the compact manifold $M = \mathbb{T}^n$ will serve as a model for $B\Gamma$. Since Γ is abelian, $C^*_r(\Gamma)$ is an abelian C^*-algebra and it is thus of the form $C(M^*)$, where M^* is some compact space. In fact, it is not hard to see that M^* is again an n-torus. The universal index map is then a map from the K-theory of the torus M (or rather of its tangent bundle) to the K-theory of the torus M^*. Notice that although M and M^* are diffeomorphic, there is no canonical identification between them. Instead, each one is a sort of 'Fourier transform' of the other.

14.37 Theorem. *The assembly map*

$$\mu\colon K(TM) \longrightarrow K(M^*)$$

is given by the formula

$$\mu(\alpha) = \mathrm{Ind}(p^*(\alpha)\cdot[P]),$$

and it is an isomorphism. □

This allows us to use the index theorem for families to investigate the Baum–Connes conjecture for $\mathbb{Z}^n$. In fact, the Baum–Connes conjecture is true in this case.

14.38 Remark. As we noted above, the full K-theoretical form of the Baum–Connes conjecture for $\mathbb{Z}^n$ can be proved in the same way from the K-theoretic families index theorem. It is more elegant, however, to make use of the implicit symmetry of the construction (compare Exercise **??**): the same line bundle $\mathcal{L}$ also allows us to define a 'dual index map'

$$K(TM^*) \to K(M).$$

It can be shown that the index and the dual index maps are essentially inverse to one another (and therefore that they are both isomorphisms). In a more precise form, this is known as *Mukai duality*; it is of interest in physics.

14.39 Theorem. *The composition*

$$K(M) \xrightarrow{\mu} K(M^*) \xrightarrow{\mu^*} K(M)$$

of the assembly maps for Γ *and* Γ^* *is the identity map on* $K(M)$.

14.6 CONNECTIONS TO GEOMETRY

14.40 Theorem. *Suppose that* Γ *is a discrete subgroup of a connected Lie group. The assembly map*

$$\mu\colon K^{\mathrm{top}}(\Gamma) \longrightarrow K(C^*_\lambda(\Gamma))$$

is a split injection.

14.41 Remark. This result and its proof (see Section 14.5) has served as a model for a variety of others. For example, the same result is now known to hold for arbitrary subgroups (discrete or not) of connected Lie groups, and in particular for arbitrary linear groups.

...we shall assume for the rest of this section that Γ *is a torsion-free discrete group.*

14.42 Proposition. *If* W *is a principal* Γ*-manifold over a closed Riemannian spin-manifold* M *with positive scalar curvature, then the index of the Dirac operator in* $K(C^*_\lambda(\Gamma))$ *is zero.*

Proof. For the ordinary index, this is simply Lichnerowicz' vanishing theorem (8.34). But the same argument in fact works for the higher index. We apply the Bochner-Lichnerowicz formula to the essentially self-adjoint operator $\widetilde{D}$ which is the lift of the Dirac operator to $\widetilde{M}$. The scalar curvature of $\widetilde{M}$ is the lift of the scalar curvature of M, and is therefore bounded from below by a strictly positive constant. The Bochner-Lichnerowicz argument now shows that $\widetilde{D}^2$ is bounded from below by a positive constant, which is to say that there is an interval $(-\varepsilon, \varepsilon)$ which does not meet the spectrum of $\widetilde{D}$. But now the family of $*$-homomorphisms $\mathcal{S} \to C^*_r(G)$ defined by

$$f \mapsto f(s^{-1}D), \quad s \in [0,1]$$

gives a homotopy between the homomorphism defining the higher idnex (when $s = 1$) and the zero homomorphism (when $s = 0$). $\square$

14.43 Theorem. *If the Baum-Connes assembly map for Γ is injective, then there is no Riemannian metric of positive scalar curvature on M.*

Proof. Since the assembly map is injective, it cannot have the symbol of the Dirac operator (which is the Thom class for $K^*(TM)$) in its kernel. $\square$

14.44 Proposition. *The index of the signature operator is an oriented homotopy invariant.*

14.45 Theorem. *If the Baum-Connes assembly map for Γ is injective, then each higher signature is an oriented homotopy invariant.*

14.7 CONNECTIONS TO REPRESENTATION THEORY

rep-sec

Connections to representation theory arise when the Baum-Connes conjecture is extended from discrete groups to topological groups. The extension has been carried out for arbitrary second countable, locally compact Hausdorff groups. But for groups such as p-adic groups that typically act only trivially on smooth manifolds, more machinery is required to carry out the extension than we are prepared to survey here. For that reason we shall confine our attention to Lie groups.

In fact *we shall assume for the rest of the section that G is a connected, linear semisimple group*. The results below could be stated and proved in greater generality, but our account is made simpler by making these assumptions throughout.

We noted in Example 14.30 that the symmetric space G/K is a universal principal manifold. Copying what we did for discrete groups, we shall therefore formulate the Baum-Connes conjecture for G as follows:

ck-conj1

14.46 Conjecture. *Let K be a maximal compact subgroup of G, and let W be the symmetric space G/K. The index homomorphism*

$$\mathrm{Ind}_G \colon K(C^*_\lambda(G, T^*W)) \longrightarrow K^*(C^*_\lambda(G))$$

is an isomorphism of abelian groups.

This is consistent with the more elaborate formulation of the Baum-Connes conjecture for general locally compact groups.

Conjecture 14.46 can be simplified by introducing and analyzing Dirac-type operators on the manifold $W = G/K$, as follows. Recall from Example 14.30 that for a semisimple group there is a natural K-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and K-invariant inner product on $\mathfrak{p}$. The vector space $\mathfrak{p}$ can be viewed as the tangent space of W at the identity coset, and the inner product on $\mathfrak{p}$ determines a complete, G-invariant Riemannian metric on W.

14.47 Definition. A *Dirac-type symbol* for the pair (G, K) is a Dirac-type symbol in the sense of Definition 8.1 for the cotangent bundle of the complete Riemannian bundle $W = G/K$ in which the hermitian $\mathbb{Z}/2$-graded bundle S over W is G-equivariant, as is the Clifford multiplication action $c\colon T^*W \to \mathrm{End}(S)$.

max-compact-formulation

14.48 Remark. Because of its G-equivariance, a Dirac-type symbol for (G, K) is in fact determined by the fiber $\mathfrak{s}$ of the bundle S at the identity coset in $W = G/K$. Thus we could equally well have defined a Dirac-type symbol to consist of a finite-dimensional hermitian vector space $\mathfrak{s}$ equipped with:

(a) a Clifford multiplication action $c\colon \mathfrak{p} \to \mathrm{End}(\mathfrak{s})$, and

(b) a unitary action $\tau\colon K \to \mathrm{Aut}(\mathfrak{s})$ that is compatible with Clifford multiplication in the sense that $c(\mathrm{Ad}_k(P)) = \tau(k)c(P)\tau(k)^{-1}$ for all $k \in K$ and $P \in \mathfrak{p}$.

