Contractions of Lie groups, representations and the Mackey bijection

Nigel Higson

Department of Mathematics
Pennsylvania State University

Xiamen University
July 8, 2022

Summary

In 1975 George Mackey pointed out an “analogy” between irreducible representations of a real reductive group G and irreducible representations of its Cartan motion group (which is a sort of simplification of G).

Mackey’s point of view was opposed to the prevailing views about the classification of irreducible representations of reductive groups, and was mostly ignored.

But thanks to quite different perspective from C^*-algebra theory and noncommutative geometry, Mackey’s point of view was kept alive.

Eventually the analogy was made precise and proved, leading to new problems and perspectives.

Real Reductive Groups

This talk will be about real reductive groups and their representations.

For simplicity I shall work with subgroups $G \subseteq GL(n, \mathbb{R})$ that are

- Closed and connected
- Closed under the transpose operation.

(The class of all reductive groups is a bit larger.) I shall write

- $K = G \cap O(n)$
- $A =$ positive diagonal matrices in G.

I shall assume that A is maximal among abelian subgroups comprised of positive-definite matrices.

Basic Structure

Form the Lie algebra of G:

$$\mathfrak{g} = \{\, X : \exp(tX) \in G \ \forall t \in \mathbb{R} \,\}$$

It decomposes as a direct sum of subspaces

$$\mathfrak{g} = \{\, \text{skew-symmetric matrices} \,\} \oplus \{\, \text{symmetric matrices} \,\} = \mathfrak{k} \oplus \mathfrak{p}$$

and there is a corresponding direct product decomposition of the group (as a manifold)

$$G = K \cdot \exp[\mathfrak{p}]$$

Moreover

$$G = KAK$$

So A (or more precisely a quotient A/W by a finite Weyl group action) describes the directions towards infinity in G (mod K).

The Example of SL(2,R)

For instance, for $G = SL(2, \mathbb{R})$ we have:

- $K = SO(2) = \left\{ \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \right\}$
- $A = \left\{ \begin{bmatrix} \exp(t) & 0 \\ 0 & \exp(-t) \end{bmatrix} \right\}$
- $\mathfrak{p} = \left\{ \begin{bmatrix} a & c \\ c & -a \end{bmatrix} \right\}$.

As a manifold:

- G is a solid torus, not including the boundary surface.
- The two directions around the torus correspond to the two copies of K in $G = KAK$.
- A (or more precisely $A/W \cong [0, \infty)$) gives the distance from the central interior circle in the solid torus.
- G/K is the hyperbolic plane.

Representations

I'll be examining irreducible unitary Hilbert space representations of G and I'll use the standard notation

$$\widehat{G} = \{\text{ irreducible unitary representations, up to equivalence }\}.$$

This is the unitary dual of G. It carries a natural topology under which it is a locally compact topological space, although not necessarily a Hausdorff topological space.

Actually, I'll be examining a closed subset of the unitary dual, called the tempered dual, which I'll define later. It was the focus of Harish-Chandra's studies.

Unless G is compact or abelian, nearly all the representations in the unitary dual will be infinite-dimensional (and usually all of the representations in the tempered dual will be infinite-dimensional).

Plancherel's Theorem

Harish-Chandra focused on the problem of decomposing the regular representation $L^2(G)$ into irreducible representations, and the determinination the Plancherel formula.

Recall first Plancherel's original formula in the theory of the Fourier transform:

$$2\pi \cdot f(0) = \int_{\mathbb{R}} \widehat{f}(\xi)\, d\xi \qquad (\forall f \in C_c^\infty(\mathbb{R})),$$

where

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{ix\xi}\, dx.$$

Now the irreducible unitary representations of $G = \mathbb{R}$ are $x \mapsto e^{ix\xi}$ ($\xi \in \mathbb{R}$), and the formula above is equivalent to

$$2\pi \cdot \|f\|^2_{L^2(\mathbb{R})} = \int_{\widehat{\mathbb{R}}} |\widehat{f}(\xi)|^2\, d\xi.$$

So the (norm on) $L^2(\mathbb{R})$ has been decomposed into (norms on) irreducible subrepresentations.

