# Spaces with Vanishing $\ell^{2}$-Homology and their Fundamental Groups (after Farber and Weinberger) 

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#### Abstract

We characterize those groups which can occur as the fundamental groups of finite $C W$-complexes with vanishing $\ell^{2}$-homology (the first examples of such groups were obtained by Farber and Weinberger)


Mathematics Subject Classifications (2000). Primary: 57Q10; Secondary: 46L99.
Key words. $\ell^{2}$-homology, $L^{2}$-cohomology, zero-in-the-spectrum conjecture.

## 1. Introduction

Let $G$ be a finitely presented group. Is $G$ the fundamental group of some finite $C W$-complex whose integral homology is trivial (meaning the homology is the same as that of a point)? Elementary homological algebra provides some simple necessary conditions:

$$
H_{1}(G, \mathbb{Z})=0 \quad \text { and } \quad H_{2}(G, \mathbb{Z})=0
$$

An elegant observation of Kervaire [3] shows that these necessary conditions are also sufficient:

THEOREM 1.1. Let $G$ be a finitely presented group and suppose that $H_{1}(G, \mathbb{Z})$ and $H_{2}(G, \mathbb{Z})$ are both zero. There is a connected three-dimensional finite $C W$-complex $X$ with $\pi_{1}(X)=G$ such that $H_{k}(X, \mathbb{Z})=0$ for all $k>0$.

Actually, Kervaire's attention was focussed on homology spheres, and what he observed is this:

THEOREM 1.2. Let $G$ be a finitely presented group and suppose that $H_{1}(G, \mathbb{Z})$ and $H_{2}(G, \mathbb{Z})$ are both zero. For every dimension $n \geqslant 5$ there is a homology $n$-sphere with fundamental group $G$.

The theorems rely on a computation, due to H. Hopf [2], of the Hurewicz homomorphism in degree 2. This and some other ingredients have been used by

Farber and Weinberger [1] to construct examples of finite $C W$-complexes (three-dimensional) and smooth, closed manifolds (six-dimensional) whose universal covers have vanishing $\ell^{2}$-homology groups in all degrees. See the survey article [5] for background material on $\ell^{2}$-homology and the 'zero in the spectrum question'. ${ }^{\star}$
The purpose of this note is to indicate that the same ingredients can be used to prove direct analogues of Kervaire's theorems:

THEOREM 1.3. Let $G$ be a finitely presented group and suppose that the homology groups $H_{k}\left(G, \ell^{2}(G)\right)$ are zero for $k=0,1,2$. Then there is a connected threedimensional finite $C W$-complex $X$ with $\pi_{1}(X)=G$ such that $H_{k}\left(X, \ell^{2}(G)\right)=0$ for all $k \geqslant 0$.

THEOREM 1.4. Let $G$ be a finitely presented group and suppose that the homology groups $H_{k}\left(G, \ell^{2}(G)\right)$ are zero for $k=0,1,2$. For every dimension $n \geqslant 6$ there is a closed manifold $M$ of dimension $n$ and with $\pi_{1}(M)=G$ such that $H_{k}\left(M, \ell^{2}(G)\right)=0$ for all $k>0$.

Remark 1.5. The difference between $n \geqslant 5$ in Kervaire's theorem and $n \geqslant 6$ in ours is accounted for by the absence in the literature of a suitable handle-cancellation lemma in $\ell^{2}$-homology. Since our main interest is in $C W$-complexes we shall not consider this issue further here.

The definitions of the $\ell^{2}$-homology groups $H_{k}\left(G, \ell^{2}(G)\right)$ and $H_{k}\left(X, \ell^{2}(G)\right)$ will be reviewed in Section 2 below. It is not hard to see that the hypotheses in the theorems can not be reduced (compare [6,11.3]), so that in fact the vanishing of $H_{k}\left(G, \ell^{2}(G)\right)$ for $k=0,1,2$ is both necessary and sufficient.
$C^{*}$-algebra theory makes an appearance in the Farber-Weinberger proof, and the main novelty of our note is the more systematic use of $C^{*}$-algebras, including homology with coefficients in a $C^{*}$-algebra. At other steps in the argument (notably the construction of manifold examples from $C W$-examples) we have no improvement to offer and we shall simply refer the reader to the paper of Farber and Weinberger.