The module of smooth sections of S identifies with the space

$$\left\{\xi\colon G \xrightarrow{\text{smooth}} \mathfrak{s} \,\middle|\, f(gk) = \tau(k)^{-1}\xi(g), \ \ \forall g \in G \ \forall k \in K\right\}$$

or equivalently

$$C^\infty(W, S) \cong \left[\mathfrak{s} \otimes C^\infty(G)\right]^K,$$

and under this identification an associated Dirac-type operator is given by the formula

$$D = \sum_{i=1}^{n} c(P_i) \otimes P_i.$$

Here $\{P_i\}$ is an orthonormal basis for $\mathfrak{p}$ and each P_i acts on $C^\infty(G)$ as a left-invariant vector field. This explicit form is very convenient for computations (and henceforth we shall refer to it as *the* associated Dirac operator).

Following the approach we have taken in the rest of this book, and in particular in Chapter 8, we shall now focus on the even dimensional case.

14.49 Definition. Assume that $W = G/K$ is even-dimensional and fix an orientation on the vector space $\mathfrak{p}$. The *spin-module* $R_{\mathrm{spin}}(G, K)$ is the Grothendieck group of positively oriented Dirac symbols for the pair (G, K) (see Definition 8.13).

Using Remark 14.48 we can now recast the left-hand side of the Baum-Connes conjecture in concrete terms involving the representation theory of K. The following result is a K-equivariant formulation of the Bott periodicity theorem; we shall not give the proof, although one could be given by making minor adaptations to the argument we presented in Chapter **??**.

14.50 Proposition. *The homomorphism*

$$\mu\colon R_{\mathrm{spin}}(G,K) \longrightarrow K(C^*_\lambda(G,T^*W))$$

that associates to a Dirac-type symbol the symbol its class in K*-theory is an isomorphism of abelian groups.* □

This of course leads to the following reformulation of the Baum-Connes conjecture; it agrees with an earlier conjecture of Connes and Kasparov.

14.51 Conjecture. *Let* G *be a connected unimodular Lie group, let* K *be a maximal compact subgroup of* G*, and let* W *be the homogeneous space* G/K*. The homomorphism*

$$\mu\colon R_{\mathrm{spin}}(G,K) \longrightarrow K(C^*_\lambda(G))$$

that associates to a Dirac-type symbol the index of its associated Dirac operator is an isomorphism of abelian groups.

14.52 Proposition. *If* π *is an irreducible unitary representation of* G*, then each irreducible unitary representation of* K *occurs with at most finite multiplicity in the restriction of* π *to* K*.* □

14.53 Corollary. *If* π *is an irreducible tempered unitary representation of* G*, then the image of the associated representation of the* C^**-algebra* $C^*_\lambda(G)$ *is the* C^**-algebra of compact operators.*

Proof. □

14.54 Definition. An irreducible tempered unitary representation of G is a *discrete series representation* if the singleton set it determines in the tempered dual is both open and closed.

14.55 Proposition. *Let* G *be a linear semisimple group. If* S *is a Dirac symbol and* π *is a discrete series representation, then*

$$\big\langle \mu(S),\pi\big\rangle_G = \dim_{\mathbb{C}}\big(\mathrm{Hom}_K(S,H_\pi)\big) = \big\langle \mu^*(\pi),S\big\rangle_K.$$

14.56 Theorem. *If* π *is any discrete series representation of* G*, then there is a unique irreducible Dirac symbol* S *such that* $\mathrm{Ind}(D_S) = [\pi]$.

Proof. □

14.57 Definition. An irreducible unitary representation π of G is *square-integrable* if any one of the following equivalent conditions holds:

(a) Every matrix coefficient function $\langle v,\pi(g)w\rangle$ is a square-integrable function on G.

(b) The irreducible representation π may be realized as a subrepresentation of the regular representation.

(c) The Plancherel measure of the singleton set in the dual of G determined by π is nonzero.

14.58 Lemma. *Let S be a Dirac-type symbol for (G, K) and assume that the index in $K(C^*_\lambda(G))$ of the associated Driac-type operator is the class of a discrete series representation π. Then*

$$\dim_G(\ker_+(D_S)) - \dim_G(\ker_-(D_S)) = \deg(\pi).$$

That is, the L^2-index of D_S is equal to the formal degree of π. □

14.59 Remark. The spin module is in part so-named because it is a module over the representation ring of $K \ldots c_V(P_i) = c(P_i) \otimes I \colon S \otimes V \to S \otimes V$.

The adjoint action of K on $\mathfrak{p}$ can be viewed as a homomorphism from K to $SO(\mathfrak{p})$, and for index-theoretic purposes it is important to determine whether or not this homomorphism lifts to the spin double cover of the special orthogonal group, as in the following diagram:

$$\begin{tikzcd} & \mathrm{Spin}(\mathfrak{p}) \arrow[d] \\ K \arrow[ur, dashed] \arrow[r, "\mathrm{Ad}"'] & SO(\mathfrak{p}) \end{tikzcd}$$

14.60 Theorem. *Assume that the hermitian space V_λ carries an irreducible representation of K with highest weight λ. If D_λ is the twisted Dirac operator associated to V then*

$$\dim_G(\ker_+(D_\lambda)) - \dim_G(\ker_-(D_\lambda)) = \prod \frac{\langle \lambda, \alpha \rangle}{\langle \rho, \alpha \rangle},$$

where the product is over a set of simple positive roots for G and ρ is the associated half-sum of positive roots.

A very significant feature of Lafforgue's work is that it applies to the situation of reductive Lie groups.

14.8 GENERALIZATIONS OF THE BAUM-CONNES CONJECTURE

Let X be a smooth manifold equipped with a smooth action of a discrete group Γ, and let W be a principal Γ-space over a smooth manifold M. In Section **??** we constructed topological and analytic index maps

$$\mathrm{Ind}_{\Gamma, X} \colon K(T^*W \times_\Gamma X) \longrightarrow K(C^*_\lambda(\Gamma, X))$$

and proved them to be equal to one another.