The Abstract Plancherel Theorem

If G is a real reductive group, if π is an irreducible unitary representation, and if we define

$$\widehat{f}(\pi) = \pi(f) = \int_G f(g)\pi(g)\,dg,$$

then the abstract Plancherel theorem states that there is a unique measure μ on $\widehat{G}$ (the Plancherel measure) such that

$$f(e) = \int_{\widehat{G}} \mathrm{Trace}(\pi(f))\,d\mu(\pi) \qquad (\forall f \in C_c^\infty(G)),$$

or equivalently

$$\|f\|^2_{L^2(G)} = \int_{\widehat{G}} \|\pi(f)\|^2_{\mathrm{H.S.}}\,d\mu(\pi).$$

For instance, the original theorem of Plancherel tells us that when $G = \mathbb{R}$, the Plancherel measure is

$$d\mu(\xi) = \tfrac{1}{2\pi} d\xi$$

The Peter-Weyl Theorem and Weyl's Formula

An interesting special case is $G =$ compact Lie group. Here

$$\widehat{G} = \text{countable discrete set}$$

and according to the Peter-Weyl theorem, the Plancherel measure of each atom $\{\pi\}$ is

$$\mu(\{\pi\}) = \frac{\dim(\pi)}{\mathrm{vol}(G)}.$$

But one can go much further. Highest weight theory tells us that

$$\widehat{G} \cong \widehat{T}\big/W,$$

(where T is a maximal torus and W is the Weyl group of (G, T)) and Weyl's dimension formula tells us that

$$\dim(\pi_\varphi) = \prod_{\alpha > 0} \frac{\langle \varphi + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$$

The Plancherel measure therefore becomes very explicit ...

Harish-Chandra's Plancherel Formula

Harish-Chandra derived an explicit Plancherel formula for every real reductive group. For example if $G = SL(2, \mathbb{R})$, and if $f \in C_c^\infty(G)$, then

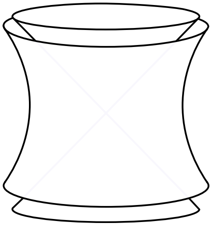
$$2\pi^2 f(e) = \sum_{n \neq 0} \mathrm{Trace}(\pi_n(f)) \cdot |n| + \frac{1}{2}\int_0^\infty \mathrm{Trace}(\pi_{\mathrm{even},\xi}(f))\xi \tanh(\pi\xi/2)\, d\xi + \frac{1}{2}\int_0^\infty \mathrm{Trace}(\pi_{\mathrm{odd},\xi}(f))\xi \coth(\pi\xi/2)\, d\xi$$

(although this case was known prior to Harish-Chandra).

So in this case, $L^2(G)$ decomposes into a discrete series of representations parametrized by $n \neq 0$ and even and odd continuous series, each parametrized by $\xi \geq 0$.

Harish-Chandra's Methods

An important role in Harish-Chandra's approach is played by the geometry of G at infinity, as illustrated here for $SL(2,\mathbb{R})$:



The hyperboloid is a 2D cartoon sketch of $SL(2,\mathbb{R})$, which is the 3D variety

$$\{\, ad - bc = 1 \,\}.$$

As a geometric space, $SL(2,\mathbb{R})$ is asymptotic to the cone

$$\{\, ad - bc = 0 \,\}.$$

Each irreducible representation π gives rise to matrix coefficient functions $g \mapsto \langle v, \pi(g^{-1})v\rangle$ on G, and the asymptotics of these functions are studied.

The same asymptotics are used in the derivation of the Plancherel formula.

Classification of Irreducible Representations

The tempered dual of G is the support of the Plancherel measure in $\widehat{G}$ (recall that the unitary dual has the structure of a topological space).

Harish-Chandra's Plancherel formula is a statement in measure theory, and sets of Plancherel measure zero can be ignored. But the techniques used to prove the Plancherel formula are foundational to the exact classification of the irreducible representations in the tempered dual.

For example when $G = SL(2, \mathbb{R})$, the tempered dual looks like

$$\{ n \in \mathbb{Z} : n \neq 0 \} \sqcup [0, \infty) \sqcup [0, \infty)$$

...except that the point 0 in the "odd" copy of $[0, \infty)$ must be replaced by a "double point" $\{0_+, 0_-\}$ because the odd continuous series representation $\pi_{\text{odd},0}$ decomposes into two irreducible subrepresentations (so the tempered dual is not Hausdorff in this case).

