On Perrot's Index Cocycles

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Introduction

I am going to discuss the following 2013 paper of Denis Perrot:

Pseudodifferential extension and Todd class

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Abstract

Let M be a closed manifold. Wodzicki shows that, in the stable range, the cyclic cohomology of the associative algebra of pseudodifferential symbols of order ≤ 0 is isomorphic to the homology of the cosphere bundle of M. In this article we develop a formalism which allows to calculate that, under this isomorphism, the Radul cocycle corresponds to the Poincaré dual of the Todd class. As an immediate corollary we obtain a purely algebraic proof of the Atiyah–Singer index theorem for elliptic pseudodifferential operators on closed manifolds.

The paper calls to mind many things . . . Kasparov's Dirac operator, his index theorem for pseudodifferential operators, Getzler's local computation for the Levi-Civita Dirac operator, the Mathai-Quillen explicit Chern character computations, the Connes-Moscovici local index formula . . .

But it is rather different from all of these, I believe.

An Overview

What did Perrot do?

In a nutshell . . .

- Perrot built two cohomologous periodic cyclic cocycles for the algebra of order zero complete pseudodifferential symbols. He used the familiar JLO technique in cyclic theory, but in a new and completely algebraic context (no Hilbert space operators, no Hilbert space traces).
- The two cocycles are extremely similar in appearance (which helps explain why they are cohomologous). But Perrot showed that one is *exactly* the Radul cocycle that gives the analytic index, whereas the other descends to the quotient algebra of principal symbols. It is a de Rham current, and it is *exactly* the Poincaré dual of the Todd form that appears in the Atiyah-Singer formula.

What am I going to do in this talk?

I shall follow a simplified route to some, but not all, of Perrot's results (the simplifications come in part from aiming to do less than Perrot).

I shall concentrate on the descended cocycle for $C^\infty(S^*M)$, and view the rest of Perrot's work as the solution of a lifting problem.

- I shall construct an algebra $\mathcal{C}(T^*M)$ that essentially consists of linear partial differential operators on a Clifford algebra bundle over T^*M . . .
- . . . except that it includes an indeterminate ε, and is comprised of **infinite-order differential operators**.
- I shall construct a canonical **supertrace**

$$\mathrm{STr}\colon \mathcal{C}(T^*M) \longrightarrow \mathbb{C}$$

(To be more precise, STr is defined on a bimodule over $\mathcal{C}(T^*M)$ that is free and singly generated as a left module.)

This is all due to Perrot, who does the constructions in the more involved context of complete pseudodifferential symbols.

Differential operators acting on spinors

- I shall use the generalized tangent bundle

$$GTM = T^*M \oplus TM$$

and its canonical nondegenerate but indefinite bilinear form to construct a bundle of Clifford algebras over T^*M.

- Given an affine connection ∇ on the tangent bundle of M, I shall construct a canonical Dirac operator $D \in \mathcal{C}(T^*M)$. It is not an elliptic operator! In fact in local coordinates,

$$\boxed{D^2 = \varepsilon \sum \partial_{x_i} \partial_{p_i} + \text{lower terms}}$$

- Combining this with a natural morphism of algebras

$$\rho: C^\infty(S^*M) \longrightarrow \mathcal{C}(T^*M)$$

one obtains a triple (A, STr, D) for $A = C^\infty(S^*M)$.

What else am I going to do in this talk?

- I shall construct from the triple a cyclic cocycle Ψ^{∇} of JLO type for the algebra $C^{\infty}(S^*M)$. The exponential $\exp(D^2)$ that appears in the JLO formula is defined as a power series.
- I shall sketch Perrot's computation that the above cyclic cocycle is in fact a de Rham current, namely

$$\alpha \longmapsto \int_{S^*M} \alpha \wedge \mathrm{Todd}(T_{\mathbb{C}}M, \nabla)$$

  This involves

  - A remarkable rescaling property for Ψ^{∇} using the tangent groupoid, and
  - Schur's well-known formula for the derivative of the exponential map to obtain the Todd class.

Pseudodifferential operators

Before turning to infinite-order differential operators and the like, I shall make some introductory remarks about (compactly supported, classical) pseudodifferential operators on a smooth manifold M.