## 2. Homological Preliminaries

Let $Z$ be a connected $C W$-complex and let $G$ be the fundamental group of $Z$. Form the cellular chain complex for the universal cover of $Z$,

$$
\begin{equation*}
C_{0}(\tilde{Z}) \stackrel{b}{\longleftarrow} C_{1}(\tilde{Z}) \stackrel{b}{\longleftarrow} C_{2}(\tilde{Z}) \stackrel{b}{\longleftarrow} \cdots, \tag{2.1}
\end{equation*}
$$

which is a complex of projective (in fact free) right $\mathbb{Z}[G]$ modules. The corresponding

[^0]cellular homology groups $H_{*}(\widetilde{Z})$ are right $\mathbb{Z}[G]$-modules too, although not of course projective.

If $V$ is any left module over $\mathbb{Z}[G]$ then let us denote by $H_{*}(Z, V)$ the homology of the tensor product complex

$$
\begin{equation*}
C_{0}(\tilde{Z}) \otimes_{\mathbb{Z}[G]} V \stackrel{b}{\longleftrightarrow} C_{1}(\tilde{Z}) \otimes_{\mathbb{Z}[G]} V \stackrel{b}{\longleftrightarrow} C_{2}(\tilde{Z}) \otimes_{\mathbb{Z}[G]} V \stackrel{b}{\longleftrightarrow} \cdots \tag{2.2}
\end{equation*}
$$

EXAMPLE 2.1. If $Z$ is a model for the classifying space $B G$ then $H_{*}(Z, V)$ is isomorphic to the group homology of the module $V: H_{*}(B G, V) \cong H_{*}(G, V)$. Indeed, if $Z=B G$ then the cellular chain complex (2.1) for $\widetilde{Z}$ is a free resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$.

There is an obvious coefficient homomorphism

$$
h_{n}: H_{n}(\widetilde{Z}) \otimes_{\mathbb{Z}[G]} V \longrightarrow H_{n}(Z, V)
$$

and we shall need the following simple observation concerning its behaviour in degree 2 :

LEMMA 2.2. There is an exact sequence

$$
H_{2}(\tilde{Z}) \otimes_{\mathbb{Z}[G]} V \xrightarrow{h_{2}} H_{2}(Z, V) \longrightarrow H_{2}(G, V) \longrightarrow 0
$$

In particular, if $H_{2}(G, V)=0$ then the coefficient homomorphism is surjective in degree 2.

Proof. This is essentially the computation of Hopf to which we referred in the introduction. We shall prove the lemma in the case where $Z$ is 2 -dimensional, which is the only case we shall need in this note. The cellular chain complex for $Z$ may be prolonged in higher degrees to obtain a projective resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ :


Now the group $H_{2}(Z, V)$ is equal to $\operatorname{Kernel}\left(b_{2} \otimes 1_{V}\right)$, while the group $H_{2}(\widetilde{Z}) \otimes_{\mathbb{Z}[G]} V$ is equal to $\operatorname{Kernel}\left(b_{2}\right) \otimes_{\mathbb{Z}[G]} V$, which is equal to $\operatorname{Image}\left(b_{3}\right) \otimes_{\mathbb{Z}[G]} V$. Therefore the image of $H_{2}(\widetilde{Z}) \otimes_{\mathbb{Z}[G]} V$ within $H_{2}(Z, V)=\operatorname{Kernel}\left(b_{2} \otimes 1_{V}\right)$ is $\quad \operatorname{Image}\left(b_{3} \otimes 1_{V}\right)$. The quotient by this image is therefore

$$
H_{2}(Z, V) /\left[H_{2}(\widetilde{Z}) \otimes_{\mathbb{Z}[G]} V\right]=\operatorname{Kernel}\left(b_{2} \otimes 1_{V}\right) / \operatorname{Image}\left(b_{3} \otimes 1_{V}\right)=H_{2}(G, V)
$$

as required.