14.61 Definition. Let Γ be a discrete, torsion-free group. The topological K-group $K^{\mathrm{top}}(\Gamma, X)$ is the abelian group generated by all symbols (W, α) where W is a principal Γ-manifold and $\alpha \in K(T^*W \times_\Gamma X)$, subject to the relations

(a) If W is a principal Γ-manifold, and if $\alpha_1, \alpha_2 \in K(T^*W \times_\Gamma X)$, then

$$[W, \alpha_1] + [W, \alpha_2] = [W, \alpha_1 + \alpha_2]$$

in the group $K^{\mathrm{top}}(\Gamma, X)$.

(b) If $f\colon W_1 \to W_2$ is a smooth morphism principal of Γ-manifolds, and if $f_!(\alpha_1) = \alpha_2$, then

$$[W_1, \alpha_1] = [W_2, \alpha_2]$$

in the group $K^{top}(\Gamma, X)$.

14.62 Definition. The *assembly map*

$$\mu\colon K^{top}(\Gamma, X) \to K(C^*_\lambda(\Gamma, X))$$

is the homomorphism that associates to a class $[W, \alpha]$ in $K^{top}(\Gamma, X)$ the image of $\alpha \in K(TW \times_\Gamma X)$ under the index homomorphism

$$\mathrm{Ind}^t_\Gamma\colon K(TW \times_\Gamma X) \to K(C^*_\lambda(\Gamma, X)).$$

14.63 Example. Suppose that $\Gamma = \mathbb{Z}$ and that X is the unit circle, equipped with the rotation action

$$n \cdot z = e^{2\pi i \theta n} z,$$

where θ is a fixed element of $\mathbb{R}$.

14.64 Example. Simple, projectionless C^*-algebras

14.65 Generalized Baum-Connes Conjecture. *If Γ is any torsion-free group and X is any smooth Γ-manifold, then the assembly map*

$$\mu\colon K^{top}(\Gamma, X) \to K(C^*_\lambda(\Gamma, X))$$

is an isomorphism of abelian groups.

We have seen that one can interpret various kinds of generalized index theorems as a manifestation of a correspondence between the topology and the analysis of objects associated to smooth groupoids, expressed by the Baum-Connes conjecture. This method is extremely powerful but it has recently become clear that it also has certain limitations. In fact, there are examples of foliations for which the Baum-Connes conjecture fails. This is not the place to go into detail about the construction of these examples, but the basic idea is rather simple and involves a fundamental point which has already appeared both in our discussion of tensor products (Chapter 2) and in our discussion of group C^*-algebras (Chapter 12).

The point is that in general there may be *more than one way* to complete a given $*$-algebra to a C^*-algebra. The group (or groupoid) algebras appearing in Baum-Connes conjecture use completions relative to the regular representation of that group (or groupoid). The disadvantage of these completions is that they are not functorial: if $\phi\colon G \to H$ is a group homomorphism, the representation of G induced by ϕ from the regular representation of H may be entirely unrelated to the regular representation of H itself. Indeed, it is possible to give specific examples (related to Kazhdan's property T) where this lack of functoriality can be detected and analyzed.

14.66 Definition. A discrete group Γ has *property* T if there is a projection $p \in C^*(\Gamma)$ with the property that in each unitary representation p acts as the orthogonal projection onto the G-fixed vectors. The element p is called the *Kazhdan projection*.

14.67 Lemma. *If Γ has property T, then any net of matrix coefficient functions that converges pointwise to 1 in fact converges uniformly to 1.*

Proof. Suppose that $\phi_\alpha(g) = \langle v_\alpha, \pi_\alpha(g)v_\alpha\rangle$. The function ϕ_α extends to a functional on $C^*(G)$ using the formula

$$\phi_\alpha(f) = \langle v_\alpha, \pi_\alpha(f)v_\alpha\rangle \qquad (f \in C^*(G)),$$

and if p is the Kazhdan projection, then $\phi_\alpha(p) \to 1$. It follows that $\|\pi_\alpha(p)v_\alpha\| \to 1$, or in other words that v_α minus its orthogonal projection onto the G-fixed vectors for the representation π_α converges to zero. The lemma follows easily from this. □

The lemma shows that no infinite property T group has the Haagerup property.

On the other hand, the left hand side of the conjecture — the K-theory of BG — involves no analytic questions about representations and is completely functorial. Thus the conjecture itself predicts that $K(C^*_r(G))$ should depend functorially on G, but this would be a functoriality for which one could not give an analytic explanation. The counterexamples that are now known to various versions of the Baum-Connes conjecture use constructions based on property T to show that no such inscrutable functoriality can exist.

Such constructions deepen the mystery surrounding the Baum-Connes conjecture. Since we now know that it is not universally true, the enormously wide scope of its validity becomes still more intriguing.

14.9 NOTES

The relationship between the higher index and the families index, when Γ is abelian, was worked out in Lusztig's thesis [**?**]. There is a nice account of Mukai duality in Chapter 3 of [**?**].

The Baum–Connes conjecture appears first in [**?**] and the definitive account of the conjecture is [**?**]. Kasparov had independently arrived at similar ideas, motivated partly by problems in differential topology where an analog of the Baum-Connes assembly map makes an appearance. There are various counterparts to the Baum–Connes conjecture in this purely topological context. For the history see [**?**].

For counterexamples to various forms of the Baum-Connes conjecture see [**?**].

Chapter Fifteen

Appendix: Bivariant Theories

15.1 A FUNCTORIAL PROPERTY

funct-prop-sec

We saw in Section **??** that if $\mathcal{E}$ is a graded Hilbert module over A, then a graded homomorphism $\phi\colon \mathcal{S} \to \mathcal{K}(\mathcal{E})$ gives rise to an element of $K(A)$. In this section we are going to generalize that construction as follows.

s-map-prop

15.1 Construction. Let A and B be trivially graded C^*-algebras and let $\mathcal{E}$ be a graded Hilbert module over A. A graded $*$-homomorphism $\phi\colon \mathcal{S} \otimes B \to \mathcal{K}(\mathcal{E})$ determines a K-theory map $\phi_*\colon K(B) \to K(A)$, with the following properties:

(i) The correspondence $\phi \mapsto \phi_*$ is functorial with respect to composition with $*$-homomorphisms $B_1 \to B$.

(ii) The map ϕ_* depends only on the homotopy class of ϕ.