They include the (compactly supported) smoothing operators as an ideal, and so there is an extension of algebras

$$0 \longrightarrow \mathrm{PSDO}^{-\infty}(M) \longrightarrow \mathrm{PSDO}(M) \overset{\sigma_{\mathrm{comp}}}{\longrightarrow} \mathcal{S}(M) \longrightarrow 0$$

in which $\mathrm{PSDO}^{-\infty}(M)$ is the smoothing operators, and the quotient algebra $\mathcal{S}(M)$ is called the algebra of complete symbols on M.

The algebra of complete symbols, and simpler algebras that are derived from it, will be the main focus here.

Order, filtration, associated graded algebra

The algebra PSDO(M) carries an increasing filtration, of course, given by the integer-valued pseudodifferential order.

The algebra $\mathcal{S}(M)$ inherits this filtration from PSDO(M), and there are natural isomorphisms

$$\mathcal{S}^k(M)\big/\mathcal{S}^{k-1}(M) \xrightarrow{\cong} \Big\{ s\colon {T^*}'M \overset{C^\infty}{\to} \mathbb{C} \\ : s(t\xi) = t^k s(\xi)\ \forall t > 0,\ \forall \xi \in {T^*}'M \Big\}$$

(the prime denotes removal of the zero section from the cotangent bundle; I'll mostly omit the prime from now on).

The algebra $\mathcal{S}(M)$ is noncommutative, but the associated graded algebra

$$\mathcal{S}_{\mathrm{gr}}(M) = \oplus_k \mathcal{S}^k(M)\big/\mathcal{S}^{k-1}(M)$$

is commutative, with the obvious pointwise multiplication.

In summary: $\mathcal{S}(M)$ is a deformation of the algebra $\mathcal{S}_{\mathrm{gr}}(M)$ of smooth, polyhomogeneous functions on T^*M.

The residue trace and the Radul cocycle

Perrot uses two crucial structures on $\mathcal{S}(M)$. The first is the residue trace or Wodzicki residue

$$\mathrm{ResTr} : \mathcal{S}(M) \longrightarrow \mathbb{C}$$

which doesn't really need to be explained to this audience.

Let me just recall that if T has order $-n$, where $n = \dim(M)$, then

$$\mathrm{ResTr}(T) = (2\pi)^{-n} \int_{S^*M} \sigma_{-n}(T)$$

after identifying order $-n$ homogeneous functions on T^*M with top-forms on S^*M.

The second is the Radul cocycle, which is the cyclic 1-cocycle

$$\mathrm{Radul}\big(A, B\big) = \mathrm{ResTr}\big(A \cdot \delta(B)\big)$$

Here δ is the outer derivation $\mathrm{ad}_{\log(\Delta)}$ on $\mathcal{S}(M)$. The Radul cocycle is related to index theory via the formula

$$\mathrm{Index}(T) = \mathrm{Radul}\big(T^{-1}, T\big)$$

(Perrot proved a more precise result prior to his paper.)

Differential Operators on the Cotangent Bundle

Polyhomogeneous functions and differential operators

Let M be a smooth manifold and denote by

$$\mathrm{Diff}_{\mathrm{poly}}(T^*M) \subseteq \mathrm{End}_{\mathbb{C}}\big(C^\infty(T^{*\prime}M)\big)$$

the algebra of linear partial differential operators on T^*M (minus the zero section) whose coefficient functions are polyhomogeneous in the fiber direction.

To be a bit more precise, introduce the Euler vector field

$$(Ef)(\alpha) = \frac{d}{dt}\Big|_{t=0} f(e^t\alpha), \qquad E = \sum p_i \partial_{p_i}$$

and define $\mathrm{Diff}_{\mathrm{poly}}(T^*M)$ to be the direct sum of the integer eigenspaces for the adjoint action of E on all differential operators.

Remark. The order zero part of $\mathrm{Diff}_{\mathrm{poly}}(T^*M)$ is the associated graded algebra $\mathcal{S}_{\mathrm{gr}}(M)$ from before.