If $A$ is an auxiliary ring, and if $V$ is equipped with a right $A$-module structure which commutes with the given left $\mathbb{Z}[G]$-module structure then the complex (2) is a complex of right $A$-modules, and the homology groups $H_{*}(Z, V)$ have the structure of right $A$-modules. Moreover the coefficient homomorphism is right $A$-linear.

We shall use this observation in the proof of the main theorem, where we shall take $V$ to be a ring $A$ (considered as a right module over itself), and where the $\mathbb{Z}[G]$-module structure on $V$ will come from left multiplication by an embedded copy of $G$ in the group of invertible elements of $A$.

## 3. Analytic Preliminaries

We continue to denote by $Z$ a connected $C W$-complex with fundamental group $G$. The left regular representation of $G$ on the Hilbert space $\ell^{2}(G)$ provides $\ell^{2}(G)$ with the structure of a left $\mathbb{Z}[G]$-module, and the homology groups $H_{k}\left(Z, \ell^{2}(G)\right)$ are the $\ell^{2}$-homology groups to which we referred in the introduction. They are denoted in a variety of different ways in the literature.

Notice that $\ell^{2}(G)$ is a Banach space completion of the complex group algebra $\mathbb{C}[G]$, or the real group algebra $\mathbb{R}[G]$ if we are using real coefficients.* In the following section it will be very convenient to work with completions which are not merely Banach spaces but Banach algebras. For our purposes the best choice is the reduced $C^{*}$-algebra of $G$, denoted $C_{r}^{*}(G)$, which is the norm-completion of $\mathbb{C}[G]$ in its left regular representation as bounded operators on $\ell^{2}(G)$. The group $G$ embeds into the group of invertible elements of $C_{r}^{*}(G)$, and left multiplication by $G$ gives $C_{r}^{*}(G)$ the structure of a left $\mathbb{Z}[G]$-module. Hence we can form the groups $H_{k}\left(Z, C_{r}^{*}(G)\right)$ as in the previous section. They are right $C_{r}^{*}(G)$-modules.

The purpose of this section is to prove the following result:
THEOREM 3.1. Suppose that $Z$ is a connected $C W$-complex with fundamental group $G$ and finitely many cells in dimensions 0 through $n$. The following are equivalent:

- The homology groups $H_{k}\left(Z, C_{r}^{*}(G)\right)$ are zero in degrees 0 through $n$.
- The homology groups $H_{k}\left(Z, \ell^{2}(G)\right)$ are zero in degrees 0 through $n$.

The argument is an exercise in functional analysis, and to set the proper context let us fix a unital $C^{*}$-algebra $A$ and a complex

$$
E_{0} \stackrel{b}{\longleftarrow} E_{1} \stackrel{b}{\longleftarrow} E_{2} \stackrel{b}{\longleftarrow} \cdots
$$

comprised of Hilbert $A$-modules and bounded, adjointable Hilbert $A$-module maps (here and below, see Lance's text [4], especially Section 3, for information on Hilbert module theory).

LEMMA 3.2. The homology groups of the above complex vanish in degrees 0 through $n$ if and only if the 'Laplace' operator $\Delta=b b^{*}+b^{*} b$ is invertible on the spaces $E_{0}$ through $E_{n}$.

[^1]Proof. Assume first $\Delta$ is surjective in degree $k$. For arbitrary $x \in \operatorname{Kernel}(b)$ choose $y$ with $\Delta y=x$. Then $0=b \Delta y=b b b^{*} y+b b^{*} b y=b b^{*} b y$. Therefore

$$
\begin{equation*}
0=\left\langle b y, b b^{*} b y\right\rangle=\left\langle b^{*} b y, b^{*} b y\right\rangle \tag{3}
\end{equation*}
$$

and so $b^{*} b y=0$. Consequently, $x=b b^{*} y$, so that $x$ lies in the image of $b$ and the $k$-th homology group vanishes.