(iii) If $A = B$, if $\mathcal{E} = A$ (with no odd-graded part) and if $\phi\colon \mathcal{S} \otimes B \to \mathcal{K}(\mathcal{E})$ is of the form

$$\phi(f \otimes a) = f(0)a \in A = \mathcal{K}(\mathcal{E}),$$

then $\phi_*\colon K(A) \to K(A)$ is the identity.

One way to give the construction is to use the ring that we introduced when we discussed relative K-theory in Chapter 2. Let J be an ideal in a ring R, and let $\phi_0, \phi_1\colon B \to R$ be two ring homomorphisms which are equal, modulo J. The pair (ϕ_0, ϕ_1) determines a ring homomorphism $\phi\colon B \to D_J(R)$. Recall now that there is a split exact sequence

$$0 \longrightarrow K(J, R) \longrightarrow K(D_J(R)) \longrightarrow K(R) \longrightarrow 0$$

By composing the induced map $\phi_*\colon K(C) \to K(D_A(B))$ with the projection $K(D_A(B)) \to K(A)$ we obtain from the pair (ϕ_0, ϕ_1) a homomorphism

$$\phi_*\colon K(C) \to K(A).$$

With this construction in hand, let us proceed to the proof of the asserted properties.

Proof. Let $\mathcal{F} = \overline{\phi[\mathcal{S} \otimes A] \cdot \mathcal{E}}$. Each operator in the image of the graded homomorphism $\phi\colon \mathcal{S} \otimes A \to \mathcal{K}(\mathcal{E})$ maps the Hilbert submodule $\mathcal{F} \subseteq \mathcal{E}$ into itself, and is a compact operator on $\mathcal{F}$. It follows that ϕ determines a graded homomorphism $\phi\colon \mathcal{S} \otimes A \to \mathcal{K}(\mathcal{F})$. This in turn determines homomorphisms

$$\phi_{\mathcal{S}}\colon \mathcal{S} \to \mathcal{B}(\mathcal{F}) \quad \text{and} \quad \phi_A\colon A \to \mathcal{B}(\mathcal{F})$$

(this follows from Exercise **??** and the fact that $\mathcal{S}$ and A belong to the multiplier algebra of $\mathcal{S} \otimes A$). From $\phi_{\mathcal{S}}$ we obtain a pair of projections P_0 and P in $\mathcal{B}(\mathcal{F})$, using exactly the same formulas we derived in Section **??**: thus we pass from $\phi_{\mathcal{S}}$ to a unitary U, then to the self-adjoint unitary $(U\varepsilon)$, and then to the projections $P_1 = \frac{1}{2}(U\varepsilon + I)$ and $P_0 = \frac{1}{2}(\varepsilon + I)$. These projections *commute* with the image of the $*$-homomorphism ϕ_A. So the maps

$$\phi_0 \colon : a \mapsto P_0\phi_A(a) \quad \text{and} \quad \phi_1 \colon a \mapsto P_1\phi_A(a)$$

are both $*$-homomorphisms. The difference of ϕ_0 and ϕ_1 maps A into $\mathcal{K}(\mathcal{F})$, so, as we noted before the proof began, the pair (ϕ_0, ϕ_1) determines a homomorphism from $K(A)$ to $K(\mathcal{K}(\mathcal{F}))$. If we follow with the map $K(\mathcal{K}(\mathcal{F})) \cong K(A)$, we obtain a map

$$\phi_* \colon K(B) \to K(A).$$

The verification of the properties listed is left to the reader. □

A more elegant but less concrete construction of this functoriality is given in Section 15.3.

15.2 A NEW DESCRIPTION OF C*-ALGEBRA K-THEORY

new-desc-k-sec

In the remaining two sections of this chapter we shall use the algebra $\mathcal{S}$ to explore K-theory for C*-algebras in a bit more detail, and in fact provide a new description of K-theory based on maps from $\mathcal{S}$. The description puts the constructions of this chapter into a clearer perspective, but it will not otherwise be used in these notes.

15.2 Definition. For C*-algebras A and B let $[A, B]$ denote the space of homotopy classes of $*$-homomorphisms from A to B (grading-preserving, if A and B are graded).

Let $H = H_0 \oplus H_1$ be a graded Hilbert space with separable, infinite-dimensional even and odd parts, and let $\mathcal{K} = \mathcal{K}(H)$, graded as in Example **??**. Then $A \otimes \mathcal{K}$ can be realized as the C*-algebra of compact endomorphisms of the graded Hilbert A-module $A \otimes H$. Thus any $*$-homomorphism from $\mathcal{S}$ to $A \otimes \mathcal{K}$ gives rise to a class in $K(A)$ by Construction **??**.

We are going to show that this procedure gives an *isomorphism* between $K(A)$ and $[\mathcal{S}, A \otimes \mathcal{K}]$. (The proof will work directly with $A \otimes \mathcal{K}$, and will not require its realization as the algebra of compact operators on a Hilbert module.)

diff-K-constr

15.3 Proposition. *With notation as above, Construction* **??** *gives a natural isomorphism* $[\mathcal{S}, A \otimes \mathcal{K}] \cong K(A)$.

Proof. Denote by $\mathrm{Map}(\mathcal{S}, A \otimes \mathcal{K})$ the space of graded $*$-homomorphisms from $\mathcal{S}$ into $A \otimes \mathcal{K}$, equipped with the topology of pointwise convergence. Thus a path in $\mathrm{Map}(\mathcal{S}, A \otimes \mathcal{K})$ is a homotopy of graded $*$-homomorphisms from $\mathcal{S}$ to A, and there is a natural isomorphism

$$[\mathcal{S}, A \otimes \mathcal{K}] \cong \pi_0(\mathrm{Map}(\mathcal{S}, A \otimes \mathcal{K}))$$

between $[\mathcal{S}, A \otimes \mathcal{K}]$ and the set of path components of $\mathrm{Map}(\mathcal{S}, A \otimes \mathcal{K})$.

As we did in Section **??** let us use the Cayley transform

$$x \mapsto \frac{x+i}{x-i},$$

to identify $\mathcal{S} = C_0(\mathbb{R})$ with the algebra of continuous functions on the unit circle $\mathbb{T}$ which vanish at $1 \in \mathbb{T}$. An element of $\mathrm{Map}(\mathcal{S}, A \otimes \mathcal{K})$ can be extended to a unital $*$-homomorphism ϕ from $C(\mathbb{T})$ into the multiplier algebra of $A \otimes \mathcal{K}$. If $u \in C(\mathbb{T})$ denotes the standard generator $u(z) = z$, then the unitary $U = \phi(u)$ is equal to 1, modulo $A \otimes \mathcal{K}$ and satisfies the relation $\alpha(U) = U^*$, where α is the grading automorphism. Conversely, every unitary U which is equal to 1, modulo $A \otimes \mathcal{K}$, and which satisfies the relation $\alpha(U) = U^*$, determines an element of $\mathrm{Map}(\mathcal{S}, A \otimes \mathcal{K})$.