Infinite-order differential operators on the cotangent bundle

I'm going to define an algebra $\mathcal{A}(T^*M)$ with

$$\mathrm{Diff}_{\mathrm{poly}}(T^*M)[\varepsilon] \subseteq \mathcal{A}(T^*M) \subseteq \mathrm{Diff}_{\mathrm{poly}}(T^*M)[[\varepsilon]]$$

In fact, I'll work with the obvious sheaf of such algebras.

I'm going to do this because I want to form operators like

$$\exp(\varepsilon\Delta) \in \mathcal{A}(T^*U)$$

associated to coordinate neighborhoods $U \subseteq M$, where

$$\Delta = \sum \partial_{x_i}\partial_{p_i}$$

Remark. In reality $\exp(\varepsilon\Delta)$ will actually lie in a bimodule over $\mathcal{A}(T^*U)$ that I shall describe presently.

Caution. The operator Δ above is not invariantly defined (it depends on the choice of coordinates).

The exact definition of the algebra

$$\mathcal{A}(T^*M) \subseteq \mathrm{Diff}_{\mathrm{poly}}(T^*M)$$

is actually not that important, and there are various options. The constraints are essentially that

- $\mathcal{A}(T^*M)$ needs to be large enough to be invariant under the adjoint action of $\exp(\varepsilon\Delta)$ (in a coordinate chart), with $\Delta = \sum \partial_{x_i}\partial_{p_i}$ as above.
- $\mathcal{A}(T^*M)$ needs to be small enough to allow for the definition of a trace functional

$$\mathrm{Tr}\colon \mathcal{A}(T^*M) \longrightarrow \mathbb{C}$$

(although in reality the trace will be defined on the bimodule just mentioned)

Some filtrations

There are various increasing filtrations

$$\cdots \subseteq \mathrm{Diff}_{\mathrm{poly}}(T^*M)_k \subseteq \mathrm{Diff}_{\mathrm{poly}}(T^*M)_{k+1} \subseteq \cdots \qquad (k \in \mathbb{Z})$$

on the algebra $\mathrm{Diff}_{\mathrm{poly}}(T^*M)$ associated to:

- the total vertical order—given by the adjoint action of E.
- the PDO order—the usual order as a differential operator.
- the horizontal PDO order—the modified partial differential operator order in which vertical derivatives are given order 0.

Here is how these work in local coordinates:

	x_i	∂_{x_i}	p_i	∂_{p_i}
total vertical order	0	0	1	-1
PDO order	0	1	0	1
horizontal PDO order	0	1	0	0

Another helpful filtration, and the definition of $\mathcal{A}(T^*M)$

Definition

$$\begin{aligned}\text{helpful order}(D) = 2 \cdot \text{total vertical order}(D) \\ + \text{PDO order}(D) \\ + 2 \cdot \text{horizontal PDO order}(D).\end{aligned}$$

Examples

$$\text{helpful order}(\Delta) = 2, \quad \text{but also}$$

$$\text{helpful order}(\partial_{p_i}) = -1 \quad \text{and} \quad \text{helpful order}(\partial_{x_i}) = 3$$

Definition

We define $\mathcal{A}(T^*M) \subseteq \mathrm{Diff}_{\mathrm{poly}}(T^*M)[[\varepsilon]]$ by

$$\sum_k D_k \varepsilon^k \in \mathcal{A}(T^*M) \quad \Leftrightarrow \quad \exists N \; \forall k : \; \text{helpful order}(D_k) \leq k+N$$

Example (Nonexample?) $\exp(\varepsilon\Delta)$ does not belong to $\mathcal{A}(T^*U)$.

$\mathcal{A}(T^*M)$ is large enough ...

Lemma

For $\Delta = \sum \partial_{x_i} \partial_{p_i}$ in a coordinate neighborhood,

$$\text{helpful order}\big(\text{ad}_\Delta(D)\big) \le \text{helpful order}\big(D\big) + 1$$

Theorem

- *The subalgebra $\mathcal{A}(T^*U) \subseteq \text{Diff}_{poly}(T^*U)[[\varepsilon]]$ is invariant under the automorphisms*

$$\text{Ad}_{\exp(t\varepsilon\Delta)} = \exp(t\varepsilon\,\text{ad}_\Delta)$$

- *If Δ and Δ' are defined using any two different coordinate systems on U, then*

$$\exp(-\varepsilon\Delta')\exp(\varepsilon\Delta) \in \mathcal{A}(T^*U)$$

Proof For the second point, $\Delta - \Delta'$ has helpful order 0.