If the homology groups in degree $k$ and $k-1$ vanish then the ranges of $b_{k+1}: E_{k+1} \rightarrow E_{k}$ and $b_{k}: E_{k} \rightarrow E_{k-1}$ are closed since they coincide with the kernels of the succeeding differentials in our complex. From [4, Theorem 3.2] and its proof we obtain an orthogonal decomposition

$$
E_{k}=\operatorname{Image}\left(b_{k}^{*}\right) \oplus \operatorname{Image}\left(b_{k+1}\right)=\operatorname{Kernel}\left(b_{k}\right)^{\perp} \oplus \operatorname{Kernel}\left(b_{k+1}^{*}\right)^{\perp}
$$

By the Open Mapping Theorem the operator $b_{k}$, and hence the operator $b_{k}^{*} b_{k}$, is bounded below on the first summand, while $b_{k+1} b_{k+1}^{*}$ is bounded below on the second (note that since Image $\left(b_{k+1}\right)$ is closed, so is Image $\left(b_{k+1}^{*}\right)$ ). So the self-adjoint operator $\Delta$ is bounded below on $E_{k}$ and is therefore invertible.

Suppose now that $A$ is represented faithfully and non-degenerately on a Hilbert space $H$ (we have in mind the regular representation of $C_{r}^{*}(G)$ on $\ell^{2}(G)$ ). Then we can form the Hilbert spaces $E_{k} \otimes_{A} H$ by completing the algebraic tensor product over $A$ with respect to the norm associated to the inner product

$$
\left\langle e_{1} \otimes v_{1}, e_{2} \otimes v_{2}\right\rangle_{E \otimes_{A} H}=\left\langle v_{1},\left\langle e_{1}, e_{2}\right\rangle_{E} v_{2}\right\rangle_{H}
$$

The Hilbert spaces so obtained assemble to form a complex of Hilbert spaces and bounded linear maps.

LEMMA 3.3. Let $T$ be a bounded and adjointable operator on $E_{k}$. The operators $T$ on $E_{k}$ and $T \otimes I_{H}$ on $E_{k} \otimes_{A} H$ have the same spectrum.

Proof. The map $T \mapsto T \otimes I$ is an injective homomorphism of $C^{*}$-algebras and so preserves spectrum.
Putting the two lemmas together we obtain the following result:
LEMMA 3.4. Suppose given a complex of Hilbert $A$-modules

$$
E_{0} \stackrel{b}{\longleftarrow} E_{1} \stackrel{b}{\longleftarrow} E_{2} \stackrel{b}{\longleftarrow} \cdots
$$

and suppose that $H$ is a Hilbert space equipped with a faithful and non-degenerate representation of $A$. Then the following are equivalent:

- The above complex has no homology in degrees 0 through $n$.
- The Hilbert module tensor product of the above complex with H has no homology in degrees 0 through $n$.

Now if $A$ is a $C^{*}$-algebra with unit and if $E$ is a finitely generated and projective module over $A$ (in the usual sense of algebra) then $E$ may be given the structure of a Hilbert $A$-module, and this structure is unique up to unitary isomorphism (this is a generalization of the well-known fact that a complex vector bundle on a compact space has an essentially unique Hermitian structure). See [4]. Moreover all $A$-linear maps between such modules are automatically bounded and adjointable. Finally, $E$ is a finitely generated projective $A$-module, and if $A$ is represented faithfully and non-degenerately on $H$, then the algebraic and Hilbert module tensor products $E \otimes_{A} H$ agree. These observations allow us to formulate a more algebraic version of the previous result:

LEMMA 3.5. Let $A$ be a unital $C^{*}$-algebra. Suppose given a complex of $A$-modules

$$
E_{0} \stackrel{b_{1}}{\longleftarrow} E_{1} \stackrel{b_{2}}{\longleftarrow} E_{2} \stackrel{b_{3}}{\longleftarrow} \cdots
$$

(in the ordinary sense of algebra) for which $E_{0}$ through $E_{n}$ are finitely generated and projective. Suppose that $H$ is a Hilbert space equipped with a faithful non-degenerate representation of $A$. Then the following are equivalent:

- The above complex has no homology in degrees 0 through $n$.
- The algebraic tensor product of the above complex over $A$ with $H$ has no homology in degrees 0 through $n$.