From a unitary U of this type we obtain a projection P_1 by the formula

$$P_1 = \tfrac{1}{2}(U\varepsilon + 1)$$

is a self-adjoint projection (here ε denotes the multiplier $\varepsilon\colon a \otimes k \mapsto a \otimes \varepsilon k$, where the last ε denotes the grading operator on H). The projection P_1 has the property that $P_1 - P_0 \in A \otimes \mathcal{K}$, where P_0 is the projection $\frac{1}{2}(\varepsilon + 1)$. These computations can be reversed to construct a unitary from the projection P_1.

Putting together the observations of the last two paragraphs, we see that the space $\mathrm{Map}(\mathcal{S}, A \otimes \mathcal{K})$ is homeomorphic to the space of projections P_1 such that $P_1 - P_0 \in \mathcal{K}$. But we noted in Lemma **??** that the set of components of this space of projections is isomorphic to $K(A)$, and this proves the result. $\square$

15.4 Exercise. (For readers familiar with the rudiments of Kasparov's KK-theory.) Show that a map $[\mathcal{S}, A \otimes \mathcal{K}] \to KK(\mathbb{C}, A)$ can also be constructed in the following way. Identify $A \otimes \mathcal{K}$ with the compact operators on a universal graded Hilbert A-module $\mathcal{E}$ (see Exercise **??**). Given a $*$-homomorphism $\phi\colon \mathcal{S} \to \mathcal{K}_A(\mathcal{E})$, let $\mathcal{E}_\phi$ be the Hilbert submodule $\phi[\mathcal{S}]\mathcal{E}$. Then ϕ extends to a homomorphism from the bounded, continuous functions on $(-\infty, \infty)$ to the bounded operators on $\mathcal{E}_\phi$. Let $F \in \mathcal{B}(\mathcal{E}_\phi)$ be the operator corresponding to the odd function $x \mapsto x(1+x^2)^{-1/2}$. Verify that F determines a Kasparov cycle for $KK(\mathbb{C}, A) = K(A)$.

15.3 THE HOMOTOPY CATEGORY OF GRADED C*-ALGEBRAS

gtp-sec

In this final section we shall place K-theory into a still more general context. The *homotopy category of graded* C^*-*algebras* has objects the graded C^*-algebras and morphisms between A and B the homotopy classes of graded homomorphisms from A to B. This category has two interesting elaborations, both involving the notion of (spatial) tensor product of graded C^*-algebras.

15.5 Definition. The *graded tensor product* of two graded algebras A and B is the algebraic tensor product $A \otimes B$ over $\mathbb{C}$, equipped with the multiplication determined by the formula

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{\partial b_1 \partial a_2} (a_1 a_2) \otimes (b_1 b_2)$$

involving homogeneous elements of A and B (here ∂ is the grading degree, defined on homogeneous elements by $\alpha(x) = \partial(x) \cdot x$). The formula extends by linearity to define the multiplication on $A \otimes B$. The graded tensor product is a graded algebra with grading automorphism

$$\alpha(a \otimes b) = \alpha(a) \otimes \alpha(b).$$

If A and B are graded $*$-algebras, then the graded tensor product is a $*$-algebra with $*$-operation

$$(a \otimes b)^* = (-1)^{\partial a \partial b}(a^* \otimes b^*)$$

on homogeneous elements $a \in A$ and $b \in B$.

In order to avoid confusion with ordinary tensor products, from now on we shall use the notation $A \mathbin{\widehat{\otimes}} B$ to denote the graded tensor product of graded algebras. Note that if one of A or B is trivially graded, then $A \mathbin{\widehat{\otimes}} B = A \otimes B$ as graded algebras (the grading on $A \otimes B$ is given by the same formula as the grading on $A \mathbin{\widehat{\otimes}} B$).

The graded tensor product of two graded C^*-algebras may be equipped with a compatible norm and completed so as to obtain a C^*-algebra. As in the trivially graded situation, there is in general more than one way of doing this, and in this book we shall concentrate on the *spatial* norm, which is defined by a small variation of the procedure used in Section **??** to define the spatial tensor product of trivially graded C^*-algebras. The *graded tensor product* $H_A \mathbin{\widehat{\otimes}} H_B$ of two graded Hilbert spaces is the usual tensor product $H_A \otimes H_B$ with the usual norm and the grading given by the operator $\varepsilon(v_A \otimes v_B) = \varepsilon(v_A) \otimes \varepsilon(v_B)$. If T_A and T_B are bounded operators on H_A and H_B, respectively, then $T_A \mathbin{\widehat{\otimes}} T_B$ is the operator given by the formula

$$(T_A \mathbin{\widehat{\otimes}} T_B)(v_A \mathbin{\widehat{\otimes}} v_B) = (-1)^{\partial T_B \partial v_A} T_A v_A \mathbin{\widehat{\otimes}} T_B v_B.$$

To define the spatial norm on the graded tensor product of two graded C^*-algebras A and B, fix faithful, graded representations of A and B on graded Hilbert spaces H_A and H_B, and represent the graded tensor product $A \mathbin{\widehat{\otimes}} B$ on $H_A \mathbin{\widehat{\otimes}} H_B$ by the formula

$$\pi(a \mathbin{\widehat{\otimes}} b) = \pi_A(a) \mathbin{\widehat{\otimes}} \pi_B(b).$$

The spatial norm is the operator norm in this representation. It does not depend on the choice of faithful representations π_A and π_B, and is functorial in A and B.

As in the trivially graded case, if A and B are graded C^*-algebras, then from now on we shall denote by $A \mathbin{\widehat{\otimes}} B$ the completion of the algebraic graded tensor product in the spatial norm.

15.6 Exercise. Check that the formula given above for π gives a graded homomorphism from $A \mathbin{\widehat{\otimes}} B$ into $\mathcal{B}(H_A \mathbin{\widehat{\otimes}} H_B)$.