The bimodule $\mathcal{A}(T^*M) \cdot \exp(\varepsilon\Delta)$

From the previous theorem, the linear space

$$\mathcal{A}(T^*U) \cdot \exp(\varepsilon\Delta) \subseteq \mathrm{Diff}_{\mathrm{poly}}(T^*U)[[\varepsilon]]$$

is independent of the choice of coordinates and an $\mathcal{A}(T^*M)$-bimodule.

There is therefore a canonical bimodule

$$\mathcal{A}(T^*M) \cdot \exp(\varepsilon\Delta) \subseteq \mathrm{Diff}_{\mathrm{poly}}(T^*M)[[\varepsilon]]$$

and the next goal is to define a scalar-valued trace on this bimodule.

Remark To be precise, the trace is defined on operators compactly supported in the M-direction; I mostly won't mention this again.

The trace functional

The trace is built in several stages:

- Construction of an integral on polyhomogeneous functions, inspired by the residue trace on pseudodifferential symbols.
- Local coordinate-based construction of a scalar-valued Gaussian integral on constant-coefficient differential operators of the form $\varepsilon^k p(\partial_{x_i}, \partial_{p_j}) \exp(\varepsilon\Delta)$.
- Combination of the above to obtain $\mathcal{A}(T^*U) \to \mathbb{C}$ (on elements compactly supported in the base direction).
- Proof of coordinate-independence of this functional, and construction of a global functional by partitions of unity.
- Proof of the trace property.

Trace on polyhomogeneous functions

There is an obvious scalar trace functional ResTr on

$$C^\infty_{\mathrm{poly}}(T^*M) = \mathcal{S}_{\mathrm{gr}}(M)$$

that mimics (apes) the noncommutative residue:

- Select the order $-n$ component f_{-n} (which may be zero, of course). Here $n = \dim(M)$.
- View f_{-n} as a top-degree form on S^*M.
- Integrate this top-degree form over S^*M.

Gaussian integrals

If A is a positive-definite $2n \times 2n$ matrix, and if $p\colon \mathbb{R}^{2n} \to \mathbb{C}$ is a polynomial function, then

$$\int_{\mathbb{R}^{2n}} p(w) \exp\left(-\tfrac{1}{2}\langle w, Aw\rangle\right) dw = \frac{(2\pi)^n}{\sqrt{\det(A)}} \left(\exp\left(\tfrac{1}{2}\langle \partial_w, A^{-1}\partial_w\rangle\right) p\right)(0).$$

So define (ignoring a multiplicative constant)

$$\int \varepsilon^k p(\partial_{x_i}, \partial_{p_j}) \exp\left(\varepsilon \Delta\right) = \varepsilon^{k-n} \left(\exp\left(\varepsilon^{-1} \partial_{x_i} \partial_{p_i}\right) p\right)(0)$$

This value lies in $\mathbb{C}[\varepsilon^{-1}, \varepsilon]$.

Since every element of $\mathrm{Diff}_{\mathrm{poly}}(T^* U)[[\varepsilon]]$ can be written as a sum of terms $\varepsilon^k f_{\alpha\beta k} \partial_x^\alpha \partial_p^\beta \exp(\varepsilon \Delta)$, the above extends $^{(*)}$ to a Gaussian integral morphism

$$\mathcal{A}(T^* U) \cdot \exp(\varepsilon \Delta) \longrightarrow C^\infty_{\mathrm{poly}}(T^* U)[\varepsilon^{-1}, \varepsilon]]$$

A technical detail, and the definition of the trace

(∗) There is a small problem with the above definition: infinitely many of the terms from a sum

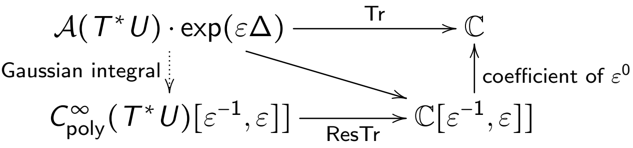
$$\sum \varepsilon^k f_{\alpha\beta k} \partial_x^\alpha \partial_p^\beta \exp(\varepsilon\Delta) \in \mathcal{A}(T^*M)$$

could contribute to a single power of ε in $C^\infty_{\text{poly}}(T^*U)[\varepsilon^{-1}, \varepsilon]]$.