Proof. Suppose first that the complex $E_{*}$ has zero homology in degrees 0 through $n$. The range of the differential $b_{n}: E_{n} \rightarrow E_{n-1}$ is closed, since it is the kernel of $b_{n-1}$, and so by [4, Theorem 3.2] the module $E_{n}$ splits as a direct sum of $\operatorname{Kernel}\left(b_{n}\right)$ and its orthogonal complement. In particular, the module $\operatorname{Kernel}\left(b_{n}\right)$ is a direct summand of a finitely generated module, and is therefore finitely generated itself. It follows that Image $\left(b_{n+1}\right)$ is finitely generated, and so we can find a finitely generated free module $E_{n+1}^{\prime}$ and a module map $f: E_{n+1}^{\prime} \rightarrow E_{n+1}$ for which the complex

$$
E_{0} \stackrel{b_{1}}{\longleftarrow} E_{1} \stackrel{b_{2}}{\longleftarrow} \ldots \cdots \stackrel{b_{n}}{\longleftarrow} E_{n} \stackrel{b_{n+1} f}{\longleftarrow} E_{n+1}^{\prime}
$$

has vanishing homology in degrees 0 through $n$. Since all the terms which appear now have Hilbert module structures we can appeal to Lemma 3.4 to conclude that the tensor product of the displayed complex by $H$ has vanishing homology in degrees 0 through $n$. This implies the same vanishing result for the tensor product of the original complex by $H$.

Suppose, conversely, that the algebraic tensor product complex $E_{*} \otimes_{A} H$ has vanishing homology in degrees 0 through $n$. Then by the previous lemma, the homology of $E_{*}$ vanishes at least though degree $n-1$, and so by the same argument as above, the module $\operatorname{Kernel}\left(b_{n}\right)$ is still finitely generated. We can therefore find $f: E_{n+1}^{\prime} \rightarrow E_{n+1}$ as in the first part of the proof, but so that the tensor product complex involving $E_{n+1}^{\prime}$ has vanishing homology in degrees 0 through $n$. Once again, Lemma 3.4 now applies to complete the proof.

Proof of Theorem 3.1. The algebraic tensor product of the module $C_{k}(\widetilde{Z}) \otimes_{\mathbb{Z}[G]} C_{r}^{*}(G)$ over $C_{r}^{*}(G)$ with the Hilbert space $\ell^{2}(G)$ is $C_{k}(\widetilde{Z}) \otimes_{\mathbb{Z}[G]} \ell^{2}(G)$. So the theorem is an immediate consequence of the last lemma.

Here are the consequences of the theorem that we shall need:
COROLLARY 3.6. If $G$ is a finitely presented discrete group then the following are equivalent:

- $H_{k}\left(G, \ell^{2}(G)\right)=0$ for $k=0,1,2$,
- $H_{k}\left(G, C_{r}^{*}(G)\right)=0$ for $k=0,1,2$.

Remark 3.7. It is very easy to find examples of groups to which the proposition above applies. As Farber and Weinberger note, any threefold direct product of finitely presented non-amenable groups will do the job.

COROLLARY 3.8. If $Z$ is a finite $C W$-complex with fundamental group $G$ then the following are equivalent:

- $H_{k}\left(Z, \ell^{2}(G)\right)=0$ for all $k \geqslant 0$,
- $H_{k}\left(Z, C_{r}^{*}(G)\right)=0$ for all $k \geqslant 0$.


## 4. Proof of the Main Theorems

We shall prove the following result:
THEOREM 4.1. Let $G$ be a finitely presented group and suppose that the homology groups $H_{k}\left(G, C_{r}^{*}(G)\right)$ are zero for $k=0,1,2$. Then there is a connected, 3-dimensional, finite $C W$-complex $X$ with $\pi_{1}(X)=G$ such that $H_{k}\left(X, C_{r}^{*}(G)\right)=0$ for all $k \geqslant 0$.

In view of the results in the previous section, this is equivalent to Theorem 1.3. Theorem 1.4 follows from Theorem 1.3 by a regular neighborhood construction. We refer the reader to the Farber-Weinberger paper [1] for details.

We shall work with the real group $C^{*}$-algebra below, although we note that the real and complex cases of Theorem 4.1 are easily derived from one another.