Mackey's 1975 Proposal

- G = real reductive group
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (the Lie algebra of G)
- $G_0 = K \ltimes \mathfrak{p}$ (this semidirect product is called the Cartan motion group).

As we'll see, G_0 is a sort of degeneration, or limiting version, of G. It is a much more elementary group, thanks to its large abelian normal subgroup $\mathfrak{p}$.

". . . the physical interpretation suggests that there ought to exist a 'natural' one to one correspondence between almost all of the unitary representations of G_0 and almost all the unitary representations of G—in spite of the rather different algebraic structures of these groups."

Mackey's 1975 Proposal, Continued

The irreducible unitary representations of G_0 are easy to determine (Mackey did this, much earlier):

$$\widehat{G}_0 = \Big(\bigsqcup_{\xi \in \mathfrak{p}} \widehat{K}_\xi \Big) \Big/ K.$$

Here $K_\xi = \{ k \in K : k\xi k^{-1} = \xi \}$, and to $\sigma \in \widehat{K}_\xi$ one associates the representation of G_0 obtained from the representation

$$(k, X) \longmapsto \exp(i \operatorname{Trace}(X \cdot \xi))\sigma(k)$$

of the subgroup $K_\xi \ltimes \mathfrak{p}$ by unitary induction.

In contrast, the unitary representation theory—and even the tempered representation theory—of G is quite complicated, as we have seen for $SL(2, \mathbb{R})$. The tempered dual was not fully determined when Mackey made his proposal.

Typical Representations

Mackey observed that “typical” tempered representations of G and G_0 closely correspond to one another, both in parametrization and in basic form.

For example, for $G = SL(2, \mathbb{R})$, the representations of G_0 in the subset

$$\Big(\bigsqcup_{\xi \in \mathfrak{p}, \xi \neq 0} \widehat{K}_\xi\Big)\Big/ K \subseteq \Big(\bigsqcup_{\xi \in \mathfrak{p}} \widehat{K}_\xi\Big)\Big/ K$$

correspond to the continuous series representations $\pi_{\text{even},\xi}$ and $\pi_{\text{odd},\xi}$ with $\xi \neq 0$ very closely: same parametrization, same Hilbert spaces, and even the same actions of the common subgroup K of G and G_0 on these Hilbert spaces.

But apart from these explorations, Mackey admitted that

> *We have not yet ventured to formulate a precise conjecture . . .*

Contraction of a Lie Group

Let me say something (only a little) about the physics perspective that motivated Mackey.

Let G be a Lie group and let H be a closed subgroup. The contraction of G along H, studied in physics, is the Lie group

$$G_H = H \ltimes \mathrm{Lie}(G)/\mathrm{Lie}(H).$$

The Cartan motion group is a special case.

The contraction group is a first-order approximation of G near H; geometrically G_H is the normal bundle of H in G.

In the reductive group/Cartan motion group case,

- $G \sim$ isometries of the symmetric space G/K
- $G_0 \sim$ isometries of $\mathfrak{p}$

But G/K and $\mathfrak{p}$ resemble one another at small length scales, hence (perhaps?) the representation theories of G and G_0 should resemble one another too ...

Convolution C*-Algebras

If G is any Lie group, then $C_c^\infty(G)$ is an algebra under convolution:

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g)\, dh,$$

and the same formula defines a representation of this algebra as bounded (left) convolution operators on $L^2(G)$.

The reduced group C^*-algebra $C_r^*(G)$ the norm-closure of this algebra of bounded convolution operators on $L^2(G)$.

In the reductive group case, every tempered irreducible representation of G determines an irreducible representation of $C_r^*(G)$ by the formula

$$\pi(f)v = \int_G f(g)\, \pi(g) v \, dg \qquad (v \in H_\pi),$$

and every irreducible representation of $C_r^*(G)$ comes from a unique tempered irreducible representation of G.

The Role of Reduced C*-Algebra

The reduced C^*-algebra $C_r^*(G)$ is important because it captures the topological structure of the tempered dual.