That's because the Gaussian integral for constant coefficient operators involves possibly large negative powers of ε.

But because $\mathcal{A}(T^*M)$ is small enough (as measured by the helpful order), this only happens among terms with $\text{order}(f_{\alpha\beta k}) \to -\infty$.

So if we integrate the coefficient functions and take the coefficient of ε^0 we obtain a well-defined functional



Fundamental properties of the trace

Theorem

- *The trace functional is independent of the choice of local coordinates.*
- *The trace functional is actually a trace.*

The second item in the theorem is proved using this fact: if

$$h(\partial_{x_i}, \partial_{p_j}) = p(\partial_{x_i}, \partial_{p_j}) \exp(\varepsilon\Delta)$$

and if h is a partial derivative, then the Gaussian integral of $p(\partial_{x_i}, \partial_{p_j}) \exp(\varepsilon\Delta)$ is zero. This, in turn, is proved by Wick rotation to ordinary Gaussian integrals.

We may now use partitions of unity to define the trace functional globally, on $\mathcal{A}(T^*M) \cdot \exp(\varepsilon\Delta)$.

Clifford algebra of the generalized tangent bundle

As I mentioned earlier, the generalized tangent bundle over M is

$$GTM = T^*M \oplus TM$$

It carries a canonical nondegenerate but indefinite bilinear form.

We can in any case form the bundle of Clifford algebras $\mathrm{Cliff}(GTM)$ over M. Pull it back to T^*M.

We can now form the algebra

$$\mathrm{Diff}_{\mathrm{poly}}(T^*M, \mathrm{Cliff}(GTM))$$

of differential operators acting on sections of $\mathrm{Cliff}(GTM)$ over T^*M (not on spinors!) whose coefficient functions are polyhomogeneous in the fiber direction.

We can now repeat the constructions that we have just sketched in the scalar case . . .

Infinite-order operators on the Clifford algebra bundle

We now form the subalgebra

$$\mathcal{C}(T^*M) \subseteq \mathrm{Diff}_{\mathrm{poly}}\big(T^*M, \mathrm{Cliff}(GTM)\big)[[\varepsilon]]$$

and the bimodule

$$\mathcal{C}(T^*M)\cdot\exp(\varepsilon\Delta) \subseteq \mathrm{Diff}_{\mathrm{poly}}\big(T^*M, \mathrm{Cliff}(GTM)\big)[[\varepsilon]]$$

using exactly the same prescriptions (involving the helpful order) as in the scalar case.

Remark. Think of $\mathcal{C}(T^*M)\cdot\exp(\varepsilon\Delta)$ as something like

$$\mathcal{C}(T^*M)\cdot\exp(\varepsilon\Delta) \approx \text{Operators on } L^2\big(T^*M, \mathrm{Cliff}(GTM)\big)$$

This is where Kasparov's version of the Dirac operator T^*M lives. More on Dirac operators in a moment.

Inserting the supertrace on the Clifford algebra into the previous construction we obtain a supertrace

$$\mathrm{STr}\colon \mathcal{C}(T^*M)\cdot\exp(\varepsilon\Delta) \longrightarrow \mathbb{C}$$

Connections and the Dirac operator

If you're still with me after all that algebra . . . it's time for some geometry.

Fix an affine connection ∇ on TM over M. It determines an affine connection on T^*M over M.

The connection induces an identification of the tangent bundle for T^*M with the generalized tangent bundle for M, pulled back to T^*M. So we can form a Dirac operator (not elliptic, of course!), which we do, with one modification:

$$D = \varepsilon(c \cdot \nabla)_{\text{horiz}} + (c \cdot \nabla)_{\text{vert}}$$

Remark The Clifford multiplication operators are defined using right multiplications on the Clifford algebra, arranged with signs to graded commute with left multiplications. This is exactly what Kasparov does.