NOTATION 4.2. In the following, $G$ will denote a fixed finitely presented group for which $H_{k}\left(G, C_{r}^{*}(G)\right)=0$, for $k=0,1,2$. All tensor products will be taken over the ring $\mathbb{Z}[G]$ (in particular they will be algebraic - no Hilbert module tensor products will be involved).

Nearly all of the argument below follows that of Farber and Weinberger [1], which in turn is inspired by the argument of Kervaire mentioned earlier.

LEMMA 4.3. There exists a connected finite 2-complex $Y$ with fundamental group $G$ for which $H_{2}\left(Y, C_{r}^{*}(G)\right)$ is a finitely generated and free $C_{r}^{*}(G)$-module.

Proof. Let $Z$ be a finite 2-dimensional $C W$-complex with $\pi_{1}(Z)=G$ (for instance the presentation 2 -complex of our finitely presented group $G$ ). Then in view of our assumptions on $G$, it follows that $H_{0}\left(Z, C_{r}^{*}(G)\right)$ and $H_{1}\left(Z, C_{r}^{*}(G)\right)$ are zero (compare the proof of Lemma 2.2). Therefore, in view of the exact sequence

$$
\begin{array}{r}
0 \stackrel{b}{\longleftarrow} C_{0}(\widetilde{Z}) \otimes C_{r}^{*}(G) \stackrel{b}{\longleftarrow} C_{1}(\widetilde{Z}) \otimes C_{r}^{*}(G) \stackrel{b}{\longleftarrow} C_{2}(\widetilde{Z}) \\
\otimes C_{r}^{*}(G) \stackrel{b}{\longleftarrow} H_{2}\left(Z, C_{r}^{*}(G)\right) \stackrel{b}{\longleftarrow} 0,
\end{array}
$$

the module $H_{2}\left(Z, C_{r}^{*}(G)\right)$ is finitely generated and stably free (note that all the modules except perhaps the rightmost one are actually free). By wedging $Z$ with finitely many 2 -spheres we obtain a finite 2 -complex $Y$ with fundamental group $G$ for which $H_{2}\left(Y, C_{r}^{*}(G)\right)$ is a free $C_{r}^{*}(G)$-module, as required.

Define $j: H_{2}(\tilde{Y}) \rightarrow H_{2}(\tilde{Y}) \otimes C_{r}^{*}(G)$ by associating to the class $x \in H_{2}(\tilde{Y})$ the elementary tensor $x \otimes 1 \in H_{2}(\tilde{Y}) \otimes C_{r}^{*}(G)$. Let us continue to denote by $h: H_{2}(\widetilde{Y}) \otimes C_{r}^{*}(G) \rightarrow H_{2}\left(Y, C_{r}^{*}(G)\right)$ the coefficient map considered in Section 2.

PROPOSITION 4.4. The image of the composition

$$
h \circ j: H_{2}(\widetilde{Y}) \rightarrow H_{2}\left(Y, C_{r}^{*}(G)\right)
$$

contains a basis for the free $C_{r}^{*}(G)$-module $H_{2}\left(G, C_{r}^{*}(G)\right)$.
Proof. Since by Lemma 2.2 the coefficient map $h$ is surjective, we can certainly find a basis $b_{1}, \ldots b_{d}$ for $H_{2}\left(Y, C_{r}^{*}(G)\right)$ in the image of $h$. Let us do so and write the basis elements as

$$
b_{k}=\sum_{i=1}^{n_{k}} h\left(x_{i, k} \otimes a_{i, k}\right)=\sum_{i=1}^{n_{k}} h\left(x_{i, k} \otimes 1\right) a_{i, k}
$$

with $x_{i, k} \in H_{2}(\tilde{Y})$ and $a_{i, k} \in C_{r}^{*}(G)$ (in the displayed formula we have used the fact that $h$ is $C_{r}^{*}(G)$-linear). Since $H_{2}\left(Y, C_{r}^{*}(G)\right)$ is a free and finitely generated module over $C_{r}^{*}(G)$ it has a Hilbert $C_{r}^{*}(G)$-module structure. The module multiplication operation

$$
H_{2}\left(Y, C_{r}^{*}(G)\right) \times C_{r}^{*}(G) \rightarrow H_{2}\left(Y, C_{r}^{*}(G)\right)
$$