For example, the Fourier transform extends to an isomorphism of C^*-algebras

$$C_r^*(\mathbb{R}) \cong C_0(\mathbb{R})$$

(on the right-hand side is the C^*-algebra, under pointwise multiplication, of continuous functions on $\mathbb{R}$ that vanish at infinity).

It follows from the Harish-Chandra theory (and a bit more) that there is a similar but much more complicated Fourier isomorphism of C^*-algebras for any reductive G.

(If the tempered dual happens to be a Hausdorff space, then the target C^*-algebra is Morita equivalent to the C^*-algebra of continuous functions, vanishing at infinity, on the tempered dual.)

K-Theory and the Connes-Kasparov Isomorphism

In view of the above, one might ask if reduced C^*-algebra can be used to probe the topological structure of the tempered dual?

Connes and Kasparov did just this, and formulated a conjecture in the language of K-theory for C^*-algebras. I shall not say much about K-theory, except that

$$A \underset{\text{Morita}}{\simeq} C_0(X)$$

$$\Rightarrow \quad K_*(A) \cong \text{Atiyah-Hirzebruch } K\text{-theory of } X.$$

So K-theory is a kind of cohomology theory for C^*-algebras that extends the K-cohomology for locally compact Hausdorff spaces invented by Atiyah and Hirzebruch.

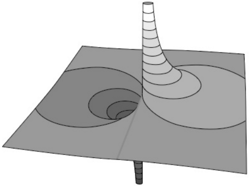
The Connes-Kasparov conjecture was formulated using Dirac operators and index theory, and was subsequently proved using index-theory techniques.

Smooth Family of Groups Associated to a Contraction

Now let me return to Mackey's program and contraction groups.

Associated to any contraction group G_H there is a smooth family of Lie groups, parametrized by $t \in \mathbb{R}$, interpolating between the groups G_H and G:

- $G_t = G$ for all $t \neq 0$
- $G_0 = G_H$.



The smooth family associated to the one-element subgroup of the circle group. All horizontal slices are copies of the circle group, except for the middle slice, which is a copy of its Lie algebra.

This is a special case of a very general construction in geometry, called the deformation to the normal cone, which applies here because G_H is the normal bundle for the submanifold H of G.

Continuous Field of Reduced C*-Algebras

In the case considered by Mackey, where G is reductive and $H = K$, the collection of reduced group C^*-algebras $\{C^*_r(G_t)\}_{t\in\mathbb{R}}$ carries the structure of a continuous field of C^*-algebras.

Now the K-theory groups in any continuous field of C^*-algebras may be assembled into a sheaf of abelian groups over the parameter space (the K-theory groups are the stalks). And about ten years after Mackey made his proposal, Connes observed that the Connes-Kasparov conjecture—now a theorem—can be reformulated using the above continuous field, as follows:

Theorem (Connes-Kasparov Isomorphism)

The continuous field of group C^*-algebras $\{C^*_\lambda(G_t)\}_{t\in\mathbb{R}}$ has constant K-theory.

Connes-Kasparov Versus Mackey

The theorem obviously has some relationship to Mackey's proposal.

But can this precise cohomological statement be reconciled with Mackey's rather vague analogy (a correspondence "almost everywhere" between individual representations of the groups G and G_0)?

There is an awkward tension between the two:

The cohomological Connes-Kasparov isomorphism (as stated above) is about the homotopy types of the tempered duals of G and G_0 and doesn't imply any kind of bijection between the two, whereas Mackey's analogy, as he envisaged it, is some kind of measure-theoretic equivalence, which doesn't imply any kind of homotopy or cohomological equivalence . . .

This prompts one to look more closely at Mackey's computations . . .

The Case of Complex Groups

Let's consider the case of complex reductive groups, where:

- The tempered dual is easy to understand, because all the tempered irreducible representations are of one type—they are all minimal principal series representations and all those minimal principal series representations are irreducible (in particular, the tempered dual is a Haudorff space).
- The dual of G_0 is also easy to understand, because each K_ξ is a connected compact group, whose irreducible representations may be classified by highest weights.