A Dirac triple

Define an algebra homomorphism

$$\rho\colon C^\infty(S^*M) \longrightarrow \mathcal{C}(T^*M)$$

by following these simple steps:

- View functions on S^*M as 0-homogeneous functions on T^*M.
- Use the canonical isomorphism

  $$\mathrm{Cliff}(GTM) = \mathrm{End}_{\mathbb{C}}(\Lambda^\bullet T^*M)$$

  to define the 0-form projection $\Pi \in \mathrm{Cliff}(GTM)$. View it as an order zero PDO on $\mathrm{Cliff}(GTM)$ by left multiplication.
- Define $\rho(f) = f \cdot \Pi$.

We have constructed a sort of "algebraic spectral triple" (A, STr, D) with $A = C^\infty(T^*M)$.

Cyclic Cocycles

Even version of Perrot's index cocycle

For p even and $f_0, \ldots, f_p \in C^\infty(S^*M)$, Perrot defines, using the JLO formalism,

$$\Phi_p^\nabla(f_0, \ldots, f_p) = \int_{\Delta^p} \mathrm{STr}\Big(\rho(f_0) e^{-s_0 D^2} [D, \rho(f_1)] e^{-s_1 D^2} \cdots [D, \rho(f_p)] e^{-s_p D^2}\Big)\, ds$$

or more briefly

$$\Phi^\nabla = JLO^{(A, \mathrm{STr}, D)}$$

This is a *finitely-supported*, even, periodic cyclic (b,B)-cocycle (a bit of a mouthful for most audiences, but not this one). But . . .

Theorem (Perrot)

If ∇ is a Levi-Civita connection, then for all even p and all $f_0, \ldots, f_p \in C^\infty(S^*M)$, $\Phi_p^\nabla(f_0, \ldots, f_p) \equiv 0$.

That may seem a bit anticlimactic . . . but Perrot uses this computation to prove the vanishing of the noncommutative residue in the cyclic cohomology of $S^0(M)$.

Odd version of JLO cocycle

To get a nonzero answer, we need to return to the Radul cocycle on pseudodifferential symbols, find a semiclassical counterpart, and incorporate it into the JLO formalism.

Fortunately all these steps are quite easy (for this audience).

- A derivation δ may be built from the Hamiltonian vector field of the symbol of any (genuine) Laplace operator on M.
- The supertrace is closed with respect to δ.
- The pair (STr, δ) may be incorporated into the JLO formalism (following e.g. Quillen, as pointed out by Rodsphon).
- Attached to this there is an odd JLO cocycle

$$\Psi^{\nabla} = JLO^{(A, \mathrm{STr}, D, \delta)}$$

Perrot's main theorems

The first theorem involves the development of everything that has been discussed today for pseudodifferential symbols on M in place of functions on T^*M.

Theorem

If ∇ is any torsion-free connection, then the pullback of Ψ^∇ along the principal symbol homomorphism

$$\mathcal{S}^0(M) \longrightarrow C^\infty(S^*M)$$

is cohomologous to the Radul cocycle.

The second theorem is a remarkable direct computation:

Theorem

If ∇ is a Levi-Civita connection, then for all odd p and all $f_0, \ldots, f_p \in C^\infty(S^*M)$,

$$\Psi_p^\nabla(f_0, \ldots, f_p) \equiv \int_{S^*M} f_0 df_1 \ldots df_p \cdot \mathrm{Todd}(T_{\mathbb{C}}M, \nabla)$$

Lie algebra action on pseudodifferential symbols

My focus is on the evaluation of Ψ^{∇}. But I shall say a few words about pseudodifferential symbols and the index theorem.

Recall that the order filtration on the classical pseudodifferential symbols has associated graded algebra $\mathcal{S}_{\text{gr}}(M) = C^{\infty}_{\text{poly}}(T^*M)$.

To do index theory, we need to deform from $\mathcal{S}_{\text{gr}}(M)$ to $\mathcal{S}(M)$. I'll use crossed product algebras to indicate how this is done.