15.7 Exercise. Let H_A and H_B be graded Hilbert spaces. Prove that the graded homomorphism $\mathcal{K}(H_A) \mathbin{\widehat{\otimes}} \mathcal{K}(H_B) \to \mathcal{K}(H_A \mathbin{\widehat{\otimes}} H_B)$ is an isomorphism of graded C^*-algebras.

compact-multiply

Returning to the homotopy category of graded C^*-algebras, let F denote the functor on this category of graded tensor product with $\mathcal{K} = \mathcal{K}(H)$, where H is a fixed graded Hilbert space H with separable, infinite-dimensional even and odd parts. There is a natural identification $F^2(A) \cong F(A)$ given by the two isomorphisms

$$\mathcal{K}(H) \mathbin{\widehat{\otimes}} \mathcal{K}(H) \cong \mathcal{K}(H \mathbin{\widehat{\otimes}} H) \cong \mathcal{K}(H).$$

The first is given by Exercise 15.7. The second is induced by any grading-preserving unitary isomorphism $H \mathbin{\widehat{\otimes}} H \cong H$ (there is a unique such isomorphism, up to homotopy). As a result of the identification, we can form a new category whose objects are graded C^*-algebras and in which the morphisms from A to B are the homotopy classes of graded homomorphisms from A to $F(B)$. The composition of two morphisms in this category,

$$\phi \colon A \to F(B) \quad \text{and} \quad \psi \colon B \to F(C),$$

is given by the formula

$$A \xrightarrow{\phi} F(B) \xrightarrow{F(\psi)} F^2(C) \xrightarrow{\cong} F(C)$$

15.8 Exercise. Prove that this composition law is associative. What are the identity morphisms in this category?

The second elaboration is similar but involves the algebra $\mathcal{S}$ in place of $\mathcal{K}$. Denote by S the functor of graded tensor product by $\mathcal{S}$. We are going to describe a natural transformation $\Delta_A \colon S(A) \to S^2(A)$ (not an isomorphism), using which we can define a category whose objects are the graded C^*-algebras and in which the morphisms from A to B are the homotopy classes of graded homomorphisms from $S(A)$ to B. The composition of two morphisms in this category,

$$\phi \colon S(A) \to B \quad \text{and} \quad \psi \colon S(B) \to C,$$

is given by the formula

$$S(A) \xrightarrow{\Delta_A} S^2(A) \xrightarrow{S(\phi)} S(B) \xrightarrow{\psi} C$$

Associativity of this composition law will follow from the commutativity of the diagram

$$\begin{tikzcd}
S(A) \arrow[r, "\Delta_A"] \arrow[d, "\Delta_A"'] & S^2(A) \arrow[d, "\Delta_{S(A)}"] \\
S^2(A) \arrow[r, "S(\Delta_A)"'] & S^3(A)
\end{tikzcd}$$

We shall discuss the existence of identity morphisms in a moment.

The natural transformation $\Delta_A \colon S(A) \to S^2(A)$ is defined by the formula

$$\Delta_A = \Delta \mathbin{\widehat{\otimes}} 1 \colon \mathcal{S} \mathbin{\widehat{\otimes}} A \to \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S} \mathbin{\widehat{\otimes}} A,$$

where $\Delta \colon \mathcal{S} \to \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S}$ is the graded homomorphism given by the following lemma.

delta-lemma

15.9 Lemma. *There is a unique graded homomorphism* $\Delta\colon \mathcal{S} \to \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S}$ *such that*

$$\Delta(e^{-x^2}) = e^{-x^2} \mathbin{\widehat{\otimes}} e^{-x^2} \quad \textit{and} \quad \Delta(xe^{-x^2}) = xe^{-x^2} \mathbin{\widehat{\otimes}} e^{-x^2} + e^{-x^2} \mathbin{\widehat{\otimes}} xe^{-x^2}.$$

Proof. Uniqueness follows from the fact that the functions e^{-x^2} and xe^{-x^2} generate the C^*-algebra $\mathcal{S}$. To prove existence, note first that the algebra generated by e^{-x^2} and xe^{-x^2} is dense in $\mathcal{S}$ and consists of all functions of the form $p(x)e^{-x^2}$, where $p(x)$ is a polynomial, and that the formula

$$\Delta(p(x)e^{-x^2}) = p(x \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} x)e^{-x^2}$$

defines a graded homomorphism from this algebra into $\mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S}$. It suffices to prove that the formula extends by continuity to all of $\mathcal{S}$. To do this, let us represent $\mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S}$ on the Hilbert space $L^2(\mathbb{R}) \mathbin{\widehat{\otimes}} L^2(\mathbb{R})$. If we denote by X the unbounded operator of multiplication by x (with domain the compactly supported functions in $L^2(\mathbb{R})$, then X is essentially self-adjoint, as is the operator $X \mathbin{\widehat{\otimes}} I + I \mathbin{\widehat{\otimes}} X$ on $L^2(\mathbb{R}) \mathbin{\widehat{\otimes}} L^2(\mathbb{R})$ (its domain is the algebraic tensor product of the domain of X with itself). The operator $\Delta(p(x)e^{-x^2})$ is equal to

$$p(X \mathbin{\widehat{\otimes}} 1 + I \mathbin{\widehat{\otimes}} X)e^{-(X \mathbin{\widehat{\otimes}} I + I \mathbin{\widehat{\otimes}} X)^2},$$

which is equal to $f(X \mathbin{\widehat{\otimes}} I + I \mathbin{\widehat{\otimes}} X)$, where $f(x) = p(x)e^{-x^2}$. Its norm is therefore no more than the supremum norm of f. $\square$

15.10 Exercise. Show that the algebra $\mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S}$ is isomorphic to the algebra of matrix-valued functions on the quarter-plane, $f\colon (\mathbb{R}^+)^2 \to M_2(\mathbb{C})$, having the properties that for each x the value $f(x, 0)$ belongs to the 2-dimensional subalgebra of matrices of the form $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$, and for each y the value $f(0, y)$ belongs to the 2-dimensional subalgebra of matrices of the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. (Hint: First give a similar description of $\mathcal{S}$ itself as functions on the half-line with values in a certain graded algebra.) Use this description of $\mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S}$ to give another proof of Lemma 15.9.