Parameters for Representations (a Calculation)

- G = complex reductive group.
- M = centralizer of A in K. This is a maximal torus in K.
- $\widehat{G}_\lambda = (\widehat{M} \times \mathfrak{a})/W$ (principal series construction). Here W is the Weyl group of (K, M).
- $\widehat{G}_0 = \left(\bigsqcup_{\xi \in \mathfrak{p}} \widehat{K}_\xi\right)/K$ (Mackey's formula) $= \left(\bigsqcup_{\xi \in \mathfrak{a}} \widehat{K}_\xi\right)/W$
- W_ξ = isotropy group of $\xi \in \mathfrak{a}$ in W = Weyl group of K_ξ
- By the above and by highest weight theory,

$$\widehat{G}_0 \cong \Big(\bigsqcup_{\xi \in \mathfrak{a}} \widehat{M}/W_\xi\Big)/W = \big(\widehat{M} \times \mathfrak{a}\big)/W \cong \widehat{G}_\lambda .$$

So the duals can be placed in an exact bijection with one another!

Mackey and Connes-Kasparov, Together

The above bijection is not a homeomorphism. It does not by itself explain the Connes-Kasparov isomorphism, and so far it is nothing more than a coincidence of parametrizations.

However, it may be considerably elaborated upon ...

Theorem

Let G be a complex reductive group. The continuous field of C^*-algebras $\{C_r^*(G_t)\}_{t\in\mathbb{R}}$ is assembled from constant fields of commutative C^*-algebras by Morita equivalences, extensions.

This implies both the Connes-Kasparov isomorphism in K-theory and a Mackey-type bijection (the bijection is the same as the one calculated above).

About the Proof of the Theorem for Complex Groups

The constant fields of commutative C^*-algebras that appear as building blocks in the theorem above may be understood as follows.

The irreducible representations of K may be partially ordered by their highest weights, and each irreducible tempered representation of G or G_0 has, upon restriction to K, a unique minimal representation K (which appears with multiplicity one).

The irreducible representations with a given minimal representation σ of K constitute a locally closed subset X_σ of the tempered dual. Being locally closed, X_σ is locally compact.

There is one constant field for each σ, and its fiber is $C_0(X_\sigma)$.

The rest of the proof involves the asymptotic analysis of matrix coefficient functions, but near $K \subseteq G$, not near infinity, as in the Harish-Chandra theory.

Afgoustidis's Theorem

I proved the theorem for complex groups around fifteen years ago. About five years ago, Alexandre Afgoustidis proved the theorem in full generality, for all real reductive groups, in this thesis:

Theorem

Let G be any real reductive group. The continuous field of C^*-algebras $\{C_r^*(G_t)\}_{t\in\mathbb{R}}$ is assembled from constant fields of commutative C^*-algebras by Morita equivalences, extensions.

The proof uses David Vogan's theory of minimal K-types in place of the (much simpler) highest weight ordering for complex groups used in my proof.

An interesting fact: Afgoustidis was inspired not by physics or by C^*-algebra K-theory, but by questions in mathematical biology!

Summary

So there is an actual, precise Mackey bijection for every real reductive group. It may be the most economical way to describe the tempered dual, for any G, and it is certainly the fastest way to prove the Connes-Kasparov isomorphism.

But important questions remain, most prominently the problem of understanding and proving the Mackey bijection conceptually, in a way that does not require a complete understanding of the tempered dual of G to begin with.

So far this remains a mystery, but there is some progress to report on understanding the continuous field $\{C_r^*(G_t)\}_{t\in\mathbb{R}}$ better, or at least differently, and perhaps this is a start ...

Recent Progress

Theorem (Joint work with Angel Roman)

Let G be a complex reductive group. There is a morphism of C^*-algebras

$$\alpha\colon C_r^*(G_0) \longrightarrow C_r^*(G)$$

such that

- *The continuous field $\{C_r^*(G_t)\}_{t\in\mathbb{R}}$ is the mapping cone field associated to this morphism, and*
- *For every irreducible tempered representation of G, the representation $\pi \circ \alpha$ of G_0 includes the corresponding representation of G_0 under the Mackey bijection as its lowest component.*

The theorem is presumably correct for all real groups (this is work in progress with Roman and Pierre Clare).

Thank You!