Denote by $\mathfrak{g}$ the Lie algebra of first order (scalar) differential operators on M (I shall ignore Clifford algebras here, for simplicity). It is filtered by order and there is an (inner!) action

$$\mathfrak{g} \times \mathcal{S}(M) \longrightarrow \mathcal{S}(M)$$

that is compatible with the order filtrations. So there is induced action of $\mathfrak{g}_{\text{gr}}$ on $\mathcal{S}_{\text{gr}}(M)$. One computes that $\mathfrak{g}_{\text{gr}}$ acts by vector fields on T^*M of total vertical order 0 or -1.

Crossed product algebra

Now form the crossed product algebras (twisted tensor products)

$$\mathcal{S}(M) \rtimes \mathcal{U}(\mathfrak{g}) \quad \text{and} \quad \mathcal{S}_{\mathrm{gr}}(M) \rtimes \mathcal{U}(\mathfrak{g}_{\mathrm{gr}})$$

There are obvious representations

$$\mathcal{S}(M) \rtimes \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}(M)) \quad \text{and} \quad \mathcal{S}_{\mathrm{gr}}(M) \rtimes \mathcal{U}(\mathfrak{g}_{\mathrm{gr}}) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}_{\mathrm{gr}}(M))$$

and the image of the second is $\mathrm{Diff}_{\mathrm{poly}}(T^*M)$. Accordingly we set

$$\mathcal{L}(M) = \mathrm{Image}\big(\mathcal{S}(M) \rtimes \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S}(M))\big)$$

(Perrot's notation) and define

$$\mathcal{D}(M) \subseteq \mathcal{L}(M)[[\varepsilon]]$$

(Perrot's notation again), in exact analogy with $\mathcal{A}(T^*M)$ earlier.

Bimodule, trace, Dirac operators and cocycles

Perrot defines a bimodule from $\mathcal{D}(M)$ and a trace, and lifts everything to the Clifford algebra context (in which $\mathcal{S}(M)$ becomes the algebra of symbols of operators acting on sections of $\Lambda^* T^* M$).

- The bad news . . .

The Dirac operator associated to a connection ∇ is not canonically defined in $\mathcal{S}(M) \rtimes \mathcal{U}(\mathfrak{g})$. This is already a problem in $\mathcal{S}_{\mathrm{gr}}(M) \rtimes \mathcal{U}(\mathfrak{g}_{\mathrm{gr}})$ (N.B. One divides by the ideal generated by all

$$Af \otimes X - A \otimes fX \qquad (f \in C^\infty_{\mathrm{poly}}(T^* M))$$

to obtain differential operators from the crossed product.) Instead, Perrot uses local coordinates and partitions of unity.

Proof of the index theorem

- The good news . . .

There are two Dirac operators! This is suggested by the canonical isomorphism

$$\mathcal{S}(M) \rtimes \mathcal{U}(\mathfrak{g}) \cong \mathcal{S}(M) \otimes \mathcal{U}(\mathfrak{g})$$

for inner actions—which gives two copies of $\mathfrak{g}$ in the crossed product.

The second copy of $\mathfrak{g}$, coming from the right-hand side in the isomorphism above, actually commutes with $\mathcal{S}(M)$.

It follows that the commutator terms $[D, a]$ in the JLO formula are zero, for the corresponding Dirac operator, and so the JLO cycocle associated to the second Dirac operator collapses to its lowest term, which is a cyclic 1-cocycle—the Radul cocycle.

To prove an index theorem, Perrot (i) shows that the JLO-type cocycles for the two Dirac operators are cohomologous, and (ii) computes the cocycle for the first Dirac operator . . .

Scale-Invariance of the Dirac Operator

Rescaled Clifford algebra bundle on the tangent groupoid

Now I shall return to Ψ^∇ (the other cocycle, in effect). The computation of Ψ^∇ is greatly simplified by a remarkable scale-invariance property of the square of the Dirac operator.

Denote by $\mathbb{M} \rightrightarrows M \times \mathbb{R}$ the tangent groupoid for M, and denote by $\mathbb{T}^*\mathbb{M}$ the fiberwise cotangent bundle for the source fibers. It is a smooth family of cotangent bundles over $M \times \mathbb{R}$ via the source map:

$$\mathbb{T}^*\mathbb{M} \longrightarrow M \times \mathbb{R}$$

The rescaled generalized tangent bundle RTM over $M \times \mathbb{R}$ is the bundle whose smooth sections are smooth families of sections s_t of $GTM = TM \oplus T^*M$ over M with s_0 a section of T^*M alone. Pull it back to $\mathbb{T}^*\mathbb{M}$ and form the Clifford algebra bundle.