15.11 Proposition. *Denote by* $e\colon \mathcal{S} \to \mathbb{C}$ *the graded homomorphism* $e(f) = f(0)$. *The diagrams*

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{S}\widehat{\otimes}\mathcal{S} \arrow[d, "1\widehat{\otimes}\Delta"] \\
\mathcal{S}\widehat{\otimes}\mathcal{S} \arrow[r, "\Delta\widehat{\otimes}1"'] & \mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}\mathcal{S}
\end{tikzcd}
\qquad \textit{and} \qquad
\begin{tikzcd}
\mathcal{S}\widehat{\otimes}\mathcal{S} \arrow[r, "1\widehat{\otimes}e"] \arrow[d, "e\widehat{\otimes}1"'] & \mathcal{S} \\
\mathcal{S} & \mathcal{S} \arrow[l, "="] \arrow[u, "="'] \arrow[ul, "\Delta"]
\end{tikzcd}$$

commute.

Proof. It suffices to check commutativity on the generators e^{-x^2} and xe^{-x^2}, which is a simple computation. $\square$

The first diagram gives the information we need to conclude that the composition law in the category constructed from the functor S is associative. The second diagram shows that the graded homomorphism $e\colon S(A) \to A$ is the identity morphism for A in this category. Moreover, the correspondence which maps $f\colon A \to B$ to the composition

$$\mathcal{S}\widehat{\otimes}A \xrightarrow{e\widehat{\otimes}\phi} B$$

is a functor from the ordinary homotopy category into our new category.

If we combine our two elaborations of the homotopy category, we obtain a new category $\mathfrak{C}$ in which the objects are graded C^*-algebras and the morphisms from A to B are the homotopy classes of graded morphisms from $S(A)$ to $F(B)$. By the results of Section 15.2, the K-theory functor has a very simple description in terms of this category:

$$K(A) = \mathrm{Hom}_{\mathfrak{C}}(\mathbb{C}, A)$$

(strictly speaking we have not defined $K(A)$ for nontrivially graded C^*-algebras; a good approach is to define $K(A)$ using the above identity). The important functorial property exhibited in Section 15.1 is now just a consequence of the way we have defined the composition law in the category $\mathfrak{C}$.

We shall return to these category-theoretic ideas at the end of the next chapter, where we shall enrich the category $\mathfrak{C}$ to incorporate homotopy classes of *asymptotic morphisms*. This provides a powerful tool for the computation of C^*-algebra K-theory groups.

15.4 ASYMPTOTIC MORPHISMS

The purpose of this section is to present the notion of *asymptotic morphism* between C^*-algebras. We shall show that an asymptotic morphism from A to B induces an ordinary homomorphism of groups from $K(A)$ to $K(B)$, and as an application we shall formulate and prove the Bott periodicity theorem. At the end of the chapter we shall briefly sketch the construction of a bivariant K-theory for C^*-algebras based on the notion of asymptotic morphism.

asymp-morphism-def

15.12 Definition. Let A and B be C^*-algebras. An *asymptotic morphism* from A to B, denoted $\phi\colon A \rightsquigarrow B$, is a family of functions $\phi_t\colon A \to B$, where $t \in (0,1]$, such that

(i) For each $a \in A$ the map $t \mapsto \phi_t(a) \in B$ is continuous and bounded.

(ii) For all $a, a_1, a_2 \in A$ and $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\lim_{t\to 0} \left\{ \begin{array}{c} \phi_t(a_1 a_2) - \phi_t(a_1)\phi_t(a_2) \\ \phi_t(\lambda_1 a_1 + \lambda_2 a_2) - \lambda_1 \phi_t(a_1) - \lambda_2 \phi_t(a_2) \\ \phi_t(a^*) - \phi_t(a)^* \end{array} \right\} = 0.$$

15.13 Example. Ordinary $*$-homomorphisms give rise asymptotic morphisms: if ϕ is a $*$-homomorphism we can set $\phi_t = \phi$, for all t. We shall give a nontrivial example of an asymptotic morphism in the next section.

Asymptotic morphisms may be composed on the left or right with $*$-homomorphisms so as to obtain new asymptotic moprhisms. The composition of two asymptotic morphisms is a more complicated operation that we shall study later in this chapter.

15.14 Definition. Two asymptotic morphisms ϕ and ψ, from A to B, are *asymptotically equivalent* if

$$\lim_{t\to 0} \|\phi_t(a) - \psi_t(a)\| = 0,$$

for every $a \in A$. The asymptotic morphisms ϕ_0 and ϕ_1 are *asymptotically homotopic* if there is an asymptotic morphism from A to $C([0,1];B)$ from which ϕ_0 and ϕ_1 can be recovered by composing with the $*$-homomorphisms $C([0,1],B) \to B$ given by evaluation at 0 and 1.

Asymptotic morphisms are important because they induce maps on K-theory, as follows. Given an asymptotic morphism $\phi\colon A \rightsquigarrow B$, there is a natural extension of ϕ to an asymptotic morphism $\phi\colon A^+ \rightsquigarrow B^+$ between C^*-algebras with units adjoined. If $p \in M_n(A^+)$ is a projection matrix with entries in A^+, then let us denote by $\phi_t(p)$ the matrix obtained by applied ϕ_t entrywise to p. It is not a projection matrix, but

$$\lim_{t\to 0} \|\phi_t(p)^2 - \phi_t(p)\| = 0 \quad \text{and} \quad \lim_{t\to 0} \|\phi_t(p) - \phi_t(p)^*\| = 0.$$

15.15 Lemma. *There is a norm-continuous path of projection matrices* $q_t \in M_n(B^+)$ *such that* $\lim_{t\to 0} \|q_t - \phi_t(p)\| = 0$.

Proof. Let $b_t = \frac{1}{2}(\phi_t(p) + \phi_t(p)^*)$. Then b_t is self-adjoint, for all t, and in addition

$$\lim_{t\to 0} \|b_t - \phi_t(p)\| = 0 \quad \text{and} \quad \lim_{t\to 0} \|b_t^2 - b_t\| = 0.$$

We shall show that there is a norm-continuous path of projection matrices q_t such that $\lim_{t\to 0} \|b_t - q_t\| = 0$. For every neighbourhood U of $\{0,1\} \subseteq \mathbb{R}$, the spectrum of b_t is contained within U, for all sufficiently small t. In fact, for any $\varepsilon > 0$, the spectrum of b_t is contained within $[-\varepsilon, \varepsilon] \cup [1-\varepsilon, 1+\varepsilon]$ whenever $\|b_t^2 - b_t\| < \varepsilon - \varepsilon^2$. It follows that if $h\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $h(x) = 0$ near 0 and $h(x) = 1$ near 1, then for all sufficiently small t, the element $p_t = h(b_t)$ is a projection matrix, and morever

$$\lim_{t\to 0} \|b_t - p_t\| = 0,$$

as required. □

Let $\phi\colon A \rightsquigarrow B$ be an asymptotic morphism. We may define a homomorphism $\phi_*\colon K(A^+) \to K(B^+)$ by mapping the class of a projection matrix $p \in M_n(A^+)$ to the class of any of the projection matrices q_t given by the lemma above (by Lemma **??** the K-theory class of q_t depends only on p). The homomorphism ϕ_* restricts to a map of $K(A) \cong \ker(K(A^+) \to K(\mathbb{C}))$ into the K-theory group $K(B) \cong \ker(K(B^+) \to K(\mathbb{C}))$.