is continuous and the set of bases for $H_{2}\left(Y, C_{r}^{*}(G)\right)$ is open within the set of ordered $d$-tuples of elements in $H_{2}\left(Y, C_{r}^{*}(G)\right)$. Therefore since $\mathbb{Q}[G]$ is dense in the real group $C^{*}$-algebra $C_{r}^{*}(G)$, we can replace the elements $a_{i, k} \in C_{r}^{*}(G)$ by sufficiently close $a_{i, k}^{\prime} \in \mathbb{Q}[G]$ in such a way that the elements

$$
b_{k}^{\prime}=\sum_{i=1}^{n_{k}} h\left(x_{i, k} \otimes 1\right) a_{i, k}^{\prime}
$$

constitute a basis for $H_{2}\left(Y, C_{r}^{*}(G)\right)$. We have now shown that the map

$$
H_{2}(\tilde{Y}) \otimes \mathbb{Q}[G] \rightarrow H_{2}\left(Y, C_{r}^{*}(G)\right)
$$

has within its image a basis for $H_{2}\left(Y, C_{r}^{*}(G)\right)$. But multiplication with a non-zero real number is an automorphism of $H_{2}\left(Y, C_{r}^{*}(G)\right)$. So multiplying with the common denominator of all coefficients of all $a_{i, k}^{\prime} \in \mathbb{Q}[G]$ we obtain basis elements

$$
b_{k}^{\prime \prime}=\sum_{i=1}^{n_{k}} h\left(x_{i, k} \otimes 1\right) a_{i, k}^{\prime \prime}
$$

with $a_{i, k}^{\prime \prime} \in \mathbb{Z}[G]$. Moreover, since

$$
h\left(x_{i, k} \otimes 1\right) \cdot a_{i, k}^{\prime \prime}=h\left(x_{i, k} \otimes 1 \cdot a_{i, k}^{\prime \prime}\right)=h\left(x_{i, k} \otimes a_{i, k}^{\prime \prime} \cdot 1\right)=h\left(x_{i, k} \cdot a_{i, k}^{\prime \prime} \otimes 1\right)
$$

the elements $b_{k}^{\prime \prime}$ lie in the image of $h \circ j$ as required.
Proof of Theorem 4.1. Choose elements $v_{1}, \ldots, v_{d} \in H_{2}(\tilde{Y})$ which are sent by $h \circ j$ to a basis for $H_{2}\left(Y, C_{r}^{*}(G)\right)$. By the Hurewicz Isomorphism Theorem each $v_{k}$ is represented by a map $S^{2} \rightarrow Y$. Let us use these maps to attach $d$ 3-cells to $Y$, and let us denote by $X$ the three-dimensional $C W$-complex obtained in this way. By construction,

$$
H_{0}\left(X, C_{r}^{*}(G)\right)=0 \quad \text { and } \quad H_{1}\left(X, C_{r}^{*}(G)\right)=0
$$

Moreover, the attaching maps are chosen exactly in such a way that the differential $b: C_{3}(\widetilde{X}) \otimes C_{r}^{*}(G) \rightarrow C_{2}(\widetilde{X}) \otimes C_{r}^{*}(G)$ gives an isomorphism between $C_{3}(\widetilde{X}) \otimes C_{r}^{*}(G)$ and the space

$$
H_{2}\left(Y, C_{r}^{*}(G)\right)=\operatorname{Kernel}\left(b \otimes 1: C_{2}(\tilde{X}) \otimes C_{r}^{*}(G) \rightarrow C_{1}(\tilde{X}) \otimes C_{r}^{*}(G)\right)
$$

Thus we have killed all homology in degree 2 without creating any new homology in degree 3. Therefore we achieve $H_{k}\left(X, C_{r}^{*}(G)\right)=0$ for all $k$, as required.

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[^0]:    *The question, or conjecture, was that no such examples of the sort that Farber and Weinberger constructed could exist.

[^1]:    *For the purposes of this note it will be convenient to work over the reals. We note that the real cases of the results in this section follow immediately from their complex counterparts by complexification.