Theorem

Perrot's Dirac $(D^\nabla)^2$, repeated on each fiber $\mathbb{T}^*\mathbb{M}_{(m,t)}$ with $t \neq 0$, extends to a smooth equivariant family over all fibers of $\mathbb{T}^*\mathbb{M}$.

Computation of Perrot's cocycle

The JLO integrand is assembled from $\exp(-sD^2)$ and $[D, a]$ (and $\delta(a)$ and $\delta(D)$), and all share the same scale invariance.

So the computation of Ψ^∇ can be reduced to a computation at $t = 0$, involving operators on the linear spaces T^*T_m that are translation invariant in the T_m-direction.

The ingredients of the computation are

$$(D^\nabla)^2_m = \varepsilon\Delta + R_m$$

where the operator R_m is formed from the curvature of ∇ at m, along with

$$[D, f]_m = \varepsilon(df)_{\mathrm{horiz},m} + (df)_{\mathrm{vert},m}$$

(and δ-terms). Under the trace,

$$\exp(\varepsilon\Delta + R_m) = \mathrm{Todd}(R_m) \cdot \exp(\varepsilon\Delta)$$

by a variation on the usual Lie theory computation (Schur, 1891).

Thank You!

References

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Satz 2. *Sobald die Grössen* $c^{\mathfrak{c}}_{\mathfrak{a},\mathfrak{b}}$ *die Bedingungen* $c^{\mathfrak{c}}_{\mathfrak{a},\mathfrak{b}} = -c^{\mathfrak{c}}_{\mathfrak{b},\mathfrak{a}}$ *und:*

$$\sum_{\mathfrak{c}=1}^{r} \left(c^{\mathfrak{c}}_{\mathfrak{b},\mathfrak{b}_1} c^{\mathfrak{a}}_{\mathfrak{b}_2,\mathfrak{c}} + c^{\mathfrak{c}}_{\mathfrak{b}_1,\mathfrak{b}_2} c^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}} + c^{\mathfrak{c}}_{\mathfrak{b}_2,\mathfrak{b}} c^{\mathfrak{a}}_{\mathfrak{b}_1,\mathfrak{c}} \right) = 0 \tag{25}$$

erfüllen, sind die Componenten der infinitesimalen Transformationen der Parametergruppe von obiger Zusammensetzung in ihrer kanonischen Form gegeben durch die Reihen:

$$\omega^{\mathfrak{b}}_{\mathfrak{a}}(u) = \sum_{m=0}^{\infty} \lambda_m U^{(m)}_{\mathfrak{a},\mathfrak{b}}, \tag{34}$$

wo:

$$U^{(0)}_{\mathfrak{a},\mathfrak{b}} = \delta_{\mathfrak{a},\mathfrak{b}}, \quad U^{(1)}_{\mathfrak{a},\mathfrak{b}} = \sum_{\mathfrak{c}=1}^{r} c^{\mathfrak{a}}_{\mathfrak{b},\mathfrak{c}} u_{\mathfrak{c}},$$

$$U^{(m)}_{\mathfrak{a},\mathfrak{b}} = \sum_{\mathfrak{c}_1,\mathfrak{c}_2,\cdots,\mathfrak{c}_{m-1}=1}^{r} U^{(1)}_{\mathfrak{a},\mathfrak{c}_1} U^{(1)}_{\mathfrak{c}_1,\mathfrak{c}_2} \cdots U^{(1)}_{\mathfrak{c}_{m-2},\mathfrak{c}_{m-1}} U^{(1)}_{\mathfrak{c}_{m-1},\mathfrak{b}},$$

ferner:

$$\lambda_0 = 1, \quad \lambda_1 = -\frac{1}{2}, \quad \lambda_{2q} = (-1)^{q+1} \frac{B_{2q-1}}{(2q)!} \quad \text{und} \quad \lambda_{2q+1} = 0.$$

- Schur, *Zur Theorie der endlichen Transformationsgruppen.* Math. Ann. **38** (1891).