15.16 Definition. We call $\phi_*\colon K(A) \to K(B)$, defined as above, the homomorphism *induced* by the asymptotic morphism $\phi\colon A \rightsquigarrow B$.

asympt-prop **15.17 Proposition.** *The* K*-theory map* $\phi\colon K(A) \to K(B)$ *associated to an asymptotic morphism* $\phi\colon A \rightsquigarrow B$ *has the following properties:*

(i) *The construction of* $\phi\colon K(A) \to K(B)$ *is functorial with respect to composition with* $*$*-homomorphisms* $A_1 \to A$ *and* $B \to B_1$.

(ii) *The* K*-theory map* $\phi\colon K(A) \to K(B)$ *depends only on the asymptotic homotopy class of* ϕ.

(iii) *If each* $\phi_t\colon A \to B$ *is actually a* $*$*-homomorphism, then* $\phi\colon K(A) \to K(B)$ *is the map induced by the* $*$*-homomorphism* $\phi_1\colon A \to B$.

□

15.5 CATEGORIES AND BIVARIANT THEORIES

The proof in the preceding section is best viewed in the context of a suitable category which includes "generalized" morphisms between C^*-algebras. Let us assume that we have a category with the following features:

(a) The objects are C^*-algebras.[1] Every $*$-homomorphism $\phi\colon A \to B$ determines in a functorial way a morphism from A to B, which depends only on the homotopy class of ϕ. (Thus there is a functor from the homotopy category of C^*-algebras into our category, which is the identity on objects.)

(b) The category has a natural product operation, so that morphisms $\sigma_1\colon A_1 \to B_1$ and $\sigma_2\colon A_2 \to B_2$ may be multiplied in a functorial way to produce a morphism

$$\sigma_1 \otimes \sigma_2\colon A_1 \otimes A_2 \to B_1 \otimes B_2.$$

The product should be compatible with tensor product of $*$-homomorphisms, and should have the property that $\sigma \otimes 1\colon A \otimes \mathbb{C} \to B \otimes \mathbb{C}$ identifies with $\sigma\colon A \to B$, once $A \otimes \mathbb{C}$ is identified with A and $B \otimes \mathbb{C}$ is identified with B. It should also be compatible with the flip isomorphisms $A \otimes B \to B \otimes A$ in the natural way.

(c) A morphism $\sigma\colon A \to B$ induces in a functorial way a homomorphism from $K(A)$ to $K(B)$, which is the standard induced homomorphism when σ is determined by a $*$-homomorphism. (Thus there K-theory functor should factor through our category.)

[1] It is customary to work with *separable* C^*-algebras, but this detail need not concern us here.

With this category in hand, the rotation argument may be expressed as follows (we shall write X in place of $C_0(X)$, and $\times$ in place of $\otimes$ in this commutative context). In view of the diagram

$$\begin{tikzcd} \mathbb{R}^{2n} \times \mathrm{pt} \arrow[r, "\alpha\times 1"] & \mathrm{pt} \times \mathrm{pt} \arrow[r, "\beta\times 1"] \arrow[d, "\mathrm{flip}"', "="] & \mathbb{R}^{2n} \times \mathrm{pt} \arrow[d, "\cong"', "\mathrm{flip}"] \\ & \mathrm{pt} \times \mathrm{pt} \arrow[r, "1\times\beta"'] & \mathrm{pt} \times \mathbb{R}^{2n} \end{tikzcd}$$

to prove $\beta \circ \alpha$ is an isomorphism it suffices to show $(1 \times \beta) \circ (\alpha \times 1)$ is an isomorphism. But consider now the commuting diagram

$$\begin{tikzcd} \mathbb{R}^{2n} \times \mathrm{pt} \arrow[r, "\alpha\times 1"] \arrow[d, "1\times\beta"'] \arrow[dr, "\alpha\times\beta"] & \mathrm{pt} \times \mathrm{pt} \arrow[d, "1\times\beta"] \\ \mathbb{R}^{2n} \times \mathbb{R}^{2n} \arrow[r, "\alpha\times 1"'] & \mathrm{pt} \times \mathbb{R}^{2n} \end{tikzcd}$$

It shows that it suffices to show $(\alpha \times 1) \circ (1 \times \beta)$ is an isomorphism. Now we can use the diagram

$$\begin{tikzcd} \mathbb{R}^{2n} \times \mathrm{pt} \arrow[r, "1\times\beta"] & \mathbb{R}^{2n} \times \mathbb{R}^{2n} \arrow[r, "\alpha\times 1"] \arrow[d, "\mathrm{flip}"', "=\mathrm{id}"] & \mathrm{pt} \times \mathbb{R}^{2n} \arrow[d, "\cong"', "\mathrm{flip}"] \\ & \mathbb{R}^{2n} \times \mathbb{R}^{2n} \arrow[r, "1\times\alpha"'] & \mathbb{R}^{2n} \times \mathrm{pt} \end{tikzcd}$$

to complete the argument, bearing in mind that the left-hand flip induces the identity map in K-theory.

The morphisms constructed in Chapter **??** provide a suitable category: one sets

$$\mathrm{Hom}(A, B) = \left\{ \begin{array}{l} \text{Homotopy classes of graded asymptotic mor-} \\ \text{phisms from } \mathcal{S} \otimes A \text{ to } B \otimes \mathcal{K} \end{array} \right\}.$$

We didn't show it, but these *are* the morphism sets in a category with a suitable product. This is the E-theory category of operator K-theory.

15.6 NOTES

The picture of K-theory using the algebra $\mathcal{S}$ is sketched in Higson's notes [**?**], which mostly dwell on more advanced topics than these notes. The same source also develops more fully the category-theoretic point of view sketched in Section 15.3. (Actually there is a small error in [**?**] in the discussion of $[\mathcal{S}, A \otimes \mathcal{K}]$. Exercise: find it.)